# On the inclusion submodule graph of a module 

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#### Abstract

Let $R$ be a ring with identity and $M$ be a unitary left $R$ module. The inclusion submodule graph of a module $M$, denoted by $\operatorname{In}(M)$, is an undirected simple graph whose vertex set $V(\operatorname{In}(M))$ is a set of all nontrivial submodules of $M$ and there is an edge between two distinct vertices $X$ and $Y$ if and only if $X \subset Y$ or $Y \subset X$. In this paper, we investigate connections between the graph-theoretic properties of $\operatorname{In}(M)$ and some algebraic properties of modules. In particular, we consider several properties of the graph $\operatorname{In}(M)$, such as connectivity, diameter and girth. Also we obtain some independent sets and universal vertices of this graph. We characterize some modules for which the inclusion submodule graphs are connected, complete and null. Finally, we study the clique number and the chromatic number of $\operatorname{In}(M)$.


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## 1. Introduction

The investigation of graphs associated with algebraic structures is an interesting subject and very important for mathematicians. Their aim is to translate the properties of the graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about the graphs, which is written in the preface of the book Algebraic Graph Theory, see 8]. Many fundamental papers devoted to graphs assigned to rings and modules have appeared recently, see for example [1, 2, 3, 6, ,9, 11] in the ring theory and [4, 5, 12, 13, 14, 17] in the module theory. Most properties of a ring and a module are connected to a behavior of its ideals and its submodules, respectively. In 2015, the inclusion ideal graph of a ring $R$, denoted by $\operatorname{In}(R)$, was introduced in [2. By studying on the sketched idea in this work, we define the inclusion submodule graph of a module. Our main goal is to search for the connection between the algebraic properties of a module and the graph theoretic properties of the graph associated with it. Throughout this paper $R$ denotes a ring with identity and all modules are unitary left $R$-modules and all graphs are simple. The inclusion submodule graph of an $R$-module $M$, denoted by $\operatorname{In}(M)$ is defined as the graph

[^0]with the vertex set $V(\operatorname{In}(M))$ whose vertices are in one to one correspondence with all nontrivial submodules of $M$ and two distinct vertices $X$ and $Y$ are adjacent if and only if $X \subset Y$ or $Y \subset X$. By a nontrivial submodule of $M$ we mean a nonzero proper left submodule of $M$. Two distinct submodules $N$ and $K$ of an $R$-module $M$ are comparable if $N \subset K$ or $K \subset N$. An $R$-module $M$ is called uniserial if any two submodules are comparable. A submodule $N$ of an $R$-module $M$ is called small in $M$ (we write $N \ll M$ ) if, for every submodule $X \subseteq M, N+X=M$ implies that $X=M$. A nonzero $R$-module $M$ is called hollow if every proper submodule of $M$ is small. If $X$ is a maximal (minimal) submodule of $M$, we write $X \leq^{\max } M\left(X \leq^{\min } M\right)$. For a module $M$, we use $\operatorname{Max}(M)$ and $\operatorname{Min}(M)$ to denote the set of all the maximal submodules and the set of all the minimal submodules of $M$, respectively. The radical of an $R$ module $M$, denoted by $\operatorname{Rad}(M)$, is the intersection of all maximal submodules of $M$. A submodule $K$ of nonzero $R$-module $M$ is said to be essential in $M$ (we write $K \unlhd M)$ if $K \cap L \neq(0)$ for every nonzero proper submodule $L$ of $M$. If every nonzero submodule of $M$ is essential, then $M$ is called a uniform module. A nonzero $R$-module $M$ is called local if it has a unique maximal submodule that contains all other proper submodules. The socle of an $R$-module $M$, denoted by $\operatorname{Soc}(M)$, is the sum of all simple submodules of $M$. An $R$-module $M$ is said to be semisimple if $\operatorname{Soc}(M)=M$. A nonzero $R$-module $M$ is called indecomposable if it is not a direct sum of two nonzero submodules. For an $R$-module $M$, the length of $M$ is the length of composition series of $M$, denoted by $l_{R}(M)$. An $R$-module $M$ has finite length if $l_{R}(M)<\infty$. For a ring $R$, we denote the Jacobson radical of $R$ by $J(R)$. The ring of all endomorphisms of an $R$-module $M$ is denoted by $\operatorname{End}_{R}(M)$.

