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Conformal Ricci soliton in para-Sasakian manifolds

Shyam Kishor[1], **Pushpendra Verma**[23]

Abstract. The object of the present paper is to study M-projective curvature tensor, pseudo projective curvature tensor, Ricci curvature tensor in para-Sasakian manifold admitting conformal Ricci soliton. We have studied M-projective semi symmetric para-Sasakian manifolds admitting a conformal Ricci soliton. We have found that an M-projective Ricci symmetric para-Sasakian manifold admitting a conformal Ricci soliton is a quadratic equation. We have proved that a pseudo projective semi symmetric para-Sasakian manifold admitting a conformal Ricci soliton is an η-Einstein manifold. We have also studied Ricci pseudo projectively symmetric para-Sasakian manifold.

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1. Introduction

The Ricci flow concept and its proof was introduced by Hamilton [9] in the year 1982. It was degined to answer the Thurston geometric conjecture, according to it each closed three manifold admits a geometric decomposition. Categorization of all compact manifolds with positive curvature operator in fourth dimension was done by Hamilton [5]. After which, the Ricci flow became one of the powerful tools to study Riemannian manifolds, especially in the manifolds having positive curvatures.

The Ricci flow is presented as

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

for a compact Riemannian manifold $\widetilde{M}$ with Riemannian metric g. Ricci soliton has come as the limit of the solutions of Ricci flow. The solution for the Ricci flow is known as a Ricci soliton in case it moves only by a one parameter group of diffeomorphism and scaling. Ramesh Sharma [6] began the study of the Ricci soliton for a compact manifold and later it was studied by M. M. Tripathi [3],

[1]Department of Mathematics and Astronomy, University of Lucknow, India
e-mail: skishormath@gmail.com

[2]Department of Mathematics and Astronomy, University of Lucknow, India
e-mail: pushpendra140690@gmail.com

[3]Corresponding author

Bejan, Crasmareanu [2] analysed the Ricci soliton in contact metric manifolds. The Ricci soliton equation is presented as

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is a Ricci tensor, g is a Riemannian metric, X is a vector field and λ is a scalar.

A.E. Fischer [4] in the year 2005 established a new concept known as conformal Ricci flow, a variation of the classical Ricci flow equation which has revised the unit volume constrain of that equation to a scalar curvature constraint. After that a conformal geometry has played a prevalent role to constraint the scalar curvature and equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the equations resulting from this are known as conformal Ricci flow equations. The new equations are presented as

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \tag{1.3}$$

and $R(g) = -1$, where p is a scalar non-dynamical field (time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

N. Basu and A. Bhattacharyya [1] in 2015 brought the notion of conformal Ricci soliton and the equation given as follows

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.4}$$

The above equation is the generalization of the Ricci soliton equation and it also assures the conformal Ricci flow equation.

A Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is Levi-Civita connection on the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi symmetric spaces and their generalization. A Riemannian manifold is said to be semi symmetric if its curvature tensor R satisfies $R(X,Y).R = 0$ for all $X, Y \in T\widetilde{M}$, where $R(X,Y)$ acts on R as a derivation.

In this paper, we have studied M-projective curvature tensor, pseudo projective curvature tensor, Ricci curvature tensor in a para-Sasakian manifold admitting a conformal Ricci soliton. We have studied para-Sasakian manifold admitting a conformal Ricci soliton and $R(\xi, X).\widetilde{Q} = 0$, and proved that it is an η-Einstein manifold. We have proved that a para-Sasakian manifold admitting a conformal Ricci soliton and $\widetilde{Q}(\xi, X).S = 0$, must be a quadratic equation. We have found that a para-Sasakian manifold admitting a conformal Ricci soliton and $R(\xi, X).\widehat{V} = 0$, is an η-Einstein manifold. We have studied a para-Sasakian manifold admitting a conformal Ricci soliton and $\widehat{V}(\xi, X).S = 0$, and proved that it is an η-Einstein manifold.

2. Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$ dimensional connected almost metric manifold with an almost contact metric structure (φ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ is a covariant vector field, η is a 1-form and g is compatible Riemannian metric such that

$$\varphi^2(X) = X - \eta(X)\xi, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta o \varphi = 0, \tag{2.3}$$

$$\eta(\varphi X) = 0, \; rank(\varphi) = 2n, \tag{2.4}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.5}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$

for all $X, Y \in \chi(\widetilde{M})$.

