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Generalization of 2-absorbing quasi primary ideals

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Abstract. In this article, we introduce and study the concept of ϕ-2-absorbing quasi primary ideals in commutative rings. Let R be a commutative ring with a nonzero identity and $L(R)$ be the lattice of all ideals of R. Suppose that $\phi : L(R) \to L(R) \cup \{\emptyset\}$ is a function. A proper ideal I of R is called a ϕ-2-absorbing quasi primary ideal of R if $a, b, c \in R$ and whenever $abc \in I - \phi(I)$, then either $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. In addition to giving many properties of ϕ-2-absorbing quasi primary ideals, we also use them to characterize von Neumann regular rings.

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1. Introduction

In this article, we focus only on commutative rings with a nonzero identity and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. $L(R)$ denotes the lattice of all ideals of R. Let I be a proper ideal of R, the set $\{r \in R \mid rs \in I \text{ for some } s \in R \setminus I\}$ will be denoted by $Z_I(R)$. Also the radical of I is defined as $\sqrt{I} := \{r \in R \mid r^k \in I \text{ for some } k \in \mathbb{N}\}$ and for $x \in R$, $(I : x)$ denotes the ideal $\{r \in R \mid rx \in I\}$ of R. A proper ideal I of a commutative ring R is *prime* if whenever $a_1, a_2 \in R$ with $a_1a_2 \in I$, then $a_1 \in I$ or $a_2 \in I$, [7]. In 2003, the authors of [4] said that if whenever $a_1, a_2 \in R$ with $0_R \neq a_1a_2 \in I$, then $a_1 \in I$ or $a_2 \in I$, a proper ideal I of a commutative ring R is *weakly prime*. In [11], Bhatwadekar and Sharma defined a proper ideal I of an integral domain R as *almost prime (resp. n-almost prime)* if for $a_1, a_2 \in R$ with $a_1a_2 \in I - I^2$, (resp. $a_1a_2 \in I - I^n, n \geq 3$) then $a_1 \in I$ or $a_2 \in I$. This definition can be made for any commutative ring R. Later, Anderson and Batanieh [2] introduced a concept which covers all the previous definitions in a commutative ring R as following: Let $\phi : L(R) \to L(R) \cup \{\emptyset\}$ be a function, where $L(R)$ denotes the set of all ideals of R. A proper ideal I of a commutative ring R is called *ϕ-prime* if for $a_1, a_2 \in R$ with $a_1a_2 \in I - \phi(I)$, then $a_1 \in I$ or $a_2 \in I$. They defined the map $\phi_\alpha : L(R) \to L(R) \cup \{\emptyset\}$ as follows:

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(i) $\phi_\emptyset : \phi(I) = \emptyset$ defines prime ideals.

(ii) $\phi_0 : \phi(I) = \{0_R\}$ defines weakly prime ideals.

(iii) $\phi_2 : \phi(I) = I^2$ defines almost prime ideals.

(iv) $\phi_n : \phi(I) = I^n$ defines n-almost prime ideals($n \geq 2$).

(v) $\phi_\omega : \phi(I) = \cap_{n=1}^{\infty} I^n$ defines ω-prime ideals.

(vi) $\phi_1 : \phi(I) = I$ defines any ideal.

The notion of a 2-absorbing ideal, which is a generalization of the prime ideal, was introduced by Badawi as the following: a proper ideal I of R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$, see [8]. Also, the notion is investigated in [6], [15], [19], [18] and [20]. Then, the notion of a 2-absorbing primary ideal, which is a generalization of a primary ideal, was introduced in [10] as: a proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Note that a 2-absorbing ideal of a commutative ring R is a 2-absorbing primary ideal of R. But the converse is not true. For example, consider the ideal $I = (20)$ of $\mathbb{Z}$. Since $2 \cdot 2 \cdot 5 \in I$, but $2 \cdot 2 \notin I$ and $2 \cdot 5 \notin I$, I is not a 2-absorbing ideal of $\mathbb{Z}$. However, it is clear that I is a 2-absorbing primary ideal of $\mathbb{Z}$, since $2 \cdot 5 \in \sqrt{I}$. In 2016, the authors introduced the concept of a ϕ-2-absorbing primary ideal which a proper ideal I of R is called a ϕ-2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in I - \phi(I)$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, see [9].

On the other hand, the concept of quasi primary ideals in commutative rings was introduced and investigated by Fuchs in [12]. The author called an ideal I of R as a quasi primary ideal if $\sqrt{I}$ is a prime ideal. In [16], the notion of 2-absorbing quasi primary ideals is introduced as following: a proper ideal I of R to be a 2-absorbing quasi primary if $\sqrt{I}$ is a 2-absorbing ideal of R.

In this paper, our aim to obtain some generalizations of the concept of the quasi primary ideals and 2-absorbing quasi primary ideals. For this, firstly we define the ϕ-quasi primary ideal. Let $\phi : L(R) \to L(R) \cup \{\emptyset\}$ be a function and I be a proper ideal of R. Then I is said to be a ϕ-quasi primary ideal if whenever $a, b \in R$ and $ab \in I - \phi(I)$, then $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Similarly, I is called a ϕ-2-absorbing quasi primary ideal if whenever $a, b, c \in R$ and $abc \in I - \phi(I)$, then $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. In Section 2, firstly we investigate the basic properties of a ϕ-quasi primary ideal and a ϕ-2-absorbing quasi primary. With the help of Theorem 2.4 and Theorem 2.5, we give a diagram which clarifies the place of a ϕ-2-absorbing quasi primary ideal in the lattice of all ideals $L(R)$ of R, see Figure 1. In Proposition 2.9, we give a method for constructing ϕ-2-absorbing quasi primary ideals in commutative rings. Also, if $\phi(I)$ is a quasi primary ideal of R, we prove that I is a ϕ-2-absorbing quasi primary ideal of $R \Leftrightarrow I$ is a 2-absorbing quasi primary ideal of R, see Corollary 2.15. With Theorem 2.17, we obtain the Nakayama´s Lemma for weakly (2-absorbing) quasi primary ideals. Moreover, we examine the notion of "ϕ-2-absorbing quasi primary ideals" in $S^{-1}R$, where S is a multiplicatively closed subset of R. In Theorem 2.19, we characterize the weakly 2-absorbing quasi primary ideal of $R \propto M$, that is, the trivial ring extension, where M is an R-module. In Theorem 2.20, we describe von Neumann regular rings in terms of ϕ-2-absorbing quasi primary ideals. Finally, with the all results of the Section 3, we characterize a ϕ-2-absorbing quasi primary ideal in the direct product of finitely many commutative rings.

2. Characterization of ϕ-2-aborbing quasi primary ideals

Throughout the paper, $\phi : L(R) \to L(R) \cup \{\emptyset\}$ is a fixed function.

Definition 2.1. Let R be a ring and I be a proper ideal of R.

(i) I is said to be a ϕ-quasi primary ideal if whenever $a, b \in R$ and $ab \in I - \phi(I)$, then $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

(ii) I is said to be a ϕ-2-absorbing quasi primary ideal if whenever $a, b, c \in R$ and $abc \in I - \phi(I)$, then $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Definition 2.2. Let $\phi_\alpha : L(R) \to L(R) \cup \{\emptyset\}$ be one of the following special functions and I be a ϕ_α-quasi primary (ϕ_α-2-absorbing quasi primary) ideal of R. Then,

$\phi_\emptyset(I) = \emptyset$ is a quasi primary (2-absorbing quasi primary) ideal

$\phi_0(I) = 0_R$ is a weakly quasi primary (weakly 2-absorbing quasi primary) ideal

$\phi_2(I) = I^2$ is an almost quasi primary (almost 2-absorbing quasi primary) ideal

$\phi_n(I) = I^n$ is an n-almost quasi primary (n-almost 2-absorbing quasi primary) ideal ($n \geq$

$\phi_\omega(I) = \cap_{n=1}^{\infty} I^n$ is an ω-quasi primary (ω-2-absorbing quasi primary) ideal

$\phi_1(I) = I$ is any ideal.

Note that since $I - \phi(I) = I - (I \cap \phi(I))$, for any ideal I of R, without loss of generality, assume that $\phi(I) \subseteq I$. Let ψ_1, $\psi_2 : L(R) \to L(R) \cup \{\emptyset\}$ be two functions, if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in L(R)$, we denote $\psi_1 \leq \psi_2$. Thus clearly, we have the following order: $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$. Also, 2-almost quasi primary (2-almost 2-absorbing quasi primary) ideals are exactly almost quasi primary (almost 2-absorbing quasi primary) ideals.