Let $G=(V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$, where an edge is an unordered pair of distinct vertices of $G$. By the order of $G$ we mean the number of vertices of $G$ and we denote it by $|G|$. If $x$ and $y$ are two adjacent vertices of $G$, then we write $x-y$. The degree of a vertex $v$ in a graph $G$, denoted by $\operatorname{deg}(v)$, is the number of edges incident with $v$. The maximum degree and the minimum degree of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex $u$ is called universal if it is adjacent to all other vertices. A vertex $v$ is called isolated if it is adjacent to no other vertices of the graph. A vertex $w$ is called an ending vertex if $\operatorname{deg}(w)=1$. Let $x$ and $y$ be two distinct vertices of $G$. An $x, y$-path is a path with starting vertex $x$ and ending vertex $y$. A path of $n$ vertices is denoted by $P_{n}$. For distinct vertices $x$ and $y$, $d(x, y)$ is the least length of an $x, y$-path. If $G$ has no such path, then we define $d(x, y)=\infty$. The diameter of $G$, is diam $(G)=\sup \{d(x, y): x$ and $y$ are distinct vertices of $G\}$. A cycle in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. By $(x, y, z)$ we denote a cycle of length 3 . A cycle of $n$ vertices is denoted by $C_{n}$ and is called an $n$-cycle. The girth of a graph is the length of its shortest cycle. A graph with no cycles has infinite girth. By the null graph, we mean a graph with no edges. A graph is said to be connected if there is a path between every pair of vertices. A tree is a connected graph which does not contain a cycle. A star graph is a tree consisting of one vertex adjacent to all others. A caterpillar is a tree for which
removing the leaves and incident edges produces a path graph. An r-partite graph is one whose vertex set can be partitioned into $r$ subsets so that an edge has both ends in no subset. The complete graph of order $n$, is denoted by $K_{n}$. By a clique in a graph $G$, we mean a complete subgraph of $G$. The clique number of $G$ is $\omega(G)=\sup \left\{n: K_{n}\right.$ is a subgraph of $\left.G\right\}$. An independent set in a graph is a set of pairwise non-adjacent vertices. An independence number of $G$, written $\alpha(G)$, is the maximum size of an independent set. For a graph $G$, the chromatic number of $G$, denoted by $\chi(G)$, is defined to be the minimum number of colors which can be assigned to the vertices of $G$ such a way that every two adjacent vertices have different colors.

## 2. Connectivity of the inclusion submodule graphs

In this section, we provide some conditions under which the inclusion submodule graphs are connected, complete and null. Moreover, we introduce some independent sets and universal vertices of this graph.

Theorem 2.1. Let $M$ be an $R$-module. Then the graph $\operatorname{In}(M)$ is not connected if and only if $M$ is a direct sum of two simple $R$-modules.

Proof. Suppose that $\operatorname{In}(M)$ is not connected. Assume that $G_{1}$ and $G_{2}$ are two components of $\operatorname{In}(M)$. Let $X$ and $Y$ be two distinct nontrivial submodules of $M$ such that $X \in G_{1}$ and $Y \in G_{2}$. Since there is no $X, Y$-path, $X$ is not contained in $Y$ and $Y$ is not contained in $X$. If $X \cap Y \neq(0)$, then there is an $X, Y$-path of the form $X-X \cap Y-Y$, a contradiction. Hence, assume that $X \cap Y=(0)$. Now, if $M \neq X+Y$, then there is an $X, Y$-path of the form $X-X+Y-Y$, a contradiction. Therefore, $M=X \oplus Y$. We show that $X$ and $Y$ are minimal submodules of $M$. To see this, let $Z$ be a submodule of $M$ such that $(0) \neq Z \subseteq X$. If $Z \subset X$, then $Z$ and $X$ are adjacent vertices, which implies that $Z \in G_{1}$. Hence there is no $Z, Y$-path and by arguing as above, they are not comparable. If $M \neq Z+Y$, then there is a $Z, Y$-path of the form $Z-Z+Y-Y$, a contradiction. Otherwise, $M=Z+Y$ and by the modularity condition, we have $X=X \cap(Z+Y)=Z+X \cap Y=Z$, a contradiction. Thus $X$ is a minimal submodule of $M$. A similar argument shows that $Y$ is also a minimal submodule of $M$. But, the minimality of $X$ and $Y$ imply that they are simple $R$-modules and sine $M=X \oplus Y$, we are done.

Conversely, assume that $M$ is a direct sum of two simple $R$-modules. Let $N$ and $K$ be two arbitrary distinct vertices of $\operatorname{In}(M)$ and $N-K$ be an edge in $\operatorname{In}(M)$. Then $N \subset K$ or $K \subset N$. This implies that $N \cap K \neq(0)$. Thus $N-K$ is an edge in the intersection graph $G(M)$ of an $R$-module $M$, which is studied in [4]. Hence $\operatorname{In}(M)$ is a subgraph of $G(M)$. Thus by [4], Theorem 2.1], $G(M)$ is not connected and consequently $\operatorname{In}(M)$ is not connected. This completes the proof.

