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Existence and multiplicity results for critical and subcritical p-fractional elliptic equations via Nehari manifold method

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Abstract. In this paper, we establish the multiplicity of nonnegative solutions to the following p-fractional Laplacian problem

$$\begin{cases} (-\Delta)_p^s u = f(x,u) + \lambda g(x,u) \text{ in } \Omega \ , u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded set in $\mathbb{R}^n, n > ps$ with $s \in (0,1), \lambda$ is a positive parameter, f, g are homogeneous positive functions of degrees q and r, respectively. Using fibering maps and the Nehari manifold, we obtain some results in subcritical and critical cases.

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1. Introduction

The aim of this article is to study the existence and multiplicity of non-negative of the following p-fractional laplacian equation

$$(1.1) \qquad \begin{cases} (-\Delta)_p^s u(x) = f(x,u) + \lambda g(x,u) \text{ in } \Omega \ , u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the p-fractional Laplacian operator, which may be defined as

$$(1.2) \qquad (-\Delta)_p^s u = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{n+ps}} dy, x \in \mathbb{R}^n.$$

Here Ω is a smooth bounded set in $\mathbb{R}^n, n > ps$ with $s \in (0,1), \lambda$ is a positive parameter, the exponents p, r and q satisfy $0 < r < 1 < q \leq p_s^* - 1$ with

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$p_s^* = \frac{np}{n-sp}$, the fractional Sobolev exponent, f, g are homogeneous positive functions of degrees q and r, respectively, that is, for all $(t, x, u) \in \mathbb{R}_*^+ \times \Omega \times \mathbb{R}$

$$\begin{cases} f(x, tu) = t^q f(x, u), \\ g(x, tu) = t^r g(x, u). \end{cases}$$

Furthermore, the primitive functions

$$\begin{cases} F(x, u) = \int_0^u f(x, s) ds \geq 0, \\ G(x, u) = \int_0^u g(x, s) ds \geq 0. \end{cases}$$

are homogeneous positive functions of degrees $(q+1)$ and $(r+1)$, respectively, that is, for every $(t, x, u) \in \mathbb{R}_*^+ \times \Omega \times \mathbb{R}$

$$\begin{cases} F(x, tu) = t^{q+1} F(x, u), \\ G(x, tu) = t^{r+1} G(x, u). \end{cases}$$

In addition, f and g lead to the so-called Euler identities

$$\begin{cases} (q+1)\, F(x, u) = u f(x, u), \\ (r+1)\, G(x, u) = u G(x, u), \end{cases}$$

and, for some constants $\gamma_1, \gamma_2 > 0$, we have, for all $(x, u) \in \Omega \times \mathbb{R}$.

$$\begin{cases} F(x, u) \leq \gamma_1 \left| u \right|^{q+1}, \\ G(x, u) \leq \gamma_2 \left| u \right|^{r+1}. \end{cases} \tag{1.3}$$

In this work, we will use the Nehari manifold method to obtain the multiplicity of solutions to the problem (1.1), for the following two cases

1 The subcritical and concave cases $(0 < r < 1 < p < q < p_s^* - 1)$.

2 The critical and concave cases $(0 < r < 1 < p < q = p_s^* - 1)$.

We introduce the functional space X that we will use in the following, as follows

$$X = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ is measurable, } u \in L^p(\Omega) \text{ and } \frac{u(x) - u(y)}{|x - y|^{\frac{n+ps}{p}}} \in L^p(\mathcal{D}, dxdy) \right\},$$

where $\mathcal{D} = \mathbb{R}^{2n} \diagdown (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm

$$\|u\|_X = \left(\|u\|^p_{L^p(\Omega)} + \int\limits_{\mathcal{D}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}} .$$

Through this paper we consider the Banach space X_0 as

$$X_0 = \{u \in X : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}, \tag{1.4}$$

with the norm

$$\|u\|_{X_0} = \left(\int\limits_{\mathcal{D}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}} . \tag{1.5}$$

For all $u, v \in X_0$, we have the duality product:

(1.6)

$$\mathcal{A}(u,v) = \left\langle (-\Delta)^s_p u, v \right\rangle_{X_0} = \int\limits_{\mathcal{D}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+ps}} dxdy.$$

In recent years, fractional spaces and the corresponding fractional problems arise in many different applications, such as phase transitions [2, 22], materials science [5], water waves [14, 15], conservation laws [6], minimal surfaces [11] and so on. Before giving our main results, let us briefly recall the literature concerning related nonlinear equations involving fractional powers of the Laplace operator. Problems involving fractional Laplace operator has been given considerable attention since they arise in many physical phenomena, in probability and also in finance. For more details see, for instance [10, 12, 24] and references therein.