Proposition 2.3. *Let R be a ring and I be a proper ideal R. Let $\psi_1, \psi_2 : L(R) \to L(R) \cup \{\emptyset\}$ be two functions with $\psi_1 \leq \psi_2$.*

(i) If I is a ψ_1-quasi primary ideal of R, then I is a ψ_2-quasi primary ideal of R.

(ii) I is a quasi primary ideal $\Rightarrow$ I is a weakly quasi primary ideal $\Rightarrow$ I is an ω-quasi primary ideal $\Rightarrow$ I is an $(n+1)$-almost quasi primary ideal $\Rightarrow$ I is an n-almost quasi primary ideal $(n \geq 2)$ $\Rightarrow$ I is an almost quasi primary ideal.

(iii) I is an ω-quasi primary ideal if and only if I is an n-almost quasi primary ideal for each $n \geq 2$.

(iv) If I is a ψ_1-2-absorbing quasi primary ideal of R, then I is a ψ_2-2-absorbing quasi primary ideal of R.

(v) I is a 2-absorbing quasi primary ideal $\Rightarrow$ I is a weakly 2-absorbing quasi primary ideal $\Rightarrow$ I is an ω-2-absorbing quasi primary ideal $\Rightarrow$ I is an $(n+1)$-almost 2-absorbing quasi primary ideal $\Rightarrow$ I is an n-almost 2-absorbing quasi primary ideal $(n \geq 2)$ $\Rightarrow$ I is an almost 2-absorbing quasi primary ideal.

(vi) I is an ω-2-absorbing quasi primary ideal if and only if I is an n-almost 2-absorbing quasi primary ideal for each $n \geq 2$.

Proof. (i): It is evident.

(ii): Follows from (i).

(iii): Every ω-quasi primary ideal is an n-almost quasi primary ideal for each $n \geq 2$ since $\phi_\omega \leq \phi_n$. Now, let I be an n-almost quasi primary ideal for each $n \geq 2$. Choose elements $a, b \in R$ such that $ab \in I - \cap_{n=1}^{\infty} I^n$. Then we have $ab \in I - I^n$ for some $n \geq 2$. Since I is an n-almost quasi primary ideal of R, we conclude either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Therefore, I is an ω-quasi primary ideal.

(iv): It is evident.

(v): Follows from (iv).

(vi): Similar to (iii). □

Theorem 2.4. *(i) If $\sqrt{I} = I$, then I is a ϕ-2-absorbing quasi primary ideal of R if and only if I is a ϕ-2-absorbing ideal of R.*

(ii) If I is a ϕ-2-absorbing quasi primary ideal of R and $\phi(\sqrt{I}) = \sqrt{\phi(I)}$, then $\sqrt{I}$ is a ϕ-2-absorbing ideal of R.

(iii) If $\sqrt{I}$ is a ϕ-2-absorbing ideal of R and $\phi(\sqrt{I}) \subseteq \phi(I)$, then I is a ϕ-2-absorbing quasi primary ideal of R.

(iv) If I is a ϕ-quasi primary ideal of R and $\phi(\sqrt{I}) = \sqrt{\phi(I)}$, then $\sqrt{I}$ is a ϕ-prime ideal of R.

(v) If $\sqrt{I}$ is a ϕ-prime ideal of R and $\phi(\sqrt{I}) \subseteq \phi(I)$, then I is a ϕ-quasi primary ideal of R.

Proof. (i): It is evident.

(ii): Let I be a ϕ-2-absorbing quasi primary ideal of R. Take $a, b, c \in R$ such that $abc \in \sqrt{I} - \phi(\sqrt{I})$. Then there exists a positive integer n such that $(abc)^n = a^n b^n c^n \in I$. Since $abc \notin \phi(\sqrt{I})$ and $\phi(\sqrt{I}) = \sqrt{\phi(I)}$, we get $abc \notin \sqrt{\phi(I)}$, so $a^n b^n c^n \notin \phi(I)$. Thus, by our hypothesis, $a^n b^n \in \sqrt{I}$ or $b^n c^n \in \sqrt{I}$ or $a^n c^n \in \sqrt{I}$. Consequently, $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$.

(iii): Assume that $\sqrt{I}$ is a ϕ-2-absorbing ideal of R. Choose $a, b, c \in R$ such that $abc \in I - \phi(I)$. Since $I \subseteq \sqrt{I}$ and $\phi(\sqrt{I}) \subseteq \phi(I)$, we have $abc \in \sqrt{I} - \phi(\sqrt{I})$. Then as $\sqrt{I}$ is ϕ-2-absorbing, $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. So I is a ϕ-quasi primary ideal of R.

(iv): It is similar to (i).

(v): It is similar to (ii). □

Theorem 2.5. *(i) Every ϕ-quasi primary ideal is a ϕ-2-absorbing primary ideal.*

(ii) Every ϕ-2-absorbing primary ideal is a ϕ-2-absorbing quasi primary ideal.

(iii) Every ϕ-quasi primary ideal is a ϕ-2-absorbing quasi primary ideal.

Proof. (i): Let I be a ϕ-quasi primary ideal and choose $a, b, c \in R$ such that $abc = a(bc) \in I - \phi(I)$. Since I is a ϕ-quasi primary ideal, we conclude either $a \in \sqrt{I}$ or $bc \in \sqrt{I}$. Then we have either $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, which completes the proof.

(ii): It is clear.

(iii): It follows from (i) and (ii). □

By Theorem 2.4 and Theorem 2.5, we give the following diagram which clarifies the place of ϕ-2-absorbing quasi primary ideals in the lattice of all ideals $L(R)$ of R.

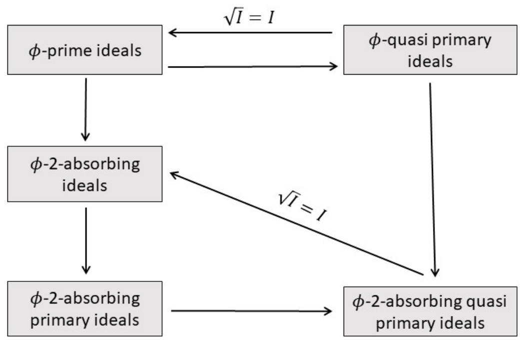

Figure 1: ϕ-2-absorbing quasi primary ideal vs other classical ϕ-ideals

Corollary 2.6. *If I is a ϕ-2-absorbing primary ideal of R and $\phi(\sqrt{I}) = \sqrt{\phi(I)}$, then $\sqrt{I}$ is a ϕ-2-absorbing ideal of R.*

Proof. It follows from Theorem 2.4(ii) and Theorem 2.5(ii). □

Proposition 2.7. *Let I be a proper ideal of R. Then,*

(i) I is a ϕ-quasi primary ideal of R if and only if $I/\phi(I)$ is a weakly quasi primary ideal of $R/\phi(I)$.

(ii) I is a ϕ-2-absorbing quasi primary ideal of R if and only $I/\phi(I)$ is a weakly 2-absorbing quasi primary ideal of $R/\phi(I)$.

Proof. (i): Suppose that I is a ϕ-quasi primary ideal of R. Let $0_{R/\phi(I)} \neq (a + \phi(I))(b + \phi(I)) = ab + \phi(I) \in I/\phi(I)$ for some $a, b \in R$. Then we have $ab \in I - \phi(I)$. Since I is a ϕ-quasi primary ideal of R, we conclude either $a \in \sqrt{I}$ or $b \in$

$\sqrt{I}$. This implies that $a+\phi(I) \in \sqrt{I}/\phi(I) = \sqrt{I/\phi(I)}$ or $b+\phi(I) \in \sqrt{I/\phi(I)}$. Therefore, $I/\phi(I)$ is a weakly quasi primary ideal of $R/\phi(I)$. Conversely, assume that $I/\phi(I)$ is a weakly quasi primary ideal of $R/\phi(I)$. Now, choose $a, b \in R$ such that $ab \in I - \phi(I)$. This yields that $0_{R/\phi(I)} \neq (a+\phi(I))(b+\phi(I)) = ab+\phi(I) \in I/\phi(I)$. Since $I/\phi(I)$ is a weakly quasi primary ideal of $R/\phi(I)$, we get either $a+\phi(I) \in \sqrt{I/\phi(I)} = \sqrt{I}/\phi(I)$ or $b+\phi(I) \in \sqrt{I}/\phi(I)$. Then we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Hence, I is a ϕ-quasi primary ideal of R.