Corollary 2.2. Let $M$ be an $R$-module. If $\operatorname{In}(M)$ is not connected, then In $(M)$ is a null graph, and as any null graph is not connected, so the converse is correct.

Example 2.3. Consider the inclusion submodule graphs of $\mathbb{Z}_{p q}, \mathbb{Z}_{p^{2}}$ and $\mathbb{Z}_{p} \oplus \mathbb{Z}_{p}$ as $\mathbb{Z}$-modules such that $p$ and $q$ are two distinct prime numbers. Since $\mathbb{Z}_{p q} \cong \mathbb{Z}_{p} \oplus \mathbb{Z}_{q}$, by Theorem 2.1, $\operatorname{In}\left(\mathbb{Z}_{p q}\right)$ and $\operatorname{In}\left(\mathbb{Z}_{p} \oplus \mathbb{Z}_{q}\right)$ are not connected and $\operatorname{In}\left(\mathbb{Z}_{p q}\right)=\operatorname{In}\left(\mathbb{Z}_{p} \oplus \mathbb{Z}_{q}\right) \cong \overline{K_{2}}$. In particular for $p=q$, we have that the only nontrivial submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^{2}}$ is $p \mathbb{Z}_{p^{2}}$ and thus $\operatorname{In}\left(\mathbb{Z}_{p^{2}}\right) \cong K_{1}$. Also, we can see easily that $\mathbb{Z}_{p} \oplus \mathbb{Z}_{p}$ as a $\mathbb{Z}$-module has exactly $p+1$ nontrivial submodules of order $p$ which are isolated vertices of $\operatorname{In}\left(\mathbb{Z}_{p} \oplus \mathbb{Z}_{p}\right)$. Consequently, $\operatorname{In}\left(\mathbb{Z}_{p} \oplus \mathbb{Z}_{p}\right) \cong \overline{K_{p+1}}$.

Corollary 2.4. Let $M$ be an $R$-module which is not simple. Then $\operatorname{In}(M)$ is connected if and only if either $M$ is not semisimple or $M=\oplus_{i=1}^{i=n} M_{i}$, where $n \geq 3$ and $M_{i}$ is a simple $R$-module, for $1 \leq i \leq n$.

Example 2.5. (1) Consider the $\mathbb{Z}$-module $\mathbb{Z}_{2} \oplus \mathbb{Z}_{4}$ which is not semisimple. The nontrivial submodules of $\mathbb{Z}_{2} \oplus \mathbb{Z}_{4}$ are $\left.\left.\langle(0,1)\rangle,<(0,2)\right\rangle,<(1,0)\right\rangle$, $<(1,1)>,<(1,2)>$ and $N=\mathbb{Z}_{2} \times\{0,2\}$, which is not cyclic. The graph $\operatorname{In}\left(\mathbb{Z}_{2} \oplus \mathbb{Z}_{4}\right)$ is a tree with four end vertices. (See Fig. 1).


Fig. 1. $\operatorname{In}\left(\mathbb{Z}_{2} \oplus \mathbb{Z}_{4}\right)$
(2) Let $p$ and $q$ are two distinct prime numbers. The $\mathbb{Z}$-modules $\mathbb{Z}_{p q^{2}}$ and $\mathbb{Z}_{p^{2} q^{2}}$ are not semisimple. We have:
(a) $\operatorname{In}\left(\mathbb{Z}_{p q^{2}}\right)$ is a tree with two end vertices and also it is isomorphic to the 2-partite graph $P_{4}$.
(b) $\operatorname{In}\left(\mathbb{Z}_{p^{2} q^{2}}\right)$ is connected. Also it is an Eulerian graph and has a Hamiltonian cycle of the length 7. (See Fig. 2).


Fig. 2. $\operatorname{In}\left(\mathbb{Z}_{p^{2} q^{2}}\right)$

Example 2.6. Consider $\mathbb{Z}_{p_{1} p_{2} p_{3}}$ as a $\mathbb{Z}$-module such that $p_{i}$ is a prime number, for $i=1,2,3$. Since $\mathbb{Z}_{p_{1} p_{2} p_{3}}=<p_{1} p_{2}>\oplus<p_{1} p_{3}>\oplus<p_{2} p_{3}>$, by Corollary 2.4 the inclusion submodule graph of $\mathbb{Z}_{p_{1} p_{2} p_{3}}$ is connected. Moreover, $\operatorname{In}\left(\mathbb{Z}_{p_{1} p_{2} p_{3}}\right) \cong C_{6}$.
Proposition 2.7. Let $M$ be an $R$-module. If $\operatorname{In}(M)$ is a connected graph, then the following statements hold:
(1) Every pair of maximal submodules of $M$ are not adjacent vertices of $\operatorname{In}(M)$ and they have nontrivial intersection and also there exists a path between them.
(2) Every pair of minimal submodules of $M$ are not adjacent vertices of In(M) and they have nontrivial sum and also there exists a path between them.