If (φ, ξ, η, g) satisfy the equations

$$d\eta = 0, \quad \nabla_X \xi = -\varphi X$$

$$(\nabla_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

then $\widetilde{M}$ is called a para-Sasakian manifold or, briefly, a P-Sasakian manifold. Especially, a P-Sasakian manifold $\widetilde{M}$ is called a special para-Sasakian manifold or briefly a SP Sasakian manifold if $\widetilde{M}$ admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.7}$$

It is known that in a para-Sasakian manifold the following relations hold [8]:

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X, \tag{2.8}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.12}$$

$$S(X, Y) = g(QX, Y), \tag{2.13}$$

for all $X, Y \in \chi(\widetilde{M})$, where R is a Riemannian curvature, S is the Ricci tensor and Q is the Ricci operator.

Now from the definition of a Lie derivative, we have

$$(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi).$$

Using (2.3)and (2.7), we have

$$(\pounds_\xi g)(X, Y) = 0. \tag{2.14}$$

Applying conformal Ricci soliton equation (1.4) in (2.14), we obtain

$$S(X, Y) = Ag(X, Y), \tag{2.15}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{n})]$, which shows that manifold is an Einstein manifold. Also

$$QX = AX, \tag{2.16}$$

$$S(X, \xi) = A\eta(X), \tag{2.17}$$

$$S(\xi, \xi) = A. \tag{2.18}$$

Using these results, we shall prove some important results on para-Sasakian manifold in the following sections.

3. Para-Sasakian manifold admitting a conformal Ricci soliton and $R(\xi, X).\widetilde{Q} = 0$

Let $\widetilde{M}$ be a $(2n+1)$ dimensional para-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ), where g is a Riemannian metric, V is a vector field and λ is scalar. The M-projective curvature tensor $\widetilde{Q}$ on $\widetilde{M}$ is defined by [7]

$$\begin{aligned} \widetilde{Q}(X, Y)Z &= R(X, Y)Z - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY]. \end{aligned} \tag{3.1}$$

Now we prove the following theorem.

Theorem 1. *If a para-Sasakian manifold admits a conformal Ricci soliton and is M-projective semi symmetric i.e., $R(\xi, X).\widetilde{Q} = 0$, then the manifold is an η-Einstein manifold where $\widetilde{Q}$ is an M-projective curvature tensor and $R(\xi, X)$ is the derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let $\widetilde{M}$ be a $(2n+1)$ dimensional para-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Putting $Z = \xi$ in (3.1), we have

$$\begin{aligned} \widetilde{Q}(X,Y)\xi &= R(X,Y)\xi - \frac{1}{4n}[S(Y,\xi)X - S(X,\xi)Y \\ &\quad + g(Y,\xi)QX - g(X,\xi)QY], \end{aligned} \tag{3.2}$$

using (2.2), (2.10), (2.16) and (2.17) in (3.2), we get

$$\widetilde{Q}(X,Y)\xi = (1 + \frac{2A}{4n})[\eta(X)Y - \eta(Y)X], \tag{3.3}$$

considering

$$b_1 = 1 + \frac{2A}{4n},$$

therefore, equation (3.3) becomes

$$\widetilde{Q}(X,Y)\xi = b_1[\eta(X)Y - \eta(Y)X], \tag{3.4}$$

and

$$g(\widetilde{Q}(X,Y)\xi, Z) = b_1[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)], \tag{3.5}$$

which implies

$$\eta(\widetilde{Q}(X,Y)Z) = b_1[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.6}$$

Now we recall that the para-Sasakian manifold admits a conformal Ricci soliton and is M-projective semi symmetric i.e., $R(\xi, X).\widetilde{Q} = 0$ holds in $\widetilde{M}$, which implies

$$\begin{aligned} & R(\xi,X)(\widetilde{Q}(Y,Z)W) - \widetilde{Q}(R(\xi,X)Y,Z)W \\ & -\widetilde{Q}(Y,R(\xi,X)Z)W - \widetilde{Q}(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{3.7}$$

for all vector field X, Y, Z on $\widetilde{M}$.