(ii): Similar to (i). □

In the following, we characterize all quasi primary and 2-absorbing quasi primary ideals in factor rings $R/\phi(I)$. Since the proof is similar to that of the previous proposition (i), we omit the proof.

Proposition 2.8. *Let I be a proper ideal of R. Then,*

(i) I is a quasi primary ideal of R if and only if $I/\phi(I)$ is a quasi primary ideal of $R/\phi(I)$.

(ii) I is a 2-absorbing quasi primary ideal of R if and only $I/\phi(I)$ is a 2-absorbing quasi primary ideal of $R/\phi(I)$.

Now, we give a method for constructing ϕ-2-absorbing quasi primary ideals in commutative rings.

Proposition 2.9. *Let P_1, P_2 be ϕ-quasi primary ideal of a ring R. Then the following statements hold:*

(i) If $\phi(P_1) = \phi(P_2) = \phi(P_1 \cap P_2)$, then $P_1 \cap P_2$ is a ϕ-2-absorbing quasi primary ideal of R.

(ii) If $\phi(P_1) = \phi(P_2) = \phi(P_1P_2)$, then P_1P_2 is a ϕ-2-absorbing quasi primary ideal of R

Proof. (i): Let $abc \in P_1 \cap P_2 - \phi(P_1 \cap P_2)$ for some $a, b, c \in R$. Then we have $abc \in P_1 - \phi(P_1)$. As P_1 is a ϕ-quasi primary ideal, we conclude either $a \in \sqrt{P_1}$ or $b \in \sqrt{P_1}$ or $c \in \sqrt{P_1}$. Similarly, we get either $a \in \sqrt{P_2}$ or $b \in \sqrt{P_2}$ or $c \in \sqrt{P_2}$. Without loss generality, we may assume that $a \in \sqrt{P_1}$ and $b \in \sqrt{P_2}$. Then $ab \in \sqrt{P_1}\sqrt{P_2} \subseteq \sqrt{P_1} \cap \sqrt{P_2} = \sqrt{P_1 \cap P_2}$. Hence, $P_1 \cap P_2$ is a ϕ-2-absorbing quasi primary ideal of R.

(ii): Similar to (i). □

Definition 2.10. Let I be a ϕ-2-absorbing quasi primary ideal of R and $a, b, c \in R$ such that $abc \in \phi(I)$. If $ab \notin \sqrt{I}$, $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$, then (a, b, c) is called a strongly-ϕ-triple zero of I. In particular, if $\phi(I) = 0$, then (a, b, c) is called a strongly-triple zero of I.

Remark 2.11. If I is a ϕ-2-absorbing quasi primary ideal of R that is not a 2-absorbing quasi primary ideal, then I has a strongly-ϕ-triple zero (a, b, c) for some $a, b, c \in R$.

Proposition 2.12. *Suppose that I is a weakly 2-absorbing quasi primary ideal of R which is not 2-absorbing quasi primary ideal, then $I^3 = 0$.*

Proof. Let I be a weakly 2-absorbing quasi primary ideal of R such that $I^3 \neq 0$. Now, we will show that I is a 2-absorbing quasi primary ideal of R. Choose $a, b, c \in R$ such that $abc \in I$. Since I is a weakly 2-absorbing quasi primary ideal, we may assume that $abc = 0$. Otherwise, we would have $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. If $abI \neq 0$, then there exists $x \in I$ such that $abx \neq 0$. Since $0 \neq abx = ab(c + x) \in I$ and I is a weakly 2-absorbing quasi primary ideal, we get either $ab \in \sqrt{I}$ or $a(c + x) \in \sqrt{I}$ or $b(c + x) \in \sqrt{I}$. If we have either $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, then we are done. So assume that $abI = 0 = acI = bcI$. On the other hand, if $aI^2 \neq 0$, then there exists $x_1, x_2 \in I$ such that $ax_1x_2 \neq 0$. Then we have $0 \neq a(b + x_1)(c + x_2) = ax_1x_2 \in I$ since $abI = acI = 0$. As I is a weakly 2-absorbing quasi primary ideal, we get either $a(b + x_1) \in \sqrt{I}$ or $a(c + x_2) \in \sqrt{I}$ or $(b + x_1)(c + x_2) \in \sqrt{I}$. Then we have $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. So assume that $aI^2 = 0$. Similarly, we may assume that $bI^2 = cI^2 = 0$. As $I^3 \neq 0$, there exist $y, z, w \in I$ such that $yzw \neq 0$. As $abI = 0 = acI = bcI = aI^2 = bI^2 = cI^2$, it is clear that $0 \neq yzw = (a + y)(b + z)(c + w) \in I$. This implies that $(a + y)(b + z) \in \sqrt{I}$ or $(a + y)(c + w) \in \sqrt{I}$ or $(b + z)(c + w) \in \sqrt{I}$ and so we have $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Hence, I is a 2-absorbing quasi primary ideal of R. □

Corollary 2.13. *If I is a weakly 2-absorbing quasi primary ideal of R which is not a 2-absorbing quasi primary ideal, then $\sqrt{I} = \sqrt{0}$.*

Theorem 2.14. *(i) Let I be a ϕ-2-absorbing quasi primary ideal of R. Then either I is a 2-absorbing quasi primary ideal or $I^3 \subseteq \phi(I)$.*

(ii) If I is a ϕ-2-absorbing quasi primary ideal of R which is not a 2-absorbing quasi primary ideal, then $\sqrt{I} = \sqrt{\phi(I)}$.

Proof. (i) Suppose that I is a ϕ-2-absorbing quasi primary ideal of R that is not a 2-absorbing quasi primary ideal. Then note that $I/\phi(I)$ is not a 2-absorbing quasi primary ideal of $R/\phi(I)$. Also by Proposition 2.7, $I/\phi(I)$ is a weakly 2-absorbing quasi primary ideal of $R/\phi(I)$. Then by Proposition 2.12, we get $(I/\phi(I))^3 = 0_{R/\phi(I)}$ and this yields $I^3 \subseteq \phi(I)$.

(ii): Suppose that I is a ϕ-2-absorbing quasi primary ideal of R which is not a 2-absorbing quasi primary ideal. Then by (i), we have $I^3 \subseteq \phi(I)$ and thus $\sqrt{I} \subseteq \sqrt{\phi(I)}$. On the other hand, since $\phi(I) \subseteq I$, we have $\sqrt{I} = \sqrt{\phi(I)}$. □

Corollary 2.15. *Suppose that I is a proper ideal of R such that $\phi(I)$ is a quasi primary ideal of R. Then the following statments are equivalent:*

(i) I is a ϕ-2-absorbing quasi primary ideal of R.

(ii) I is a 2-absorbing quasi primary ideal of R.

Proof. $(i) \Rightarrow (ii)$: Suppose that I is a ϕ-2-absorbing quasi primary ideal of R. Now, we will show that I is a 2-absorbing quasi primary ideal of R. Suppose that it is not. Then by Theorem 2.14 (ii), we have $\sqrt{I} = \sqrt{\phi(I)}$. Since $\phi(I)$ is a quasi primary ideal, we have $\sqrt{I} = \sqrt{\phi(I)}$ is a prime ideal and so I is quasi primary. Then by [16, Proposition 2.6], I is a 2-absorbing quasi primary, a contradiction.

$(ii) \Rightarrow (i)$: Directly from the definition. □

Theorem 2.16. *(i) If P is a weakly quasi primary ideal of R that is not quasi primary, then $P^2 = 0$.*

(ii) If P is a ϕ-quasi primary ideal of R that is not quasi primary, then $P^2 \subseteq \phi(P)$.

(iii) If P is a ϕ-quasi primary ideal of R where $\phi \leq \phi_3$, then P is n-almost quasi primary for all $n \geq 2$, so P is ω-quasi primary.

Proof. (i): Similar to Proposition 2.12.

(ii): Similar to Theorem 2.14 (i).

(ii): Assume that P is a ϕ-quasi primary ideal of R and $\phi \leq \phi_3$. If P is quasi primary, then P is ϕ-quasi primary for each ϕ. If P is not quasi primary, by (i), $P^2 \subseteq \phi(P)$. Also as $\phi \leq \phi_3$, we get $P^2 \subseteq \phi(P) \subseteq P^3$, so $\phi(P) = P^n$ for each $n \geq 2$. Consequently, since P is ϕ-quasi primary, P is n-almost quasi primary for all $n \geq 2$, so P is ω-quasi primary by Proposition 2.3 (iii). $\square$

Now, we give the Nakayama's Lemma for weakly (2-absorbing) quasi primary ideals as follows.