There are many works on existence of a solution for fractional elliptic equations with regular nonlinearities as

$$\begin{cases} (-\Delta)^s u = \lambda u^p + u^q \text{ in } \Omega \ , u > 0, \\ \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.7}$$

where $N > 2s, 0 < s < 1, p, q > 0, \lambda > 0$. In [19, 20] the authors studied the existence and the multiplicity of non-negative solutions to the subcritical growth problems (1.7) and the critical exponent problems are studied in [18]. Problems similar to (1.1) have been also studied in the local setting with different elliptic operators. As far as we know, the first example in this direction was given in [17] for the p-Laplacian operator. Other results, this time for the Laplacian (or essentially the classical Laplacian) operator can be found in [1, 3, 7, 13]. More generally, the case of fully nonlinear operators has been studied in [12].

Definition 1. *We say that u is a weak solution of problem* (1.1)*, if u satisfies the weak formulation*

$$\mathcal{A}(u,v)=\int_{\Omega} f(x,u)v(x)dx+\lambda\int_{\Omega} g(x,u)v(x)dx.$$

The fact that u is a weak solution is equivalent to being a critical point of the following functional

$$J_\lambda(u)=\frac{1}{p}\|u\|_{X_0}^p-\int_{\Omega} F(x,u)dx-\lambda\int_{\Omega} G(x,u)dx. \tag{1.8}$$

To simplify the calculus, we put

$$P=P(u)=\|u\|_{X_0}^p, Q=Q(u)=\int_{\Omega} F(x,u)dx \text{ and } R=R(u)=\int_{\Omega} G(x,u)dx,$$

$$P_n=P(u_n), Q_n=Q(u_n) \text{ and } R_n=R(u_n).$$

Then, we can write J_λ as follows

$$J_\lambda(u)=\frac{1}{p}P-Q-\lambda R.$$

Also, $J_\lambda\in C^1(E,\mathbb{R})$ and $J_\lambda^{'}:E\to E^{'}$ is given by

$$\langle J_\lambda'(u),u\rangle=P-(q+1)Q-\lambda(r+1)R.$$

Our first result is about the subcritical and concave case.

Theorem 1. *Let $s\in(0,1), n>ps$, and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. If $0<r<1<p<q<p_s^*-1$, then there exists $\lambda_*>0$, such that for $\lambda\in(0,\lambda_*)$ problem* (1.1) *has at least two positive solutions.*

For the critical case $(q=p_s^*-1)$, since the embedding $X_0\hookrightarrow L^{p_s^*}(\mathbb{R}^n)$ is not compact, then the energy functional does not satisfy the Palais–Smale condition globally, but it is true for the energy functional in a suitable range related to the best fractional critical Sobolev constant, that we can define by the following expression:

$$\theta_p=\inf_{v\in X_0\setminus\{0\}}\frac{\|v\|_{X_0}^p}{\|v\|_{L^{p_s^*}}^p}. \tag{1.9}$$

Theorem 2. *Let $s\in(0,1), n>ps$, and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $0<r<1<p<q=p_s^*-1$ and assume that there exists $u_0\in X_0\backslash\{0\}$ with $u_0>0$ in $\mathbb{R}^n$, such that*

$$\left(\frac{1}{p}P_0t^p-t^{p_s^*}Q_0\right)<\frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}}\theta_p^{\frac{n}{sp}}. \tag{1.10}$$

Then there exists $\lambda^*>0$ such that, for all $\lambda\in(0,\lambda^*)$, problem (1.1) *has at least two positive solutions.*

Remark 1. *The condition* (1.10) *can be guaranteed by results of [[4], Lemma 2.11], or [[21], Section 4.2].*

This paper is organized as follows. In Section 2, we give some notations and preliminaries about the Nehari manifold and fibering maps. In Section 3, we consider the subcritical an concave case, and give the proof of Theorem 1. In Section 4, we deal with the critical and concave case and prove Theorem 2.

2. Nehari manifold and fibering maps analysis

To prove Theorem 1 and Theorem 2, we will consider critical points of the function J_λ on the Banach space X_0. A good candidate for an appropriate subset of X is the so-called Nehari manifold

$$\mathcal{N}_\lambda = \{u \in X_0 \backslash \{0\}, \langle J'_\lambda(u), u\rangle_{X_0} = 0\}.$$

All critical points of J_λ must lie on $\mathcal{N}_\lambda$ as we will see below, local minimizers on $\mathcal{N}_\lambda$ are usually critical points of J_λ, and $\mathcal{N}_\lambda \subset X_0$. So we see that $u \in \mathcal{N}_\lambda$ if and only if

$$P - (q+1)Q - \lambda(r+1)R = 0. \tag{2.1}$$

It is useful to understand $\mathcal{N}_\lambda$ in terms of the stationary points of mappings of the form $\varphi_u : t \mapsto J_\lambda(tu), \forall t > 0$,

$$\varphi_u(t) = \frac{1}{p}Pt^p - Qt^{q+1} - \lambda Rt^{r+1}.$$

Such maps are known as fibering maps and were introduced by Drabek and Pohozaev in [16] and are also discussed by Brown and Zhang [10], Brown and Wu [9].