Proof. Let $\operatorname{Max}(M)=\left\{X \mid X \leq^{\max } M\right\}$ and $\operatorname{Min}(M)=\left\{Y \mid Y \leq^{\min } M\right\}$.
(1) Suppose that $X_{1}, X_{2} \in \operatorname{Max}(M)$. If $X_{1} \subset X_{2}$, then the maximality of $X_{1}$ implies that $M=X_{2}$, a contradiction. Similarly, if $X_{2} \subset X_{1}$, then $M=X_{1}$ which again yields a contradiction. Hence $X_{1}$ and $X_{2}$ are not comparable. Thus every pair of maximal submodules of $M$ are not two adjacent vertices. Now, we show that $X_{1} \cap X_{2}$ is a vertex of $\operatorname{In}(M)$. Clearly, $X_{1} \cap X_{2} \neq M$. Since $X_{i} \subset X_{1}+X_{2} \subseteq M$ for $i=1,2$, the maximality of $X_{i}$ implies that $M=X_{1}+X_{2}$. Let $X_{1} \cap X_{2}=(0)$. So $M=X_{1} \oplus X_{2}$. However, $M / X_{1} \cong X_{2}$ and $M / X_{2} \cong X_{1}$, hence $X_{1}$ and $X_{2}$ are two simple $R$-modules. Now, by Theorem 2.1, $\operatorname{In}(M)$ is not connected, a contradiction. Therefore, $X_{1} \cap X_{2} \neq(0)$ and there exists an $X_{1}, X_{2}$-path of the form $X_{1}-X_{1} \cap X_{2}-X_{2}$.
(2) Assume that $Y_{1}, Y_{2} \in \operatorname{Min}(M)$. If $Y_{1} \subset Y_{2}$, then the minimality of $Y_{2}$ implies that $Y_{1}=(0)$, a contradiction. A similar argument shows that if $Y_{2} \subset Y_{1}$, then $Y_{2}=(0)$, which again yields a contradiction. Hence $Y_{1}$ and $Y_{2}$ are not comparable. Thus every pair of minimal submodules of $M$ are not two adjacent vertices. Now, we show that $Y_{1}+Y_{2}$ is a vertex of $\operatorname{In}(M)$. Clearly, $Y_{1}+Y_{2} \neq(0)$. If $Y_{1} \cap Y_{2} \neq(0)$, since $(0) \subset Y_{1} \cap Y_{2} \subset Y_{i} \subset M$ for $i=1,2$, the minimality of $Y_{i}$ implies that $Y_{1} \cap Y_{2}=Y_{1}=Y_{2}$, a contradiction with $Y_{1} \neq Y_{2}$. Hence, $Y_{1} \cap Y_{2}=(0)$. Let us get $M=Y_{1}+Y_{2}$. Then $M=Y_{1} \oplus Y_{2}$ such that $Y_{1}$ and $Y_{2}$ are two simple $R$-modules. Hence by Theorem 2.1, $\operatorname{In}(M)$ is not connected, a contradiction. Therefore, $Y_{1}+Y_{2} \neq M$ and there is a $Y_{1}, Y_{2}$-path of the form $Y_{1}-Y_{1}+Y_{2}-Y_{2}$.

The following corollary is an immediate consequence of Proposition 2.7.
Corollary 2.8. Let $M$ be an $R$-module. Then we have:
(1) $\operatorname{Max}(M)$ and $\operatorname{Min}(M)$ are two independent sets of $\operatorname{In}(M)$.
(2) $\alpha(\operatorname{In}(M)) \geq \max \{\operatorname{card}(\operatorname{Max}(M)), \operatorname{card}(\operatorname{Min}(M))\}$.