Theorem 2.17. *(Nakayama's Lemma) Let P be a weakly 2-absorbing quasi primary (weakly quasi primary) ideal of R that is not 2-absorbing quasi primary (quasi primary) and let M be an R-module. Then the following statements hold:*

(i) $P \subseteq Jac(R)$, where $Jac(R)$ is the Jacobson radical of R.

(ii) If $PM = M$, then $M = 0$.

(iii) If N is a submodule of M such that $PM + N = M$, then $N = M$.

Proof. (i) : Suppose that P is a weakly 2-absorbing quasi primary ideal of R that is not 2-absorbing quasi primary. Then by Theorem 2.12, $P^3 = 0$. Let $x \in P$. Now, we will show that $1-rx$ is a unit of R for each $r \in R$. Note that $rx \in P$ and so $r^3x^3 = 0$. This implies that $1 = 1-r^3x^3 = (1-rx)(1+rx+r^2x^2)$. Thus $x \in Jac(R)$ and so $P \subseteq Jac(R)$.

(ii) : Suppose that $PM = M$. Then by Theorem 2.12, $P^3 = 0$ and so $M = PM = P^3M = 0$.

(iii) : Follows from (ii). $\square$

Theorem 2.18. *Let S be a multiplicatively closed subset of R and $\phi_q : L(S^{-1}R) \to L(S^{-1}R) \cup \{\emptyset\}$, defined by $\phi_q(S^{-1}I) = S^{-1}(\phi(I))$ for each ideal I of R, be a function. Then the following statements hold:*

(i) If P is a ϕ-2-absorbing quasi primary ideal of R with $S \cap P = \emptyset$, then $S^{-1}P$ is a ϕ_q-2-absorbing quasi primary ideal of $S^{-1}R$.

(ii) Let P be an ideal of R such that $Z_{\phi(P)}(R) \cap S = \emptyset$ and $Z_P(R) \cap S = \emptyset$. If $S^{-1}P$ is a ϕ_q-2-absorbing quasi primary ideal of $S^{-1}R$, then P is a ϕ-2-absorbing quasi primary ideal of R.

Proof. (i): Let $\frac{a}{s}\frac{b}{t}\frac{c}{u} \in S^{-1}P - \phi_q(S^{-1}P)$ for any $a, b, c \in R$ and $s, t, u \in S$. As $\phi_q(S^{-1}P) = S^{-1}(\phi(P))$, we get $t^*abc = (t^*a)bc \in P - \phi(P)$ for some $t^* \in S$. Since P is a ϕ-2-absorbing quasi primary ideal of R, we get $t^*ab \in \sqrt{P}$ or $t^*ac \in \sqrt{P}$ or $bc \in \sqrt{P}$. This implies that $\frac{ab}{st} = \frac{t^*ab}{t^*st} \in S^{-1}\sqrt{P} = \sqrt{S^{-1}P}$ or

$\frac{ac}{su} = \frac{t^*ac}{t^*su} \in \sqrt{S^{-1}P}$ or $\frac{bc}{tu} \in \sqrt{S^{-1}P}$. Hence $S^{-1}P$ is a ϕ_q-2-absorbing quasi primary ideal of $S^{-1}R$.

(ii): Let $abc \in P - \phi(P)$ for some $a, b, c \in R$. Then $\frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}P$. Since $Z_{\phi(P)}(R) \cap S = \emptyset$, it is clear that $\frac{a}{1}\frac{b}{1}\frac{c}{1} \notin S^{-1}(\phi(P)) = \phi_q(S^{-1}P)$. As $S^{-1}P$ is a ϕ_q-2-absorbing quasi primary ideal of $S^{-1}R$, we get either $\frac{a}{1}\frac{b}{1} \in \sqrt{S^{-1}P} = S^{-1}\sqrt{P}$ or $\frac{a}{1}\frac{c}{1} \in S^{-1}\sqrt{P}$ or $\frac{b}{1}\frac{c}{1} \in S^{-1}\sqrt{P}$. Without loss generality, we may assume that $\frac{b}{1}\frac{c}{1} \in S^{-1}\sqrt{P}$. Then we get $tbc \in \sqrt{P}$ and so $t^n b^n c^n \in P$ for some $n \in \mathbb{N}$. If $b^n c^n \notin P$, then we have $t^n \in Z_P(R) \cap S$, a contradiction. So we have $b^n c^n \in P$ and thus $bc \in \sqrt{P}$. Thus P is a ϕ-2-absorbing quasi primary ideal of R. $\square$

Let M be an R-module. *The trivial ring extension (or idealization)* $R \propto M = R \oplus M$ of M is a commutative ring with the componentwise addition and the multiplication $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$; $m, m' \in M$ [13]. Let I be an ideal of R and N is a submodule of M. Then $I \propto N = I \oplus N$ is an ideal of $R \propto M$ if and only if $IM \subseteq N$ [5]. In that case, $I \propto N$ is called a homogeneous ideal of $R \propto M$. For any ideal $I \propto N$ of $R \propto M$, the radical of $I \propto N$ is characterized as follows:

$$\sqrt{I \propto N} = \sqrt{I} \propto M$$

[5, Theorem 3.2].

Now, we characterize certain weakly 2-absorbing quasi primary ideals in trivial ring extensions.

Theorem 2.19. *Let M be an R-module and I be a proper ideal of R. Then the following statements are equivalent:*

(i) $I \propto M$ is a weakly 2-absorbing quasi primary ideal of $R \propto M$.

(ii) I is a weakly 2-absorbing quasi primary ideal of R and for any strongly triple zero (a, b, c) of I, we have $abM = 0 = acM = bcM$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $I \propto M$ is a weakly 2-absorbing quasi primary ideal of $R \propto M$. Now, we will show that I is a weakly 2-absorbing quasi primary ideal of R. Let $0 \neq abc \in I$. Then note that $(0, 0_M) \neq (a, 0_M)(b, 0_M)(c, 0_M) = (abc, 0_M) \in I \propto M$. As $I \propto M$ is a weakly 2-absorbing quasi primary ideal, we conclude either $(a, 0_M)(b, 0_M) = (ab, 0_M) \in \sqrt{I \propto M} = \sqrt{I} \propto M$ or $(ac, 0_M) \in \sqrt{I} \propto M$ or $(bc, 0_M) \in \sqrt{I} \propto M$. This implies that $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Therefore, I is a weakly 2-absorbing quasi primary ideal of R. Let (x, y, z) be a strongly triple zero of I. Then $xyz = 0$ and also $xy, xz, yz \notin \sqrt{I}$. Assume that $xyM \neq 0$. Then there exists $m \in M$ such that $xym \neq 0$. Then note that $(0, 0_M) \neq (x, 0_M)(y, 0_M)(z, m) = (0, xym) \in I \propto M$. Since $I \propto M$ is a weakly 2-absorbing quasi primary ideal, we conclude either $(x, 0_M)(y, 0_M) = (xy, 0_M) \in \sqrt{I \propto M} = \sqrt{I} \propto M$ or $(xz, xm) \in \sqrt{I} \propto M$ or $(yz, ym) \in \sqrt{I} \propto M$, a contradiction. Thus $xM = yM = zM = 0$.

$(ii) \Rightarrow (i)$: Suppose that $(0, 0_M) \neq (a, m)(b, m')(c, m'') = (abc, abm'' + acm' + bcm) \in I \propto M$ for some $a, b, c \in R$; $m, m', m'' \in M$. Then $abc \in I$. If $abc \neq 0$, then either $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. This implies that $(a,m)(b,m') \in \sqrt{I \propto M} = \sqrt{I} \propto M$ or $(a,m)(c,m'') \in \sqrt{I} \propto M$ or $(b,m')(c,m'') \in \sqrt{I} \propto M$. Now assume that $abc = 0$. If (a,b,c) is a strongly triple zero of I, then by assumption $abM = 0 = acM = bcM$ and so $(0, 0_M) = (abc, abm'' + acm' + bcm) = (a,m)(b,m')(c,m'')$ which is a contradiction. So that (a,b,c) is not strongly triple zero and this yields $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Therefore, we have $(a,m)(b,m') \in \sqrt{I \propto M}$ or $(a,m)(c,m'') \in \sqrt{I \propto M}$ or $(b,m')(c,m'') \in \sqrt{I \propto M}$. Hence, $I \propto M$ is a weakly 2-absorbing quasi primary ideal of $R \propto M$. □

Let $R_1, R_2, \ldots, R_n$ be commutative rings and $R = R_1 \times R_2 \times \cdots \times R_n$ be the direct product of those rings. It is well known that every ideal of R has the form $I = I_1 \times I_2 \times \cdots \times I_n$, where I_k is an ideal of R_k for each $1 \leq k \leq n$. Suppose that $\psi_i : L(R_i) \to L(R_i) \cup \{\emptyset\}$ is a function for each $1 \leq i \leq n$ and put $\phi := \psi_1 \times \psi_2 \times \cdots \times \psi_n$, that is, $\phi(I) = \psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_n(I_n)$. Then note that $\phi : L(R) \to L(R) \cup \{\emptyset\}$ becomes a function.