Using (2.9) in (3.7) and putting $W = \xi$, we get

$$\begin{aligned} & -g(X,\widetilde{Q}(Y,Z)\xi)\xi + \eta(\widetilde{Q}(Y,Z)\xi)X \\ & +g(X,Y)\widetilde{Q}(\xi,Z)\xi - \eta(Y)\widetilde{Q}(X,Z)\xi \\ & +g(X,Z)\widetilde{Q}(Y,\xi)\xi - \eta(Z)\widetilde{Q}(Y,X)\xi \\ & +g(X,\xi)\widetilde{Q}(Y,Z)\xi - \eta(\xi)\widetilde{Q}(Y,Z)X = 0 \end{aligned} \tag{3.8}$$

which implies

$$\begin{aligned} & -b_1\eta(Y)g(X,Z)\xi + b_1\eta(Z)g(X,Y)\xi + g(X,Y)\widetilde{Q}(\xi,Z)\xi \\ & -\eta(Y)\widetilde{Q}(X,Z)\xi + g(X,Z)\widetilde{Q}(Y,\xi)\xi - \eta(Z)\widetilde{Q}(Y,X)\xi \\ & +\eta(X)\widetilde{Q}(Y,Z)\xi - \widetilde{Q}(Y,Z)X = 0. \end{aligned} \tag{3.9}$$

Taking inner product with ξ in (3.8) and using (2.3), we get

$$\begin{aligned}
&-b_1\eta(Y)g(X,Z)+b_1\eta(Z)g(X,Y)+g(X,Y)\eta(\widetilde{Q}(\xi,Z)\xi)\\
&-\eta(Y)\eta(\widetilde{Q}(X,Z)\xi)+g(X,Z)\eta(\widetilde{Q}(Y,\xi)\xi)-\eta(Z)\eta(\widetilde{Q}(Y,X)\xi)\\
&+\eta(X)\eta(\widetilde{Q}(Y,Z)\xi)-\eta(\widetilde{Q}(Y,Z)X)=0. \qquad (3.10)
\end{aligned}$$

Using (3.4) in (3.10), we have

$$b_1[\eta(Z)g(X,Y)-\eta(Y)g(X,Z)]-\eta(\widetilde{Q}(Y,Z)X)=0. \qquad (3.11)$$

Putting $Z=\xi$ in (3.11) and using (2.3), we get

$$b_1\{g(X,Y)-\eta(Y)\eta(X)\}-\eta(\widetilde{Q}(Y,\xi)X)=0. \qquad (3.12)$$

Now from (3.1), we get

$$\begin{aligned}
\eta(\widetilde{Q}(Y,\xi)X) &= g(X,Y)-\eta(Y)\eta(X)\\
&-\frac{1}{4n}[A\eta(X)\eta(Y)-S(X,Y)\\
&+A\eta(X)\eta(Y)-Ag(X,Y). \qquad (3.13)
\end{aligned}$$

After putting (3.13) in (3.12), we get

$$\begin{aligned}
&b_1\{g(X,Y)-\eta(Y)\eta(X)\}-g(X,Y)\\
&+\eta(Y)\eta(X)+\frac{1}{4n}[A\eta(X)\eta(Y)-S(X,Y)\\
&+A\eta(X)\eta(Y)-Ag(X,Y)=0, \qquad (3.14)
\end{aligned}$$

simplifying (3.14), we get

$$\begin{aligned}
S(X,Y) &= 4n[b_1-1-\frac{A}{4n}]g(X,Y)\\
&+4n[-b_1+1+\frac{A}{4n}]\eta(X)\eta(Y),
\end{aligned}$$

the above equation can be written in the form

$$S(X,Y)=\rho g(X,Y)+\sigma\eta(X)\eta(Y), \qquad (3.15)$$

where

$$\rho=4n[b_1-1-\frac{A}{4n}],$$

and

$$\sigma=4n[-b_1+1+\frac{A}{4n}].$$

So from (3.14), we conclude that the manifold becomes an η-Einstein manifold.