Example 2.9. Consider $\mathbb{Z}, \mathbb{Z}_{p} \oplus \mathbb{Z}_{p}$ and $\mathbb{Z}_{p_{1} p_{2} p_{3}}$ as $\mathbb{Z}$-modules, where $p$ and $p_{i}$ are primes, for $i=1,2,3$. We know that $\operatorname{Max}(\mathbb{Z})=\{<p\rangle: p \in \mathbb{Z}$ and $p$ is prime $\}$ is an independent set of the graph $\operatorname{In}(\mathbb{Z})$ and $\alpha(\operatorname{In}(\mathbb{Z}))=\infty$. Also, we can see two Examples 2.3 and 2.6 that $\alpha\left(\operatorname{In}\left(\mathbb{Z}_{p} \oplus \mathbb{Z}_{p}\right)\right) \geq p+1$ and $\alpha\left(\operatorname{In}\left(\mathbb{Z}_{p_{1} p_{2} p_{3}}\right)\right)=3$.

Proposition 2.10. Let $M$ be an $R$-module and $N$ be a nontrivial submodule of $M$. Then the following statements hold:
(1) For any nontrivial submodule $K$ of $M$, if $N$ and $K$ are not comparable, then $N$ is an isolated vertex.
(2) If $N$ is a direct summand of $M$ and for every nonzero submodule $K$ of $M$, $\operatorname{Rad}(K) \neq(0)$, then there is a path of length 2 in $\operatorname{In}(M)$.
(3) If $N$ is a small submodule of $M$ and $\operatorname{Rad}(M) \neq N$, then for any nontrivial submodule $K$ of $M$, which is not comparable with $N$, there is a path of length 2 in $\operatorname{In}(M)$ and $d(N, \operatorname{Rad}(M))=1$.
(4) If $N$ is an essential submodule of $M$ such that $\operatorname{Soc}(M) \neq N$ and $|\operatorname{In}(M)| \geq$ 3 , then there is a cycle of length 3 in $\operatorname{In}(M)$ and $d(N, \operatorname{Soc}(M))=1$.

Proof. (1) Obvious.
(2) Assume that $N$ is a nonzero proper submodule of $M$ which is a direct summand of $M$ and for every nonzero submodule $K$ of $M, \operatorname{Rad}(K) \neq(0)$. Then there exists a submodule $L$ of $M$ such that $N \oplus L=M$. Hence $\operatorname{Rad}(N) \oplus$ $\operatorname{Rad}(L)=\operatorname{Rad}(M)$. Since $\operatorname{Rad}(N) \subset N$ and $L \cap \operatorname{Rad}(N) \subset L \cap N=(0)$, by the modularity condition, we obtain $N \cap \operatorname{Rad}(M)=\operatorname{Rad}(N) \subset \operatorname{Rad}(M)$. Hence, there is an $N, \operatorname{Rad}(M)$-path of the form $N-\operatorname{Rad}(N)-\operatorname{Rad}(M)$ of length 2.
(3) Since $N$ is a nontrivial small submodule of $M$, for any nontrivial submodule $K$ of $M, N+K \neq M$. As $N$ and $K$ are not comparable, there is a path of the form $N-N+K-K$ of length 2 in $\operatorname{In}(M)$. Moreover, by Proposition 9.13 of [[7], p. 120], $\operatorname{Rad}(M)$ is the sum of all small submodules of $M$, then $N \subset \operatorname{Rad}(M)$ and $d(N, \operatorname{Rad}(M))=1$.
(4) Since $N$ is an essential submodule of $M$ and since by Proposition 9.7 of [7], p. 118], $\operatorname{Soc}(M)$ is the intersection of all essential submodules of $M$, then $\operatorname{Soc}(M) \subset N$. Also, by Corollary 9.9 of [[7], p. 119], $\operatorname{Soc}(N)=N \cap \operatorname{Soc}(M) \subset$ $N, \operatorname{Soc}(M)$. Then there is a cycle of the form $(N, \operatorname{Soc}(N), \operatorname{Soc}(M))$ of length 3 in $\operatorname{In}(M)$. Moreover, $d(N, \operatorname{Soc}(M))=1$.

Corollary 2.11. Let $M$ be an $R$-module. Then $\operatorname{In}(M)$ is a connected graph if one of the following holds:
(1) The module $M$ is hollow and $\operatorname{Rad}(M) \neq M$.
(2) The module $M$ is uniform.
(3) The module $M$ is indecomposable.
(4) The module $M$ is finitely generated and $\operatorname{Rad}(M) \neq(0)$.
(5) The module $M$ is finitely cogenerated with $\operatorname{Rad}(M) \neq(0)$ and $\operatorname{Soc}(M) \neq$ (0).
(6) The module $M$ is self-injective and indecomposable.
(7) The module $M$ is self-projective with $\operatorname{Rad}(M) \neq M$ and $\operatorname{End}_{R}(M)$ is a local ring.

Proof. It is clear that each of the conditions in the corollary implies that $M$ is not a direct sum of two simple $R$-modules. Hence, the corollary is an immediate consequence of Theorem 2.1 .