Recall that a commutative ring R is said to be a *von Neumann regular ring* if for each $a \in R$, there exists $x \in R$ such that $a = a^2x$ [17]. In this case, the principal ideal (a) is a generated by an idempotent element $e \in R$. The notion of von Neumann regular rings has an important place in commutative algebra. So far, there have been many generalizations of this concept. See, for example, [14], [3] and [1]. Now, we characterize von Neumann regular rings in terms of ϕ-2-absorbing quasi primary ideals.

Theorem 2.20. *Let $R_1, R_2, \ldots, R_m$ be commutative rings and $R = R_1 \times R_2 \times \cdots \times R_m$, where $3 \leq m < \infty$. Suppose that $n \geq 2$. Then the following statements are equivalent.*

(i) Every ideal of R is a ϕ_n-2-absorbing quasi primary ideal.

(ii) $R_1, R_2, \ldots, R_m$ are von Neumann regular rings.

Proof. $(i) \Rightarrow (ii)$: Suppose that every ideal of R is a ϕ_n-2-absorbing quasi primary ideal. Now, we will show that $R_1, R_2, \ldots, R_m$ are von Neumann regular rings. Suppose not. Without loss of generality, we may assume that R_1 is not a von Neumann regular ring. Then there exists an ideal I_1 of R_1 such that $I_1^n \subsetneq I_1$. Then we can find an element $a \in I_1 - I_1^n$. Now, put $J = I_1 \times 0 \times 0 \times R_4 \times R_5 \times \cdots \times R_m$ and also $x_1 = (a, 1, 1, 1, \ldots, 1)$, $x_2 = (1, 0, 1, 1, \ldots, 1)$, $x_3 = (1, 1, 0, 1, \ldots, 1)$. Then note that $x_1x_2x_3 \in J - \phi_n(J)$. As x_1x_2, x_1x_3 and $x_2x_3 \notin \sqrt{J}$, J is not a ϕ_n-2-absorbing quasi primary ideal of R which is a contradiction. Thus $R_1, R_2, \ldots, R_m$ are von Neumann regular rings.

$(ii) \Rightarrow (i)$: Suppose that $R_1, R_2, \ldots, R_m$ are von Neumann regular rings. Then note that $I_i^n = I_i$ for any ideal I_i of R_i. Take any ideal J of R. Then $J = J_1 \times J_2 \times \cdots \times J_m$ for some ideal J_k of R_k, where $1 \leq k \leq m$. Then $J^n = J_1^n \times J_2^n \times \cdots \times J_m^n = \phi_n(J) = J_1 \times J_2 \times \cdots \times J_m = J$. This implies that $J - \phi_n(J) = \emptyset$ and so J is trivially a ϕ_n-2-absorbing quasi primary ideal of R. □

Corollary 2.21. *Let R be a ring and $n \geq 2$. Then the following statements are equivalent:*

(i) Every ideal of R^3 is a ϕ_n-2-absorbing quasi primary ideal.

(ii) R is a von Neumann regular ring.

3. ϕ-(2-absorbing) quasi primary ideals in the direct product of rings

In this section, we investigate ϕ-2-absorbing quasi primary ideal in the direct product of finitely many commutative rings.

Theorem 3.1. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings and let $\psi_i : L(R_i) \to L(R_i) \cup \{\emptyset\}$ is a function for each $i = 1, 2$. Let $\phi := \psi_1 \times \psi_2$ and let J be an ideal of R. Then J is a ϕ-quasi primary ideal of R if and only if J is in one of the following three forms:*

(i) $J = I_1 \times I_2$ such that $\psi_i(I_i) = I_i$ for $i = 1, 2$.

(ii) $J = I_1 \times R_2$ for some ψ_1-quasi primary ideal I_1 of R_1 which must be quasi primary if $\psi_2(R_2) \neq R_2$.

(iii) $J = R_1 \times I_2$ for some ψ_2-quasi primary ideal I_2 of R_2 which must be quasi primary if $\psi_1(R_1) \neq R_1$.

Proof. $\Rightarrow$: Suppose that J is a ϕ-quasi primary ideal of R. Then $J = I_1 \times I_2$ for some ideal I_1 of R_1 and some ideal I_2 of R_2. Let $xy \in I_1 - \psi_1(I_1)$. Then we have $(x, 0)(y, 0) = (xy, 0) \in J - \phi(J)$. As J is a ϕ-quasi primary ideal, we conclude either $(x, 0) \in \sqrt{J}$ or $(y, 0) \in \sqrt{J}$. Since $\sqrt{J} = \sqrt{I_1} \times \sqrt{I_2}$, we get $x \in \sqrt{I_1}$ or $y \in \sqrt{I_1}$. Hence, I_1 is a ψ_1-quasi primary ideal. Similarly, I_2 is a ψ_2-quasi primary ideal. We may assume that $J \neq \phi(J)$. Then we have either $I_1 \neq \psi_1(I_1)$ or $I_2 \neq \psi_2(I_2)$. Without loss of generality, we may assume that $I_1 \neq \psi_1(I_1)$. So there exists $a \in I_1 - \psi_1(I_1)$. Take $b \in I_2$. Then we have $(a, 1)(1, b) \in J - \phi(J)$. This implies either $(a, 1) \in \sqrt{J}$ or $(1, b) \in \sqrt{J}$. Then we get $1 \in \sqrt{I_1}$ or $1 \in \sqrt{I_2}$, that is, $I_1 = R_1$ or $I_2 = R_2$. Now, assume that $I_2 = R_2$. Now, we will show that I_1 is a quasi primary ideal provided that $\psi_2(R_2) \neq R_2$. So suppose $\psi_2(R_2) \neq R_2$. Let $xy \in I_1$ for some $x, y \in R_1$. Then we have $(x, 1)(y, 1) = (xy, 1) \in I_1 \times R_2 - \phi(I_1 \times R_2)$. As J is a ϕ-quasi primary ideal, we get either $(x, 1) \in \sqrt{J}$ or $(y, 1) \in \sqrt{J}$. Hence, $x \in \sqrt{I_1}$ or $y \in \sqrt{I_1}$. Therefore, I_1 is a quasi primary ideal.

$\Leftarrow$: Suppose that $J = I_1 \times I_2$ such that $\psi_i(I_i) = I_i$ for $i = 1, 2$. Since $\phi(I_1 \times I_2) = \psi_1(I_1) \times \psi_2(I_2) = I_1 \times I_2$, we get $I_1 \times I_2 - \phi(I_1 \times I_2) = \emptyset$ and so J is trivially a ϕ-quasi primary ideal. Let $J = I_1 \times R_2$ for some ψ_1-quasi primary ideal I_1 of R_1 which must be quasi primary if $\psi_2(R_2) \neq R_2$. First, assume that $\psi_2(R_2) = R_2$. Then note that $\phi(J) = \psi_1(I_1) \times R_2$. Let $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2) \in J - \phi(J)$ for some $x_i, y_i \in R_i$. Then we have $x_1y_1 \in I_1 - \psi_1(I_1)$. This yields that $x_1 \in \sqrt{I_1}$ or $y_1 \in \sqrt{I_1}$ since I_1 is a ψ_1-quasi primary ideal. Then we get either $(x_1, x_2) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$ or $(y_1, y_2) \in \sqrt{I_1 \times R_2}$. Hence, J is a ϕ-quasi primary ideal of R. Now, assume that $\psi_2(R_2) \neq R_2$ and I_1 is a quasi primary ideal. Then $I_1 \times R_2$ is a quasi primary ideal of R by [16, Lemma 2.2]. Hence, $J = I_1 \times R_2$ is a ϕ-quasi primary ideal of R. In the third case, one can see that J is also a ϕ-quasi primary ideal of R. $\square$

Theorem 3.2. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are rings and let $\psi_i : L(R_i) \to L(R_i) \cup \{\emptyset\}$ be a function for each $i = 1, 2, \ldots, n$. Let*

$\phi := \psi_1 \times \psi_2 \times \cdots \times \psi_n$ and let J be an ideal of R. Then J is a ϕ-quasi primary ideal of R if and only if J is in one of the following two forms:

(i) $J = I_1 \times I_2 \times \cdots \times I_n$ such that $\psi_i(I_i) = I_i$ for $i = 1, 2, \ldots, n$.