□

4. Para-Sasakian manifold admitting a conformal Ricci soliton and $\widetilde{Q}(\xi, X).S = 0$

Theorem 2. *If a para-Sasakian manifold $\widetilde{M}$ admits a conformal Ricci soliton and the manifold is M-projective Ricci symmetric i.e., $\widetilde{Q}(\xi, X).S = 0$, then the Ricci operator Q satisfies the quadratic equation $Q^2 - Q + D = 0$ for all $X \in \chi(\widetilde{M})$, where D are constants, $\widetilde{Q}$ is an M-projective curvature tensor and S is a Ricci tensor.*

Proof. Let $\widetilde{M}$ be a $(2n + 1)$ dimensional para-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). From (3.1), we can write

$$\begin{aligned} \widetilde{Q}(\xi, X)Y &= R(\xi, X)Y - \frac{1}{4n}[S(X, Y)\xi - S(\xi, Y)X \\ &\quad + g(X, Y)Q\xi - g(\xi, Y)QX]. \end{aligned} \tag{4.1}$$

Using (2.9), (2.16) and (2.17) in (4.1), we have

$$\begin{aligned} \widetilde{Q}(\xi, X)Y &= [-g(X, Y)\xi + \eta(Y)X] - \frac{1}{4n}[S(X, Y)\xi \\ &\quad - A\eta(Y)X + Ag(X, Y)\xi - \eta(Y)QX], \end{aligned} \tag{4.2}$$

and similarly, we have

$$\begin{aligned} \widetilde{Q}(\xi, X)Z &= [-g(X, Y)\xi + \eta(Y)X] - \frac{1}{4n}[S(X, Z)\xi \\ &\quad - A\eta(Z)X + Ag(X, Z)\xi - \eta(Z)QX]. \end{aligned} \tag{4.3}$$

Now we consider that the tensor derivative of S by $\widetilde{Q}(\xi, X)$ is zero i.e., $\widetilde{Q}(\xi, X).S = 0$. Then the para-Sasakian manifold $\widetilde{M}$ admitting a conformal Ricci soliton is M-projective Ricci symmetric. It gives

$$S(\widetilde{Q}(\xi, X)Y, Z) + S(Y, \widetilde{Q}(\xi, X)Z) = 0, \tag{4.4}$$

using (4.2) and (4.3) in (4.4), we get

$$\begin{aligned} &S([-g(X, Y)\xi + \eta(Y)X] - \frac{1}{4n}[S(X, Y)\xi \\ &-A\eta(Y)X + Ag(X, Y)\xi - \eta(Y)QX], Z) \\ &+S(Y, [-g(X, Y)\xi + \eta(Y)X] - \frac{1}{4n}[S(X, Z)\xi \\ &-A\eta(Z)X + Ag(X, Z)\xi - \eta(Z)QX]) = 0. \end{aligned} \tag{4.5}$$

Putting $Z = \xi$ and using (2.3) and (2.17) in (4.5), we get

$$S(X, Y) + S(Y, QX) - (A + \frac{A^2}{4n})g(X, Y) = 0, \tag{4.6}$$

which implies

$$S(X, Y) = Dg(X, Y) - S(QX, Y),$$

where $D = (A + \frac{A^2}{4n})$, which implies

$$QX = DX - Q^2X \qquad \forall\, Y \in \chi(\widetilde{M}), \tag{4.7}$$

i.e.,

$$Q^2 + Q - D = 0 \quad \forall\, X,$$

which is a quadratic equation in Q. □

5. Para-Sasakian manifold admitting a conformal Ricci soliton and $R(\xi, X).\widehat{V} = 0$

Let $\widetilde{M}$ be a $(2n+1)$ dimensional para-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The pseudo projective curvature tensor $\widehat{V}$ on $\widetilde{M}$ is defined by [7]

$$\begin{aligned}\widehat{V}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n} + b][g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{5.1}$$

Now we prove the following theorem:

Theorem 3. *If a para-Sasakian manifold admits a conformal Ricci soliton and is pseudo projective semi symmetric i.e.,* $R(\xi, X).\widehat{V} = 0$, *then the manifold is an* η*-Einstein manifold where* $\widehat{V}$ *is a pseudo projective curvature tensor and* $R(\xi, X)$ *is the derivation of the tensor algebra of the tangent space of the manifold.*

Proof. Let $\widetilde{M}$ be a $(2n+1)$ dimensional para-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Putting $Z = \xi$ in (5.1), we have

$$\begin{aligned}\widehat{V}(X,Y)\xi &= aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n} + b][g(Y,\xi)X - g(X,\xi)Y],\end{aligned} \tag{5.2}$$

using (2.2), (2.10), (2.17) in (5.2), we get

$$\widehat{V}(X,Y)\xi = [-a + bA - \frac{r}{2n+1}(\frac{a}{2n} + b)][\eta(Y)X - \eta(X)Y]. \tag{5.3}$$