Theorem 2.12. Let $M$ be an $R$-module and $\operatorname{In}(M)$ be a graph with $|\operatorname{In}(M)| \geq$ 3. Then $\operatorname{In}(M)$ is connected with at least one cycle, if one of the following holds:
(1) The module $M$ is Noetherian and contains a unique maximal submodule.
(2) The module $M$ is Artinian and contains a unique minimal submodule.

Proof. (1) Suppose that $M$ is a Noetherian $R$-module. Then $M$ has at least one maximal submodule. Moreover, every nonzero submodule of $M$ is contained in a maximal submodule. Therefore, if $M$ possesses a unique maximal submodule, say $U$, then $U$ contains every nonzero submodule of $M$. Let $N$ and $K$ be two distinct vertices of $\operatorname{In}(M)$. Hence $N \subset U$ and $K \subset U$ and thus $U$ is an adjacent vertex to both $N$ and $K$. Then there is a $N, K$-path of the form $N-U-K$, and this implies that the graph $\operatorname{In}(M)$ is connected. However, since $N, K \subseteq N+K \subset U \neq M$, if $N+K=N$ or $N+K=K$, then $N$ and $K$ are two adjacent vertices and $(N, U, K)$ is a cycle. Otherwise, there are three cycles of the forms $(N, U, N+K)$ and $(K, U, N+K)$ and also $N-U-K-N+K-N$. Therefore, $\operatorname{In}(M)$ is a connected graph with at least one cycle.
(2) Assume that $M$ is an Artinian $R$-module. Then $M$ has at least one minimal submodule. Moreover, every nonzero submodule of $M$ contains a minimal submodule. Therefore, if $M$ possesses a unique minimal submodule, say $L$, then $L$ is contained in every nonzero submodule of $M$. Let $A$ and $B$ be two distinct vertices of $\operatorname{In}(M)$. Hence $L \subset A$ and $L \subset B$ and thus $L$ is an adjacent vertex to both $A$ and $B$. Then there is an $A, B$-path, of the form $A-L-B$. Therefore, $\operatorname{In}(M)$ is a connected graph. However, since $L \subset A \cap B \subseteq A, B \neq M$, if $A \cap B=A$ or $A \cap B=B$, then $A$ and $B$ are two adjacent vertices and $(A, L, B)$ is a cycle. Otherwise, there are three cycles of the forms $(A, L, A \cap B)$ and $(B, L, A \cap B)$ and also $A-L-B-A \cap B-A$. Consequently, $\operatorname{In}(M)$ is a connected graph with at least one cycle.

Lemma 2.13. Let $R$ be a ring and $M$ be an $R$-module. Then the following hold:
(1) The graph $\operatorname{In}(M)$ is complete if and only if the module $M$ is uniserial.
(2) If $R$ has the only one left maximal ideal and every finitely generated submodule of $M$ is cyclic, then the graph $\operatorname{In}(M)$ is complete.

Proof. Part 1 is obvious and Part 2 follows from the proof of Part 1 of [14], Theorem 2.4].

Proposition 2.14. Let $M$ be an $R$-module and $\operatorname{In}(M)$ be a complete graph. Then every nontrivial submodule of $M$ is small and essential.

Proof. Suppose that $N$ is an arbitrary nontrivial submodule of $M$ and $\operatorname{In}(M)$ is a complete graph. Assume that for every submodule $X$ of $M, N+X=M$. If $X \subseteq N$, then $M=N$, which is a contradiction. However, if $N \subseteq X$, then $N+X=X$. Thus $M=X$, hence $N \ll M$. Now, we claim that $N$ is an essential submodule in $M$. To see this, if for every nonzero submodule $X$ of $M, X \subseteq N$, then $X \cap N=X \neq(0)$ and also if $N \subseteq X$, then $N \cap X=N \neq(0)$. Hence, $N \unlhd M$. Consequently, every nontrivial submodule of $M$ is small and essential.

Proposition 2.15. Let $M$ be an $R$-module such that $\operatorname{Rad}(M)$ and $\operatorname{Soc}(M)$ are two distinct vertices of $\operatorname{In}(M)$. Then the following statements hold:
(1) If $M$ is hollow, then $\operatorname{Rad}(M)$ is a universal vertex.
(2) If $M$ is uniform, then $\operatorname{Soc}(M)$ is a universal vertex.