(ii) $J = R_1 \times R_2 \times \cdots \times R_{t-1} \times I_t \times R_{t+1} \times \cdots \times R_n$ for some ψ_t-quasi primary ideal I_t of R_t which must be quasi primary if $\psi_j(R_j) \neq R_j$ for some $j \neq t$.

Proof. We use induction on n to prove the claim. If $n = 1$, the claim is clear. If $n = 2$, the claim follows from the previous theorem. Assume that the claim is true for all $n < k$ and put $n = k$. Put $R' = R_1 \times R_2 \times \cdots \times R_{k-1},\ J' = I_1 \times I_2 \times \cdots \times I_{k-1}$ and $\phi' = \psi_1 \times \psi_2 \times \cdots \times \psi_{k-1}$. Then note that $R = R' \times R_k,\ J = J' \times J_k$ and $\phi = \phi' \times \psi_k$. Then by the previous theorem, J is a ϕ-quasi primary ideal of R if and only if one of the following conditions hold: (i) $J = J' \times I_k$ such that $\phi'(J') = J'$ and $\psi_k(I_k) = I_k$ (ii) $J = J' \times R_k$ for some ϕ'-quasi primary ideal J' of R' which must be quasi primary if $\psi_k(R_k) \neq R_k$ (iii) $J = R' \times I_k$ for some ψ_k-quasi primary ideal I_k of R_k which must be quasi primary if $\phi'(R') \neq R'$. The rest follows from the induction hypothesis and [16, Theorem 2.3]. □

Theorem 3.3. *Let R_1 and R_2 be commutative rings with identity and let $R = R_1 \times R_2$. Suppose that $\psi_i : L(R_i) \to L(R_i) \cup \{\emptyset\}$ $(i = 1, 2)$ are functions such that $\psi_2(R_2) \neq R_2$ and $\phi = \psi_1 \times \psi_2$. Then the following assertions are equivalent:*

(i) $I_1 \times R_2$ is a ϕ-2-absorbing quasi primary ideal of R.

(ii) $I_1 \times R_2$ is a 2-absorbing quasi primary ideal of R.

(iii) I_1 is a 2-absorbing quasi primary ideal of R_1.

Proof. Assume that $\psi_1(I_1) = \emptyset$ or $\psi_2(R_2) = \emptyset$. Then clearly $\phi(I_1 \times R_2) = \emptyset$ so that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from [16, Theorem 2.23]. Hence suppose that $\psi_1(I_1) \neq \emptyset$ and $\psi_2(R_2) \neq \emptyset$, so $\phi(I_1 \times R_2) \neq \emptyset$.

$(i) \Rightarrow (ii)$: Suppose that $I_1 \times R_2$ is a ϕ-2-absorbing quasi primary ideal of R. A similar argument to the one we made in the proof of Theorem 3.1 shows that I_1 is a ψ_1-2-absorbing quasi primary ideal of R_1. If I_1 is 2-absorbing quasi primary, then $I_1 \times R_2$ is a 2-absorbing quasi primary ideal of R, by [16, Theorem 2.23]. If I_1 is not 2-absorbing quasi primary, then I_1 has a strongly ψ_1-triple zero (x, y, z) for some $x, y, z \in R_1$ by Remark 1. Then $(x, 1)(y, 1)(z, 1) = (xyz, 1) \in I_1 \times R_2 - \psi_1(I_1) \times \psi_2(R_2)$ since $\psi_2(R_2) \neq R_2$. This implies that $xy \in \sqrt{I_1}$ or $yz \in \sqrt{I_1}$ or $xz \in \sqrt{I_1}$, a contradiction. Thus I_1 is 2-absorbing quasi primary. Consequently, $I_1 \times R_2$ is a 2-absorbing quasi primary ideal of R.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$: Follows from [16, Theorem 2.23]. □

Theorem 3.4. *Let R_1 and R_2 be commutative rings with identity and let $R = R_1 \times R_2$. Suppose that $\psi_i : S(R_i) \to S(R_i) \cup \{\emptyset\}$ $(i = 1, 2)$ are functions and $\phi = \psi_1 \times \psi_2$. The following statements are equivalent:*

(i) $I_1 \times R_2$ is a ϕ-2-absorbing quasi primary ideal of R that is not a 2-absorbing quasi primary ideal of R.

(ii) $\phi(I_1 \times R_2) \neq \emptyset, \psi_2(R_2) = R_2$ and I_1 is a ψ_1-2-absorbing quasi primary ideal of R_1 that is not a 2-absorbing quasi primary ideal of R_1.

Proof. $(i) \Rightarrow (ii)$: Let $I_1 \times R_2$ be ϕ-2-absorbing quasi primary ideal that is not 2-absorbing quasi primary. By Theorem 3.3, since $I_1 \times R_2$ is not a 2-absorbing quasi primary ideal of R, one can see that $\phi(I_1 \times R_2) \neq \emptyset$ and $\psi_2(R_2) = R_2$. As $I_1 \times R_2$ is a ϕ-2-absorbing quasi primary ideal of R, it is clear that I_1 is a ψ_1-2-absorbing quasi primary ideal of R_1. Also, since $I_1 \times R_2$ is not a 2-absorbing quasi primary ideal of R, I_1 is not a 2-absorbing quasi primary ideal of R_1 by [16, Theorem 2.3].

$(ii) \Rightarrow (i)$: Since $\phi(I_1 \times R_2) \neq \emptyset$ and $\psi_2(R_2) = R_2$, we get $R/\phi(I_1 \times R_2) \cong R_1/\psi_1(R_1)$ and $I_1 \times R_2/\phi(I_1 \times R_2) \cong I_1/\psi_1(I_1)$. By Proposition 2.7(ii), since I_1 is a ψ_1-2-absorbing quasi primary ideal of R_1, $I_1/\psi_1(I_1)$ is a weakly 2-absorbing quasi primary ideal of $R_1/\psi_1(R_1)$. Also, as I_1 is not a 2-absorbing quasi primary ideal of R_1, then $I_1/\psi_1(I_1)$ is not a 2-absorbing quasi primary ideal of $R_1/\psi_1(R_1)$, by Proposition 2.8(ii). Thus, $I_1 \times R_2/\phi(I_1 \times R_2)$ is a weakly 2-absorbing quasi primary ideal of $R/\phi(I_1 \times R_2)$ that is not a 2-absorbing quasi primary. Consequently, again by Proposition 2.7(ii) and Proposition 2.8(ii), we obtain that $I_1 \times R_2$ is a ϕ-2-absorbing quasi primary ideal of R that is not a 2-absorbing quasi primary ideal of R. □

The following theorem is a consequence of Theorem 3.3.

Theorem 3.5. *Let R_1 and R_2 be commutative rings with a nonzero identity and let $R = R_1 \times R_2$. Then the following assertions are equivalent:*

(i) $I_1 \times R_2$ is a weakly 2-absorbing quasi primary ideal of R.

(ii) $I_1 \times R_2$ is a 2-absorbing quasi primary ideal of R.

(iii) I_1 is a 2-absorbing quasi primary ideal of R_1.

Theorem 3.6. *Let R_1 and R_2 be commutative rings with a nonzero identity and $R = R_1 \times R_2$. Let $I_1 \times I_2$ be a proper ideal of R, where I_1, I_2 are nonzero ideals of R_1 and R_2, respectively. Then the following assertions are equivalent:*

(i) $I_1 \times I_2$ is a weakly 2-absorbing quasi primary ideal of R.

(ii) $I_1 \times I_2$ is a 2-absorbing quasi primary ideal of R.

(ii) $I_1 = R_1$ and I_2 is a 2-absorbing quasi primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing quasi primary ideal of R_1 or I_1, I_2 are quasi primary of R_1, R_2, respectively.