Lemma 1. *Let $u \in X_0\backslash\{0\}$, then $tu \in \mathcal{N}_\lambda$ if and only if $\varphi'_u(t) = 0$.*

Proof. The result is an immediate consequence of the fact that

$$\varphi'_u(t) = \langle J'_\lambda(tu), u\rangle_{X_0} = \frac{1}{t}\langle J'_\lambda(tu), tu\rangle_{X_0}.$$

$\square$

Thus points in $\mathcal{N}_\lambda$ correspond to stationary points of the maps φ_u and so it is natural to divide $\mathcal{N}_\lambda$ into three subsets $\mathcal{N}^+_\lambda, \mathcal{N}^+_\lambda$ and $\mathcal{N}^0_\lambda$ corresponding to local minima, local maxima and points of inflexion of fibering maps. We have

$$\varphi'_u(t) = t^{p-1}P - (q+1)t^qQ - \lambda(r+1)t^rR, \tag{2.2}$$

$$\varphi''_u(t) = (p-1)\,t^{p-2}P - q(q+1)t^{q-1}Q - \lambda r(r+1)t^{r-1}R. \tag{2.3}$$

$u \in \mathcal{N}_\lambda$ if and only if $\varphi'_u(1) = 0$. Hence, by (2.2) we get

$$\begin{aligned} \varphi''_u(1) &= (p-1)\,P - q(q+1)Q - \lambda r(r+1)R \\ &= (p-q-1)(q+1)Q + \lambda(r+1)(p-r-1)R \\ &= (p-q-1)P + \lambda(r+1)(q-r)R \\ &= (p-r-1)P - \lambda(q+1)(r-q)Q. \end{aligned} \tag{2.4}$$

Hence, we define

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \left\{u \in \mathcal{N}_\lambda : \varphi''_u(1) > 0\right\}, \\ \mathcal{N}_\lambda^- &= \left\{u \in \mathcal{N}_\lambda : \varphi''_u(1) < 0\right\}, \\ \mathcal{N}_\lambda^0 &= \left\{u \in \mathcal{N}_\lambda : \varphi''_u(1) = 0\right\}. \end{aligned}$$

Lemma 2. *Suppose that u_0 is a local minimizer for J_λ on N_λ and that $u_0 \notin N_\lambda$, then u_0 is a critical point of J_λ.*

Proof. If u_0 is a local minimizer for J_λ on $\mathcal{N}_\lambda$, then u_0 is a solution of the minimization problem:

$$\begin{cases} \min J_\lambda(u) = J_\lambda(u_0), \\ \beta(u_0) = 0, \end{cases}$$

where

$$\beta(u) = P - (q+1)Q - \lambda(r+1)R.$$

Hence, by the theorem of Lagrangian multipliers, $\exists \delta \in \mathbb{R}$ such that

$$J'(u_0) = \delta\beta'(u_0),$$

Thus

$$\langle J'(u_0), u_0\rangle_{X_0} = \delta\langle\beta'(u_0), u_0\rangle_{X_0}. \tag{2.5}$$

Since $u_0 \in \mathcal{N}_\lambda$ and so

$$\langle\ J'(u_0), u_0\rangle_{X_0} = 0.$$

Hence

$$P_0 = (q+1)Q_0 - \lambda(r+1)R_0. \tag{2.6}$$

On the other hand,

$$\langle \beta'(u_0), u_0\rangle_{X_0} = (p-1)P_0 - q(q+1)Q_0 - \lambda r(r+1)R_0 = \varphi''_{u_0}(1).$$

Using the fact that $u_0 \notin \mathcal{N}_\lambda^0 \Leftrightarrow \varphi''_{u_0}(1) \neq 0 \Leftrightarrow \langle \beta'(u_0), u_0\rangle_{X_0} \neq 0$, by (2.5), we obtain $\delta = 0$, then $J'_\lambda(u_0) = 0$. □

In order to understand the Nehari manifold and fibering maps, let us consider the function $\psi_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_u(t) = t^{p-r-1}P - (q+1)t^{q-r}Q. \tag{2.7}$$

It is clear that, for $t > 0, tu \in N_\lambda$ if and only if

$$\psi_u(t) = \lambda(r+1)R. \tag{2.8}$$

Moreover,

$$\psi'_u(t) = (p-r-1)t^{p-r-2}P - (q+1)(q-r)t^{q-r-1}Q, \tag{2.9}$$

so we can see that if $tu \in \mathcal{N}_\lambda$, then

$$t^r\psi'_u(t) = \varphi''_u(t). \tag{2.10}$$

Hence, $tu \in \mathcal{N}_\lambda^+$ $(\mathcal{N}_\lambda^-)$ if and only if $\psi'_u(t) > 0$ $(\psi'_u(t) \leq 0)$.