Proof. (1) Suppose that $M$ is a hollow $R$-module. Then every nontrivial submodule of $M$ is small in $M$. Since, by Proposition 9.13 of [ 7 ], p. 120], $\operatorname{Rad}(M)$ is the sum of all small submodules of $M$, then every nontrivial small submodule is contained in $\operatorname{Rad}(M)$. Thus every nontrivial small submodule in $M$ is an adjacent vertex to $\operatorname{Rad}(M)$. Therefore, $\operatorname{Rad}(M)$ is a universal vertex in $\operatorname{In}(M)$.
(2) Assume that $M$ is a uniform $R$-module. Then every nontrivial submodule of $M$ is essential in $M$. Since, by Proposition 9.7 of [[7], p. 118], $\operatorname{Soc}(M)$ is the intersection of all essential submodules of $M$, then $\operatorname{Soc}(M)$ is contained in every nontrivial essential submodule of $M$, thus it is an adjacent vertex to every nontrivial essential submodule in $M$. Therefore, $\operatorname{Soc}(M)$ is a universal vertex in $\operatorname{In}(M)$.

## 3. Diameter and girth of the inclusion submodule graphs

In this section, we determine the diameter and the girth of the inclusion submodule graphs. Also, we prove that, if the graph $\operatorname{In}(M)$ is a tree, then $\operatorname{In}(M)$ is a caterpillar with $\operatorname{diam}(\operatorname{In}(M)) \leq 3$ and if the module $M$ is local, then $\operatorname{In}(M)$ is a star graph.

Theorem 3.1. Let $M$ be an $R$-module with the connected $\operatorname{graph} \operatorname{In}(M)$. Then $\operatorname{diam}(\operatorname{In}(M)) \leq 3$.

Proof. Let $A$ and $B$ be two nontrivial distinct submodules of $M$. If $A$ and $B$ are comparable, then there is an edge between $A$ and $B$ and we are done. Thus suppose that $A$ and $B$ are not comparable. Now, we consider three cases.
Case 1. Let $A+B \neq M$. Then there exists an $A, B$ - path of the form $A-A+B-B$ of length 2 , so $d(A, B)=2$. Case 2 . Let $A \cap B \neq(0)$. Then there exists an $A, B$-path of the form $A-A \cap B-B$ of length 2 , so $d(A, B)=2$. Case 3. Let $A+B=M$ and $A \cap B=(0)$. Then $M=A \oplus B$ and since $\operatorname{In}(M)$ is connected, we conclude that at least one of $A$ and $B$ should be non-maximal. To see this, let both $A$ and $B$ are maximal. Since $A \cong M / B, A$ is simple and similarity $B$ is simple and by Theorem 2.1, $\operatorname{In}(M)$ is not connected, a contradiction. So assume that $B$ is not maximal. Hence there exists a submodule $C$ of $M$ such that $B \subset C \subset M$. Then $B$ and $C$ are two adjacent vertices of $\operatorname{In}(M)$. Now, if $A$ and $C$ are comparable, then there exists an $A, B$-path of the form $A-C-B$ of length 2 , so $d(A, B)=2$. But if $A$ and $C$ are not comparable, by the modularity condition, we have $C=C \cap M=C \cap(A \oplus B)=(C \cap A) \oplus B$. Now, if $C \cap A=(0)$, then $C=B$, a contradiction with the existence of $C$. Also, if $C \cap A \neq(0)$, then there exists an $A, B$-path of the form $A-C \cap A-C-B$ of length 3 , so $d(A, B) \leq 3$. Therefore, $\operatorname{diam}(\operatorname{In}(M)) \leq 3$.

Example 3.2. By Example 2.5, $\operatorname{diam}\left(\operatorname{In}\left(\mathbb{Z}_{2} \oplus \mathbb{Z}_{4}\right)\right)=\operatorname{diam}\left(\operatorname{In}\left(\mathbb{Z}_{36}\right)\right)=3$ and by Example 2.6, $\operatorname{diam}\left(\operatorname{In}\left(\mathbb{Z}_{70}\right)\right)=3$.

Remark 3.3. Let $R$ be a ring with the graph $\operatorname{In}(R)$ and $M_{2}(R)$ be the set of 2 by 2 matrices over the ring $R$. Then the following statements hold:
(1) If $R$ is not isomorphic to $M_{2}(D)$, nor to $D_{1} \times D_{2}$, where $D, D_{1}$ and $D_{2}$ are division rings, then $\operatorname{In}(R)$ is connected with $\operatorname{diam}(\operatorname{In}(R)) \leq 3$.
(2) If $R$ is an integral domain and $|\operatorname{In}(R)|>2$, then $\operatorname{In}(R)$ is connected with $\operatorname{diam}(\operatorname{In}(R)) \leq 2$.