Proof. $(i) \Rightarrow (iii)$: Suppose that $I_1 \times I_2$ is a weakly 2-absorbing quasi primary ideal of R. If $I_1 = R_1$, by Theorem 3.5, I_2 is a 2-absorbing quasi primary ideal of R_2. Similarly, if $I_2 = R_2$, I_1 is a 2-absorbing quasi primary ideal of R_1. Thus we may assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Let us show I_2 is a quasi primary ideal of R_2. Take $x, y \in R_2$ such that $xy \in I_2$. Choose $0 \neq a \in I_1$. Then $0 \neq (a, 1)(1, x)(1, y) = (a, xy) \in I_1 \times I_2$. By our hypothesis, $(a, x) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ or $(1, xy) \in \sqrt{I_1} \times \sqrt{I_2}$ or $(a, y) \in \sqrt{I_1} \times \sqrt{I_2}$. If $(1, xy) \in \sqrt{I_1} \times \sqrt{I_2}$, a contradiction (as $I_1 \neq R_1$). Thus we obtain that $(a, x) \in \sqrt{I_1} \times \sqrt{I_2}$ or $(a, y) \in \sqrt{I_1} \times \sqrt{I_2}$. This implies that $x \in \sqrt{I_2}$ or $y \in \sqrt{I_2}$. Similarly, we can show that I_1 is a quasi primary ideal of R_1.

$(ii) \Leftrightarrow (iii)$: By [16, Theorem 2.23].

$(ii) \Rightarrow (i)$: It is clear. □

Theorem 3.7. *Let R_1 and R_2 be commutative rings with a nonzero identity and $R = R_1 \times R_2$. Then a nonzero ideal $I_1 \times I_2$ of R is weakly 2-absorbing quasi primary that is not 2-absorbing quasi primary if and only if one of the following assertions holds:*

(i) $I_1 \neq R_1$ is a nonzero weakly quasi primary ideal of R_1 that is not quasi primary and $I_2 = 0$ is a quasi primary ideal of R_2.

(ii) $I_2 \neq R_2$ is a nonzero weakly quasi primary ideal of R_2 that is not quasi primary and $I_1 = 0$ is a quasi primary ideal of R_1.

Proof. Assume that $I_1 \times I_2$ is a weakly 2-absorbing quasi primary ideal of R that is not 2-absorbing quasi primary. Suppose that $I_1 \neq 0$ and $I_2 \neq 0$. By Theorem 3.6, $I_1 \times I_2$ is 2-absorbing quasi primary, a contradiction. Thus $I_1 = 0$ or $I_2 = 0$. Without loss of generality, suppose that $I_2 = 0$. Let us prove that $I_2 = 0$ is a quasi primary ideal of R_2. Choose $x, y \in R_2$ such that $xy \in I_2$. Take $0 \neq a \in I_1$. Then $0 \neq (a,1)(1,x)(1,y) = (a,xy) \in I_1 \times I_2$. By our hypothesis, $(a,x) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ or $(1,xy) \in \sqrt{I_1} \times \sqrt{I_2}$ or $(a,y) \in \sqrt{I_1} \times \sqrt{I_2}$. Here $(1,xy) \notin \sqrt{I_1} \times \sqrt{I_2}$. Indeed, firstly observe that $I_1 \neq R_1$. If $I_1 = R_1$, then by Theorem 3.3, $I_1 \times I_2 = R_1 \times 0$ is 2-absorbing quasi primary, a contradiction. Thus we conclude that $(a,x) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ or $(a,y) \in \sqrt{I_1} \times \sqrt{I_2}$. This implies $x \in \sqrt{I_2}$ or $y \in \sqrt{I_2}$. Hence $I_2 = 0$ is quasi primary. Now, let us show that I_1 is weakly quasi primary ideal of R_1. Choose $x, y \in R_1$ such that $0 \neq xy \in I_1$. Then $0 \neq (x,1)(y,1)(1,0) = (xy,0) \in I_1 \times 0 = I_1 \times I_2$. As $I_1 \times I_2$ is weakly 2-absorbing quasi primary and $(xy,1) \notin \sqrt{I_1 \times 0}$, we have $(y,0) \in \sqrt{I_1 \times 0}$ or $(x,0) \in \sqrt{I_1 \times 0}$. This implies that $x \in \sqrt{I_1}$ or $y \in \sqrt{I_1}$. Finally, we show that I_1 is not quasi primary. Suppose that I_1 is quasi primary. As $I_2 = 0$ is a quasi primary, we have that $I_1 \times I_2$ is 2-absorbing quasi primary by [16, Theorem 2.3]. This contradicts with our assumption. Thus I_1 is not quasi primary. Conversely, assume that (i) holds. Let us prove $I_1 \times I_2$ is weakly 2-absorbing quasi primary. Let $(0,0) \neq (a_1,a_2)(b_1,b_2)(c_1,c_2) \in I = I_1 \times I_2 = I_1 \times 0$. As $a_2b_2c_2 = 0$, we get $a_1b_1c_1 \neq 0$. Since $a_2b_2c_2 \in I_2$ and I_2 is a quasi primary ideal of R_2, we get either $a_2 \in \sqrt{I_2}$ or $b_2 \in \sqrt{I_2}$ or $c_2 \in \sqrt{I_2}$. Without loss of generality, we may assume that $a_2 \in \sqrt{I_2}$. On the other hand, since $0 \neq a_1b_1c_1 = b_1(a_1c_1) \in I_1$ and I_1 is a weakly quasi primary ideal, we have either $b_1 \in \sqrt{I_1}$ or $a_1c_1 \in \sqrt{I_1}$. This implies that either $(a_1,a_2)(b_1,b_2) \in \sqrt{I_1 \times I_2}$ or $(a_1,a_2)(c_1,c_2) \in \sqrt{I_1 \times I_2}$. In other cases, one can similarly show that $(a_1,a_2)(b_1,b_2) \in \sqrt{I_1 \times I_2}$ or $(a_1,a_2)(c_1,c_2) \in \sqrt{I_1 \times I_2}$ or $(b_1,b_2)(c_1,c_2) \in \sqrt{I_1 \times I_2}$. Hence, $I_1 \times I_2$ is weakly 2-absorbing quasi primary ideal of R. Also, since I_1 is not a quasi primary ideal, $I_1 \times I_2$ is not a 2-absorbing quasi primary ideal by [16, Theorem 2.3]. □

Theorem 3.8. *Let R_1 and R_2 be commutative rings with a nonzero identity and let $R = R_1 \times R_2$. Suppose that $\psi_i : L(R_i) \to L(R_i) \cup \{\emptyset\}$ $(i = 1,2)$ are functions and $\phi = \psi_1 \times \psi_2$. Let $I = I_1 \times I_2$ be a nonzero ideal of R and $\phi(I) \neq I_1 \times I_2$. Then $I_1 \times I_2$ is ϕ-2-absorbing quasi primary that is not*

2-absorbing quasi primary if and only if $\phi(I) \neq \emptyset$ and one of the following statements holds.

(i) $\psi_2(R_2) = R_2$ and I_1 is a ψ_1-2-absorbing quasi primary ideal of R_1 that is not a 2-absorbing quasi primary ideal of R_1.

(ii) $\psi_1(R_1) = R_1$ and I_2 is a ψ_2-2-absorbing quasi primary ideal of R_2 that is not a 2-absorbing quasi primary ideal of R_2.

(iii) $I_2 = \psi_2(I_2)$ is a quasi primary ideal of R_2 and $I_1 \neq R_1$ is a ψ_1-quasi primary ideal of R_1 that is not quasi primary such that $I_1 \neq \psi_1(I_1)$ (note that if $I_1 = 0$, then $I_2 \neq 0$)

(iv) $I_1 = \psi_1(I_1)$ is a quasi primary ideal of R_1 and $I_2 \neq R_2$ is a ψ_2-quasi primary ideal of R_2 that is not quasi primary such that $I_2 \neq \psi_2(I_2)$ (note that if $I_2 = 0$, then $I_1 \neq 0$)

Proof. Suppose that $I_1 \times I_2$ is a ϕ-2-absorbing quasi primary ideal that is not 2-absorbing quasi primary. Then $\phi(I) \neq \emptyset$. Let $I_1 = R_1$. Then $\psi_1(R_1) = R_1$ and I_2 is a ψ_2-2-absorbing quasi primary ideal of R_2 that is not a 2-absorbing quasi primary ideal of R_2 by Theorem 3.4. Let $I_2 = R_2$. Then $\psi_2(R_2) = R_2$ and I_1 is a ψ_1-2-absorbing quasi primary ideal of R_1 that is not a 2-absorbing quasi primary ideal of R_1 by Theorem 3.4. Hence assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Since $\phi(I) \neq I_1 \times I_2$, we obtain that $I/\phi(I)$ is a nonzero weakly 2-absorbing quasi primary ideal of $R/\phi(I)$ that is not 2-absorbing quasi primary by Proposition 2.7(ii). Thus $I_1/\psi_1(I_1) \times I_2/\psi_2(I_2)$ is a nonzero weakly 2-absorbing quasi primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$ that is not 2-absorbing quasi primary. Then by Theorem 3.7, we know that one of the following cases holds:

Case 1: $I_1/\psi_1(I_1) = \psi_1(I_1)/\psi_1(I_1)$ is a quasi primary ideal of $R_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ is a non-zero weakly quasi primary ideal of $R_2/\psi_2(I_2)$ that is not quasi primary.