Suppose that $u \in X_0$ and $u \neq 0$. Starting from (2.7), ψ_u satisfies the following properties

(a) ψ_u has a single critical point at

$$t_{\max}(u) = \left[\frac{(p-r-1)}{(q+1)(q-r)}\frac{P}{Q}\right]^{\frac{1}{q+1-p}}.$$

(b) $\lim\limits_{t\to\infty} \psi_u(t) = -\infty$.

(c) ψ_u is strictly increasing on $(0, t_{\max}(u))$ and strictly decreasing on $(t_{\max}(u), +\infty)$.

According to (1.3), (1.9) and Hölder inequality, we get

$$Q \leq \gamma_1 |\Omega|^{\frac{p_s^*-q-1}{p_s^*}} \|u\|^{q+1}_{L^{p_s^*}} \leq \gamma_1\theta_p^{-\frac{q+1}{p}} |\Omega|^{\frac{p_s^*-q-1}{p_s^*}} P^{\frac{q+1}{p}}. \tag{2.11}$$

Likewise,

$$R \leq \gamma_2\theta_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^*-r-1}{p_s^*}} P^{\frac{r+1}{p}}. \tag{2.12}$$

Moreover,

$$\begin{aligned} &\psi_u(t_{\max}) - \lambda(r+1)R \\ &= \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q+1-p}} \left(\frac{q+1-p}{q-r}\right) P^{\frac{q-r}{q+1-p}} Q^{-\frac{p-r-1}{q+1-p}} - \lambda(r+1)R \\ &\geq \left((r+1)\gamma_2\theta_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^*-r-1}{p_s^*}}\right)(\lambda_* - \lambda)P^{\frac{r+1}{p}}, \end{aligned} \tag{2.13}$$

such that
(2.14)

$$\lambda_* = \frac{1}{\gamma_2}\left(\frac{q+1-p}{(q-r)\,(r+1)}\right)\left(\frac{p-r-1}{(q+1)(q-r)\gamma_1}\right)^{\frac{p-r-1}{q+1-p}}\left(\theta_p\,|\Omega|^{\frac{p-p_s^*}{p_s^*}}\right)^{\frac{q-r}{q+1-p}}.$$

Moreover, by (2.8) and (2.13), if $\lambda < \lambda_*$, then $\varphi_u'(t) > 0$. It seems that $(\varphi_u'(t) < 0)$ when λ is large.

Therefore $tu \in \mathcal{N}_\lambda$, for all $t > 0$. On the other hand, if λ satisfies

$$0 < \lambda(r+1)R < \psi_u(t_{\max}(u)), \tag{2.15}$$

then there exist t_1 and t_2 with $t_1 < t_{\max}(u) < t_2$, such that:

$$\psi_u(t_1) = \psi_u(t_2) = \lambda(r+1)R, \psi_u'(t_1) > 0 \text{ and } \psi_u'(t_2) < 0. \tag{2.16}$$

From (2.2) and (2.8), we have $\varphi_u'(t_1) = \varphi_u'(t_2) = 0$. By (2.10), we obtain $\varphi_u''(t_1) > 0$ and $\varphi_u''(t_2) < 0$.

These facts imply that the fibering map φ_u has a local minimum at t_1 and a local maximum at t_2 such that $t_1u \in \mathcal{N}_\lambda^+$ and $t_2u \in \mathcal{N}_\lambda^-$.

Lemma 3. *For all $\lambda \in (0, \lambda_*)$, we have $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. Suppose that $\mathcal{N}_0^\lambda \neq \emptyset$, then there exists $u_0 \in \mathcal{N}_\lambda^0$ such that $\varphi_{u_0}^{'}(1) = 0$ and $\varphi_{u_0}^{''}(1) = 0$, namely,

$$(p-r-1)P_0 + \lambda(q+1)(r-q)Q_0 = 0.$$

Then, we have

$$Q_0 = \frac{(r-p+1)}{\lambda(q+1)(r-q)}P_0,$$

consequently,

$$\begin{aligned} 0 &= \varphi_{u_0}^{'}(1) = P_0 - (q+1)Q_0 - \lambda(r+1)R_0 \\ &= P_0 - \frac{r+1-p}{r-q}P_0 - \lambda(r+1)R_0 \\ &= \frac{p-q-1}{r-q}P_0 - \lambda(r+1)R_0, \end{aligned}$$