Considering

$$\gamma = [-a + bA - \frac{r}{2n+1}(\frac{a}{2n} + b)],$$

therefore, (5.3) becomes

$$\widehat{V}(X,Y)\xi = \gamma[\eta(Y)X - \eta(X)Y], \tag{5.4}$$

and

$$g(\widehat{V}(X,Y)\xi, Z) = \gamma[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{5.5}$$

which implies

$$\eta(\widehat{V}(X,Y)Z) = \gamma[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]. \tag{5.6}$$

Now we consider that para-Sasakian manifold admits a conformal Ricci soliton and is pseudo projective semi symmetric i.e., $R(\xi, X).\widehat{V} = 0$ holds in $\widetilde{M}$, which implies

$$\begin{aligned} &R(\xi,X)(\widehat{V}(Y,Z)W) - \widehat{V}(R(\xi,X)Y,Z)W \\ &-\widehat{V}(Y,R(\xi,X)Z)W - \widehat{V}(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{5.7}$$

for all vector field X, Y, Z on $\widetilde{M}$.

Using (2.9) in (5.7) and putting $W = \xi$, we get

$$\begin{aligned} &-g(X,\widehat{V}(Y,Z)\xi) + \eta(\widehat{V}(Y,Z)\xi)X - \widehat{V}(-g(X,Y)\xi \\ &+\eta(Y)X,Z)\xi - \widehat{V}(Y,-g(X,Z)\xi + \eta(Z)X)\xi \\ &-\widehat{V}(Y,Z)(-g(X,\xi)\xi + \eta(\xi)X) = 0, \end{aligned} \tag{5.8}$$

which implies

$$\begin{aligned} &\gamma(-\eta(Z)g(X,Y)\xi + \eta(Y)g(X,Z)\xi) + g(X,Y)\widehat{V}(\xi,Z)\xi \\ &-\eta(Y)\widehat{V}(X,Z)\xi + g(X,Z)\widehat{V}(Y,\xi)\xi - \eta(Z)\widehat{V}(Y,X)\xi \\ &+\eta(X)\widehat{V}(Y,Z)\xi - \widehat{V}(Y,Z)X = 0. \end{aligned} \tag{5.9}$$

Taking inner product with ξ in (5.9) and using (2.3), we get

$$\begin{aligned} &\gamma(-\eta(Z)g(X,Y) + \eta(Y)g(X,Z)) + g(X,Y)\eta(\widehat{V}(\xi,Z)\xi) \\ &-\eta(Y)\eta(\widehat{V}(X,Z)\xi) + g(X,Z)\eta(\widehat{V}(Y,\xi)\xi) - \eta(Z)\eta(\widehat{V}(Y,X)\xi) \\ &+\eta(X)\eta(\widehat{V}(Y,Z)\xi) - \eta(\widehat{V}(Y,Z)X) = 0, \end{aligned} \tag{5.10}$$

using (5.4) in (5.10), we have

$$\gamma[\eta(Z)g(X,Y) - \eta(Y)g(X,Z)] + \eta(\widehat{V}(Y,Z)X)) = 0. \tag{5.11}$$

Putting $Z = \xi$ in (3.11) and using (2.2) and (2.3), we get

$$\gamma[g(X,Y) - \eta(Y)\eta(X)] + \eta(\widehat{V}(Y,\xi)X) = 0. \tag{5.12}$$

Now from (5.1), we get

$$\begin{aligned} \eta(\widehat{V}(Y,\xi)X) &= a[g(X,Y) - \eta(Y)\eta(X)] \\ &+b[A\eta(X)\eta(Y) - S(X,Y)] \\ &-\frac{r}{2n+1}[\frac{a}{2n} + b][\eta(X)\eta(Y) - g(X,Y)]. \end{aligned} \tag{5.13}$$

After putting (5.13) in (5.12), the equation reduces to

$$\begin{aligned} &\gamma[g(X,Y) - \eta(Y)\eta(X)] + a[g(X,Y) \\ &-\eta(Y)\eta(X)] + b[A\eta(X)\eta(Y) - S(X,Y)] \\ &-\frac{r}{2n+1}[\frac{a}{2n} + b][\eta(X)\eta(Y) - g(X,Y)] = 0. \end{aligned} \tag{5.14}$$