Case 2: $I_2/\psi_2(I_2) = \psi_2(I_2)/\psi_2(I_2)$ is a quasi primary ideal of $R_2/\psi_2(I_2)$ and $I_1/\psi_1(I_1)$ is a non-zero weakly quasi primary ideal of $R_1/\psi_1(I_1)$ that is not quasi primary.

Thus, (iii) or (iv) holds by Proposition 2.7(i) and Proposition 2.8(i).

Conversely, assume that $\phi(I) \neq \emptyset$. If (i) or (ii) holds, then $I_1 \times I_2$ is ϕ-2-absorbing quasi primary that is not 2-absorbing quasi primary by Theorem 3.4. Assume that (iii) or (iv) holds, then $I/\phi(I)$ is a non-zero weakly 2-absorbing quasi primary ideal of $R/\phi(I)$ that is not 2-absorbing quasi primary by Theorem 3.7. Thus $I_1 \times I_2$ is ϕ-2-absorbing quasi primary that is not 2-absorbing quasi primary of R by Proposition 2.7(ii) and Proposition 2.8(ii). □

Theorem 3.9. *Let R_1 and R_2 be commutative rings with a nonzero identity and I_1,I_2 be nonzero ideals of R_1 and R_2, respectively. Let $R = R_1 \times R_2$ and $\psi_i : L(R_i) \to L(R_i) \cup \{\emptyset\}$ $(i = 1, 2)$ be functions such that $\psi_1(I_1) \neq I_1$ and $\psi_2(I_2) \neq I_2$. Suppose that $\phi = \psi_1 \times \psi_2$ and $I_1 \times I_2$ is a proper ideal of R. Then the following assertions are equivalent:*

(i) $I_1 \times I_2$ is a ϕ-2-absorbing quasi primary ideal of R.

(ii) Either $I_1 = R_1$ and I_2 is a 2-absorbing quasi primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing quasi primary ideal of R_1 or I_1, I_2 are quasi

primary ideals of R_1 *and* R_2*, respectively.*

(iii) $I_1 \times I_2$ *is a 2-absorbing quasi primary ideal of* R.

Proof. Assume that $\psi_1(I_1) = \emptyset$ or $\psi_2(I_2) = \emptyset$. Then clearly $\phi(I_1 \times I_2) = \emptyset$ so that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from [16, Theorem 2.23]. Hence suppose that $\psi_1(I_1) \neq \emptyset$ and $\psi_2(I_2) \neq \emptyset$, so $\phi(I_1 \times I_2) \neq \emptyset$.

$(i) \Rightarrow (ii)$: Let $I_1 \times I_2$ be a ϕ-2-absorbing quasi primary ideal of R. Thus $I_1/\psi_1(I_1) \times I_2/\psi_2(I_2)$ is a non-zero weakly 2-absorbing quasi primary ideal of $R_1/\psi_1(I_1) \times R_2/\psi_2(I_2)$ by Proposition 2.7(ii). Then by Theorem 3.6, we know that one of the following cases holds:

Case 1: $I_1/\psi_1(I_1) = R_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ is a 2-absorbing quasi primary ideal of $R_2/\psi_2(I_2)$. Then we have $I_1 = R_1$ and I_2 is a 2-absorbing quasi primary ideal of R_2.

Case 2: $I_2/\psi_2(I_2) = R_2/\psi_2(I_2)$ and $I_1/\psi_1(I_1)$ is a 2-absorbing quasi primary ideal of $R_1/\psi_1(I_1)$. Similar to Case 1, $I_2 = R_2$ and I_1 is a 2-absorbing quasi primary ideal of R_1.

Case 3: $I_1/\psi_1(I_1)$ and $I_2/\psi_2(I_2)$ are quasi primary of $R_1/\psi_1(I_1), R_2/\psi_2(I_2)$, respectively. Then I_1, I_2 are quasi primary ideals of R_1 and R_2, respectively by Proposition 2.8(ii).

$(ii) \Rightarrow (iii)$: Assume that $I_1 = R_1$ and I_2 is a 2-absorbing quasi primary ideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing quasi primary ideal of R_1 or I_1, I_2 are quasi primary ideals of R_1 and R_2, respectively. Then by Theorem Theorem [16, Theorem 2.23], $I_1 \times I_2$ is a 2-absorbing quasi primary ideal of R.

$(iii) \Rightarrow (i)$: It is evident. □

References

[1] Alkan, M., Nicholson, W. K., and Özcan, A. c. Comorphic rings. *J. Algebra Appl. 17*, 4 (2018), 1850075, 21.

[2] Anderson, D. D., and Bataineh, M. Generalizations of prime ideals. *Comm. Algebra 36*, 2 (2008), 686–696.

[3] Anderson, D. D., Chun, S., and Juett, J. R. Module-theoretic generalization of commutative von Neumann regular rings. *Comm. Algebra 47*, 11 (2019), 4713–4728.

[4] Anderson, D. D., and Smith, E. Weakly prime ideals. *Houston J. Math. 29*, 4 (2003), 831–840.

[5] Anderson, D. D., and Winders, M. Idealization of a module. *J. Commut. Algebra 1*, 1 (2009), 3–56.

[6] Anderson, D. F., and Badawi, A. On n-absorbing ideals of commutative rings. *Comm. Algebra 39*, 5 (2011), 1646–1672.

[7] Atiyah, M. F., and Macdonald, I. G. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[8] Badawi, A. On 2-absorbing ideals of commutative rings. *Bull. Austral. Math. Soc. 75*, 3 (2007), 417–429.

[9] Badawi, A., Tekir, U., Aslankarayiğit Uğurlu, E., Ulucak, G., and Yetkin Çelikel, E. Generalizations of 2-absorbing primary ideals of commutative rings. *Turkish J. Math. 40*, 3 (2016), 703–717.

[10] Badawi, A., Tekir, U., and Yetkin, E. On 2-absorbing primary ideals in commutative rings. *Bull. Korean Math. Soc. 51*, 4 (2014), 1163–1173.

[11] Bhatwadekar, S. M., and Sharma, P. K. Unique factorization and birth of almost primes. *Comm. Algebra 33*, 1 (2005), 43–49.

[12] Fuchs, L. On quasi-primary ideals. *Acta Univ. Szeged. Sect. Sci. Math. 11* (1947), 174–183.

[13] Huckaba, J. A. *Commutative rings with zero divisors*, vol. 117 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1988.

[14] Jayaram, C., and Tekir, U. von Neumann regular modules. *Comm. Algebra 46*, 5 (2018), 2205–2217.

[15] Payrovi, S., and Babaei, S. On the 2-absorbing ideals. *Int. Math. Forum 7*, 5-8 (2012), 265–271.

[16] Tekir, U., Koç, S., Oral, K. H., and Shum, K. P. On 2-absorbing quasi-primary ideals in commutative rings. *Commun. Math. Stat. 4*, 1 (2016), 55–62.

[17] Von Neumann, J. On regular rings. *Proceedings of the National Academy of Sciences of the United States of America 22*, 12 (1936), 707–713.

[18] Yousefian Darani, A. On 2-absorbing and weakly 2-absorbing ideals of commutative semirings. *Kyungpook Math. J. 52*, 1 (2012), 91–97.

[19] Yousefian Darani, A., and Puczył owski, E. R. On 2-absorbing commutative semigroups and their applications to rings. *Semigroup Forum 86*, 1 (2013), 83–91.

[20] Yousefian Darani, A., and Soheilnia, F. 2-absorbing and weakly 2-absorbing submodules. *Thai J. Math. 9*, 3 (2011), 577–584.

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