which implies that

$$R_0 = \frac{(p-q-1)}{\lambda\,(r-q)\,(r+1)}P_0. \tag{2.17}$$

Therefore,

$$
\begin{aligned}
&\psi_{u_0}(t_{\max}) - \lambda(r+1)R_0 \\
&= \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q-p+1}} \left(\frac{q-p+1}{q-r}\right)\left(\frac{P_0^{q-r}}{Q_0^{p-r-1}}\right)^{\frac{1}{q-p}} - \lambda(r+1)R_0 \\
&= \left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{p-r-1}{q-p+1}} \left(\frac{q-p+1}{q-r}\right)\left(\frac{p-r-1}{(q+1)(q-r)}\right)^{\frac{r-p+1}{q-p+1}} P_0 \\
&\quad - \left(\frac{p-q-1}{r-q}\right) P_0 \\
&= 0.
\end{aligned} \tag{2.18}
$$

So $\psi_{u_0}(t_{\max}) - \lambda(r+1)R_0 = 0$ is a contradiction, then for any $\lambda \in (0, \lambda_*)$, we have $\mathcal{N}_\lambda^0 = \emptyset$. $\square$

Lemma 4. *J_λ is coercive and bounded from below on $\mathcal{N}_\lambda, \forall \lambda \in (0, \lambda_*)$.*

Proof. If $u \in \mathcal{N}_\lambda$, from (2.1) we have

$$Q = \frac{1}{q+1}\left(P - \lambda(r+1)R\right).$$

Therefore,

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{q+1}\right) P - \lambda\left(\frac{q-r}{q+1}\right) R.$$

By using (2.12), we obtain

$$J_\lambda(u) \geq \left(\frac{1}{p} - \frac{1}{q+1}\right) P - \gamma_2 \theta_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^*-r-1}{p_s^*}} P^{\frac{r+1}{p}}.$$

Then we conclude that the functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. $\square$

By Lemmas 1 and 2, for any $\lambda \in (0, \lambda_*)$, we know that $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$. Therefore, we define

$$\alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) \text{ and } \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u).$$

Proposition 1. *In the case where $0 < r < 1 < q < p_s^* - 1$, if $\lambda \in (0, \lambda_*)$, the functional J_λ has a minimizer u_1 in $\mathcal{N}_\lambda^+$ and satisfies*

(1) $J_\lambda(u_1) = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) \leq 0$.

(2) u_1 is a solution of problem (1.1).

Proof. Since J_λ is bounded from below on $\mathcal{N}_\lambda^+$, so there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u).$$

Thus, by Lemma 4, the sequence $\{u_k\}$ is bounded in X_0, there exists $u_1 \in X_0$ such that

$$P_k \to P_1, \text{ as } k \to \infty.$$

Moreover, by [[19], Lemma 8], up to a subsequence,

$$u_k \to u_1 \text{ in } L^\sigma(\mathbb{R}^n), u_k \to u_1 \text{ in } \mathbb{R}^n \text{ as } k \to \infty.$$

And by [[8], Theorem IV-9], there exists $l \in L^\sigma(\mathbb{R}^n)$ such that

$$|u_k(x)| \le l(x) \text{ in } \mathbb{R}^n,$$

for any $1 \le \sigma < p_s^* \, (n > sp)$. Therefore, by dominated convergence theorem, we have that

$$\begin{cases} Q_k \to Q_1, \\ \\ R_k \to R_1, \end{cases} \tag{2.19}$$

as $k \to \infty$, because

$$\begin{cases} Q_k < \gamma_1 \|u_k\|_{L^{q+1}}^{q+1}, \\ \\ R_k < \gamma_2 \|u_k\|_{L^{r+1}}^{r+1}. \end{cases}$$

Moreover, there exists t_1 such that, $t_1 u_1 \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1 u_1) < 0$. Hence, we have $\inf\limits_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0$.