Simplifying (5.14), we get

$$S(X,Y) = \frac{[\gamma + a + \frac{r}{2n+1}(\frac{a}{2n}+b)]}{b} g(X,Y) + \frac{[bA - a - D - \frac{r}{2n+1}(\frac{a}{2n}+b)]}{b} \eta(X)\eta(Y).$$

the above equation can be written in the form

$$S(X,Y) = E_1 g(X,Y) + E_2 \eta(X)\eta(Y), \tag{5.15}$$

where

$$E_1 = \frac{[\gamma + a + \frac{r}{2n+1}(\frac{a}{2n}+b)]}{b},$$

and

$$E_2 = \frac{[bA - a - D - \frac{r}{2n+1}(\frac{a}{2n}+b)]}{b}.$$

So from (5.15), we conclude that the manifold becomes an η-Einstein manifold. □

6. Para-Sasakian manifold admitting a conformal Ricci soliton and $\widehat{V}(\xi, X).S = 0$

Theorem 4. *If a para-Sasakian manifold $\widetilde{M}$ admits a conformal Ricci soliton and $\widehat{V}(\xi, X).S = 0$ holds i.e., the manifold is Ricci pseudo projectively symmetric, then the manifold is an η-Einstein manifold, where $\widehat{V}$ is a pseudo projective curvature tensor and S is a Ricci tensor.*

Proof. Let $\widetilde{M}$ be a $(2n+1)$ dimensional para-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Now the equation (5.1) can be written as

$$\widehat{V}(\xi, X)Y = aR(\xi, X)Y + b[S(X,Y)\xi - S(\xi,Y)X] - \frac{r}{2n+1}[\frac{a}{2n} + b][g(X,Y)\xi - g(\xi,Y)X], \tag{6.1}$$

and

$$\widehat{V}(\xi, X)Z = aR(\xi, X)Z + b[S(X,Z)\xi - S(\xi,Z)X] - \frac{r}{2n+1}[\frac{a}{2n} + b][g(X,Z)\xi - g(\xi,Z)X]. \tag{6.2}$$

Now we assume that the manifold is Ricci pseudo projectively symmetric i.e., $\widehat{V}(\xi, X).S = 0$ holds in $\widetilde{M}$, which gives

$$S(\widehat{V}(\xi, X)Y, Z) + S(Y, \widehat{V}(\xi, X)Z) = 0, \tag{6.3}$$

using (6.1), (6.2) in (6.3), we have

$$\begin{aligned}
&S(aR(\xi,X)Y+b[S(X,Y)\xi-S(\xi,Y)X]\\
&-\frac{r}{2n+1}[\frac{a}{2n}+b][g(X,Y)\xi-g(\xi,Y)X],Z)\\
&+S(Y,aR(\xi,X)Z+b[S(X,Z)\xi-S(\xi,Z)X]\\
&-\frac{r}{2n+1}[\frac{a}{2n}+b][g(X,Z)\xi-g(\xi,Z)X])=0.
\end{aligned} \tag{6.4}$$

Putting $Z=\xi$, we get

$$\begin{aligned}
&S(aR(\xi,X)Y+b[S(X,Y)\xi-S(\xi,Y)X]\\
&-\frac{r}{2n+1}[\frac{a}{2n}+b][g(X,Y)\xi-g(\xi,Y)X],\xi)\\
&+S(Y,aR(\xi,X)\xi+b[S(X,\xi)\xi-S(\xi,\xi)X]\\
&-\frac{r}{2n+1}[\frac{a}{2n}+b][g(X,\xi)\xi-g(\xi,\xi)X])=0,
\end{aligned} \tag{6.5}$$

on simplifying, we have

$$S(X,Y)=(\frac{aA+\frac{rA}{2n+1}[\frac{a}{2n}+b]}{aA+\frac{r}{2n+1}[\frac{a}{2n}+b]})g(X,Y)+(\frac{aA-aA^2}{aA+\frac{r}{2n+1}[\frac{a}{2n}+b]})\eta(X)\eta(Y). \tag{6.6}$$

Let $\alpha=(\frac{aA+\frac{rA}{2n+1}[\frac{a}{2n}+b]}{aA+\frac{r}{2n+1}[\frac{a}{2n}+b]})$ and $\beta=(\frac{aA-aA^2}{aA+\frac{r}{2n+1}[\frac{a}{2n}+b]})$, then teh above equation becomes

$$S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y),$$

which proves that the manifold is an η-Einstein manifold. □

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