Next, we show that $u_k \to u_1$ strongly in X_0. If not, then $\|u_1\|_{X_0} < \liminf\limits_{k\to\infty} \|u_k\|_{X_0}$. Thus, for $\{u_k\} \subset \mathcal{N}_\lambda^+$ we get

$$\begin{aligned} \lim_{k\to\infty} \varphi'_{u_k}(t_1) &= \lim_{k\to\infty} \left[t_1^{p-1} P_k - (q+1) t_1^q Q_k - \lambda(r+1) t_1^r R_k \right] \\ &> t_1^{p-1} P_1 - (q+1) t_1^q Q_1 - \lambda(r+1) t_1^r R_1 \\ &= \varphi'_{u_1}(t_1) = 0. \end{aligned}$$

That is, $\varphi'_{u_1}(t_1) > 0$ for k large enough. Since $u_k = 1.u_k \in \mathcal{N}_\lambda^+$, we can see that $\varphi'_{u_k}(t_1) < 0$ for $t \in (0, t_1)$, and $\varphi'_{u_k}(1) = 0$ for all k. Then, we must have $t_1 > 1$. On the other hand, $\varphi_{u_1}(t)$ is decreasing on $(0, t_1)$, then we get

$$J_\lambda(t_1 u_1) \le J_\lambda(u_1) < \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),$$

which is a contradiction. Hence, $u_k \to u_1$ strongly in X_0. This implies that $J_\lambda(u_k) \to J_\lambda(u_1) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u)$ as $k \to \infty$.

Namely, u_1 is a minimizer of J_λ on $\mathcal{N}_\lambda^+$. Using Lemma 2, u_1 is a solution of (1.1). □

Proposition 2. *In the case where* $0 < r < 1 < q < p_s^* - 1$, *if* $\lambda \in (0, \lambda_*)$, *the functional* J_λ *has a minimizer* u_2 *in* $\mathcal{N}_\lambda^-$ *and satisfies*

(1) $J_\lambda(u_2) = \inf\limits_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) > 0.$

(2) u_2 *is a solution of problem* (1.1).

Proof. Since J_λ is bounded from below on $\mathcal{N}_\lambda^-$ there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^-$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u).$$

By the same argument given in the proof of Proposition 1, there exists $u_2 \in X_0$ such that, up to a subsequence,

$$P_k \to P_2, Q_k \to Q_2 \text{ and } R_k \to R_2, \text{ as } k \to \infty,$$

Moreover, from the analysis of the fibering maps $\varphi_u(t)$, we know that there exists t_1, t_2 with $t_1 < t_{\max}(u) < t_2$ such that $t_1 u \in \mathcal{N}_\lambda^+, t_2 u \in \mathcal{N}_\lambda^-$ and $J_\lambda(t_1 u) < J_\lambda(tu) < J_\lambda(t_2 u)$. Next, we show that $u_k \to u_2$ strongly in X_0. If not, then $\|u_2\|_{X_0} < \liminf\limits_{k\to\infty} \|u_k\|_{X_0}$. Thus, for $\{u_k\} \subset \mathcal{N}_\lambda^-$, we have $J_\lambda(u_k) > J_\lambda(tu_k)$ for all $t > t_{\max}$, and

$$\begin{aligned}
J_\lambda(t_2 u_2) &= \frac{t_2^p}{p} P_2 - t_2^{q+1} Q_2 - \lambda t_2^{r+1} R_2 \\
&< \lim_{k\to\infty} (\frac{t_2^p}{p} P_k - t_2^{q+1} Q_k - \lambda t_2^{r+1} R_k) \\
&= \lim_{k\to\infty} J_\lambda(t_2 u_k) \le J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u),
\end{aligned}$$

which is a contradiction. Hence, $u_k \to u_2$ strongly in X_0. This implies

$$J_\lambda(u_k) \to J_\lambda(u_2) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u), k \to \infty.$$

Namely, u_2 is a minimizer if J_λ on $\mathcal{N}_\lambda^-$. Using Lemma 2, u_2 is a solution of problem (1.1). □

Proof of Theorem 1.1. By Propositions 1, 2 and Lemma 2, we get that problem (1.1) has two solutions $u_1 \in \mathcal{N}_\lambda^+$ and $u_2 \in \mathcal{N}_\lambda^-$ on X_0. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, then those two solutions are distinct. This finishes the proof. □

Proposition 3. *In the case where* $0<r<1<q=p_s^*-1$. *Let* $\{u_k\}\subset X_0$ *be a* $(PS)_c$ *sequence for* J_λ *with*

$$c<\frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}}\theta_p^{\frac{n}{ps}}-M\lambda^{\frac{p}{p-r-1}}, \tag{2.20}$$

then, there exists a subsequence of $\{u_k\}$, *which converges strongly in* X_0, *where* θ_p *is defined in* (1.2) *and* M *is a positive constant given by*

$$M=\left(\frac{p-r-1}{p}\right)\left(\frac{(r+1)\left(p_s^*-r-1\right)}{\left(p_s^*-p\right)\theta_p}\right)^{\frac{r+1}{p-r-1}}\left(\gamma_2\left|\Omega\right|^{\frac{p_s^*-r-1}{p_s^*}}\right)^{\frac{p}{p-r-1}}. \tag{2.21}$$

Proof. From Lemma 4, we see that $\{u_k\}$ is bounded in X_0. Then, up to a sequence, still denoted by $\{u_k\}$, there exists $u_*\in X_0$ such that $u_k\to u_*$ weakly in X_0 , that is

$$P_k\to P_*,\ \text{as } k\to\infty.$$

Moreover, by [[19], Lemma 8], we have that

$$u_k\to u_*\text{ weakly in } L^{p_s^*}\left(\mathbb{R}^n\right), u_k\to u_*\text{ in } L^{r+1}(\mathbb{R}^n), u_k\to u_*\text{ in } \mathbb{R}^n,$$

as $k\to\infty$, and by [[8], Theorem IV-9], there exists $l\in L^{r+1}(\mathbb{R}^n)$ such that:

$$|u_k(x)|\le l(x)\text{ in } \mathbb{R}^n,$$

for any $1\le r+1<p_s^*$. Therefore, by dominated convergence theorem, we have that

$$R_k\longrightarrow R_*,\ \text{as } k\to\infty.$$

By Brezis-Lieb Lemma [[23], Lemma 1.32], we get

$$\begin{aligned} P_k &= P(u_k-u_*)+P_*+o(1),\\ Q_k &= Q\left(u_k-u_*\right)+Q_*+o(1).\end{aligned}$$

Then,

$$\begin{aligned}&\langle J_\lambda'(u_k),u_k\rangle_{X_0}\\ &= P_k-p_s^*Q_k-\lambda(r+1)R_k\\ &= P(u_k-u_*)+P_*-p_s^*(Q\left(u_k-u_*\right)+Q_*)-\lambda(r+1)R_k+o(1)\\ &= \langle J_\lambda'(u_*),u_*\rangle_{X_0}+P(u_k-u_*)-p_s^*Q\left(u_k-u_*\right).\end{aligned}$$

By $\langle J_\lambda'(u_*),u_*\rangle_{X_0}=0$ and $\langle J_\lambda'(u_k),u_k\rangle_{X_0}\longrightarrow 0$ as $k\longrightarrow\infty$,, we know that

$$P(u_k-u_*)\longrightarrow b\text{ and } p_s^*Q\left(u_k-u_*\right)\longrightarrow b. \tag{2.22}$$

If $b = 0$, the proof is complete.
Assuming $b > 0$, by (2.11), we get

$$p_s^* Q\left(u_k - u_*\right) \leq p_s^* \gamma_1 \theta_p^{-\frac{p_s^*}{p}} \left(P(u_k - u_*)\right)^{\frac{p_s^*}{p}}.$$

Then

$$b \geq \left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}}.$$

On the other hand, we have

$$\begin{aligned}
c &= \lim_{k \longrightarrow \infty} \left(\frac{1}{p} P_k - Q_k - \lambda R_k\right) \\
&= \lim_{k \longrightarrow \infty} \left(\frac{1}{p} P(u_k - u_*) - Q\left(u_k - u_*\right) - \frac{1}{p} P_* - Q_* - \lambda R_k\right) + o(1) \\
&= J_\lambda\left(u_*\right) + b\left(\frac{1}{p} - \frac{1}{p_s^*}\right) \geq J_\lambda(u_*) + \frac{s}{n}\left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}}.
\end{aligned}$$

By the assumption that $c < \frac{s}{n}\left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}}$, we obtain $J_\lambda(u_*) < 0$. In particular, $u_* \neq 0$, and

$$Q_* > \frac{1}{p} P_* - \lambda R_*. \tag{2.23}$$

Then,

$$\begin{aligned}
c &= \lim_{k \longrightarrow \infty} J_\lambda(u_k) = \lim_{k \longrightarrow \infty} \left(J_\lambda(u_k) - \frac{1}{p}\left\langle J_\lambda'(u_k), u_k\right\rangle_{X_0}\right) \\
&= \lim_{k \longrightarrow \infty} \left(\frac{p_s^*}{p} - 1\right)\left(Q\left(u_k - u_*\right)\right) + Q_* - \lambda\left(\frac{p - r - 1}{p}\right) R_k \\
&= \frac{sp_s^*}{n}\left(Q\left(u_k - u_*\right) + Q_*\right) - \lambda\left(\frac{p - r - 1}{p}\right) R_* \\
&\geq \frac{s}{n}\left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} + \frac{sp_s^*}{n} Q_* - \lambda\left(\frac{p - r - 1}{p}\right) R_*.
\end{aligned}$$

Using (2.11) and (2.23), we obtain

$$c \geq \frac{s}{n}\left(p_s^* \gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - \lambda \gamma_2 \theta_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^* - r - 1}{p_s^*}} \left(\frac{p_s^* - r - 1}{p}\right) P_*^{\frac{r+1}{p}} + \frac{sp_s^*}{np} P_*.$$

We denote: $\eta = P_*^{\frac{1}{p}}$, and we define the function $h(\eta)$ as follow

$$h(\eta) = \lambda \gamma_2 \theta_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^* - r - 1}{p_s^*}} \left(\frac{p_s^* - r - 1}{p}\right) \eta^{r+1} - \frac{sp_s^*}{np} \eta^p.$$

We note that $h(\eta)$ attaints its maximum at

$$\eta_0 = \left(\frac{\lambda n(r+1)\left(p_s^* - r - 1\right)\gamma_2}{spp_s^*} \theta_p^{-\frac{r+1}{p}} |\Omega|^{\frac{p_s^*-r-1}{p_s^*}} \right)^{\frac{1}{p-r-1}},$$

and

$$h(\eta_0) = \sup_{\eta>0} h\left(\eta\right) = -M\lambda^{\frac{p}{p-r-1}},$$

where M is defined in (2.21). Then,

$$c \geq \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r-1}}.$$

Thus, we get a contradiction with our hypothesis. Hence, $b = 0$ and we conclude that $u_k \to u_*$ strongly in X_0. This completes the proof. □

Proposition 4. *There exists $\lambda^* > 0$ and $u_0 \in X_0$ such that, for all $\lambda \in (0, \lambda^*)$, we have*

$$\sup_{t>0} J_\lambda(tu_0) \leq \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r-1}}. \tag{2.24}$$

In particular,

$$\alpha_\lambda^- < \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r-1}}. \tag{2.25}$$

Proof. First, replacing the values of λ_* and M, we get

$$\begin{aligned} & \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r-1}} \\ & > \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - M\lambda_*^{\frac{p}{p-r-1}} \\ & > \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} \left(1 - \frac{1}{r+1}\left(\frac{p-r-1}{p_s^*-r-1}\right)^{\frac{n}{sp}}\right) > 0, \end{aligned}$$

because

$$0 < \frac{1}{r+1}\left(\frac{p-r-1}{p_s^*-r-1}\right)^{\frac{n}{sp}} < 1.$$

Then

$$\lambda < \lambda_{**} = \left(\frac{s}{nM}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}}\right)^{\frac{p-r-1}{p}}$$

By condition (1.10), $\exists u_0 \in X_0 \setminus \{0\}$ such that, for $t \geq t_0 > 0$, we have

$$\begin{aligned} J_\lambda(tu_0) &\leq \sup_{t>0}\left(\frac{1}{p}P_0 t^p - t^q Q_0 - \lambda t^{r+1} R_0\right) \\ &< \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - \lambda t_0^{r+1} R_0. \end{aligned} \tag{2.26}$$

Let $\lambda_{***} = \left(\frac{t_0^{r+1}R_0}{M}\right)^{\frac{p_s^*-r-1}{1+r}}$. For all $\lambda \in (0, \lambda_{***})$ we define

$$-\lambda t_0^{r+1} R_0 < -M\lambda^{\frac{p}{p-r-1}}.$$

Thus, we obtain that (2.24) holds. Finally, we set $\lambda^* = \min\{\lambda_*, \lambda_{**}, \lambda_{***}\}$; by the analysis of fibering maps $\varphi_u(t) = J_\lambda(tu)$, we get

$$\alpha_\lambda^- < \frac{s}{n}\left(p_s^*\gamma_1\right)^{\frac{-n}{sp_s^*}} \theta_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r-1}},$$

for $\lambda \in (0, \lambda_2)$. This completes the proof. $\square$

Proof of Theorem 1.2. By Propositions 3 and 4, there exist two sequences $\{u_k^+\}$ and $\{u_k^-\}$ in X_0, such that

$$\begin{aligned} J_\lambda(u_k^+) &\longrightarrow \alpha_\lambda^+, J'_\lambda(u_k^+) \longrightarrow 0, \\ &\text{and} \\ J_\lambda(u_k^-) &\longrightarrow \alpha_\lambda^-, J'_\lambda(u_k^-) \longrightarrow 0. \end{aligned}$$

as $k \longrightarrow \infty$.

We observe that from the analysis of fibering maps $\varphi_u(t)$, we have $\alpha_\lambda^+ < 0$. Similar to the proof of Propositions 1 and 2 and Theorem 2, problem (1.1) has two solutions $u_1 \in \mathcal{N}_\lambda^+$, $u_2 \in \mathcal{N}_\lambda^-$ in X_0 and since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- \neq \emptyset$, then these two solutions are distinct. This finishes the proof. $\square$

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