Proof. (1) It is an immediate consequence of [[2], Theorem 1].
(2) Suppose that $I$ and $J$ are two distinct ideals of integral domain $R$. If $I \subset J$ or $J \subset I$, then they are two adjacent vertices of $\operatorname{In}(R)$, so $d(I, J)=1$. We know that $I \cap J \subseteq I, J \subseteq I+J$. Now, if $I$ is not contained in $J$ and $J$ is not contained in $I$, then there exist three cases.
Case 1. $I+J \neq R$ imply that there exists an $I, J$-path of the form $I-I+J-J$, so $d(I, J)=2$. Case 2 . $I \cap J \neq(0)$ imply that there is an $I, J$-path of the form $I-I \cap J-J$, so $d(I, J)=2$. Case $3 . ~ I+J=R$ and $I \cap J=(0)$ imply that $R=I \oplus J$. Hence there is an idempotent $e$ in $R$, such that $I=R e$ and $J=R(1-e)$. Since the integral domain $R$ has no zero divisor, then $e=0$ or $e=1$, thus $I=(0)$ and $J=R$ or $I=R$ and $J=(0)$, a contradiction. Therefore, by the above arguments, $d(I, J) \leq 2$. Consequently, $\operatorname{In}(R)$ is a connected graph and $\operatorname{diam}(\operatorname{In}(R)) \leq 2$.

Proposition 3.4. Let $M$ be an $R$-module such that $\operatorname{In}(M)$ be a tree. Then the following statements hold:
(1) The graph $\operatorname{In}(M)$ is a caterpillar with $\operatorname{diam}(\operatorname{In}(M)) \leq 3$.
(2) If $M$ is a local $R$-module, then $\operatorname{In}(M)$ is a star graph.

Proof. (1) It follows form Theorem 2.1 .
(2) Suppose that $M$ is a local $R$-module. Then by 41.4 Part 2 of [16], p. 352], $M$ is hollow and by Corollary 2.11 Part $1, \operatorname{In}(M)$ is connected. But, by Proposition 2.15 Part 1, $\operatorname{Rad}(M)$ is a universal vertex of the graph $\operatorname{In}(M)$. Let $X$ and $Y$ be two distinct vertices of $\operatorname{In}(M)$ and different from $\operatorname{Rad}(M)$. Since $\operatorname{In}(M)$ is a tree, it has no cycles, thus $X$ and $Y$ are not comparable. Therefore, $\operatorname{In}(M)$ is a star graph.

Example 3.5. Suppose that $R=F[x, y] /(x, y)^{2}$, where $F$ is an infinite field and $x$ and $y$ are indeterminates. Then $I=\overline{(x, y)}, I_{x}=\overline{(x)}, I_{y}=\overline{(y)}$, and $I_{a}=\{\overline{(a x+y)} \mid 0 \neq a \in F\}$ are all nontrivial ideals of $R$. Also, $I$ is the only maximal ideal of $R$, and for every proper ideal $J$ of $R$, we have $J \subseteq I$. Since every pair of nontrivial ideals except $I$ are not comparable, $J(R)=I$ is the only universal vertex of infinite degree of the graph $\operatorname{In}(R)$. Hence, $\operatorname{In}(R)$ is an infinite star graph.

In the following theorem we show that for any $R$-module $M, \operatorname{girth}(\operatorname{In}(M)) \in$ $\{3,6, \infty\}$.

Theorem 3.6. For any module $M$, exactly one of the following three claims holds: Either the graph $\operatorname{In}(M)$ is acyclic, or the girth of $\operatorname{In}(M)$ is 3, or the girth of $\operatorname{In}(M)$ is 6 .

Proof. First we prove that if the graph $\operatorname{In}(M)$ has an isolated vertex $A$, then $\operatorname{In}(M)$ is a null graph. Assume the opposite, let $B-C$ be an edge in $\operatorname{In}(M)$, while $A$ is an isolated vertex in the same graph. Then $A$ is incomparable to both $B$ and $C$, so $A \cap B=A \cap C=(0)$ and $A+B=A+C=M$. Therefore, (0), $A, B, C, M$ form a sublattice of the submodule lattice of $M$, and this sublattice is isomorphic to the pentagon lattice $N_{5}$. This would imply that the submodule lattice of $M$ is not modular, and that is a contradiction, as the modularity of the submodule lattices was proved by Dedekind in the 1800s. The contradiction proves our claim, that either all vertices of $\operatorname{In}(M)$ are isolated, or none are.

Assume from now on that $\operatorname{In}(M)$ is not acyclic. Hence each vertex of $\operatorname{In}(M)$ is adjacent to at least one other vertex. If there are three distinct vertices $A$, $B$ and $C$ in $\operatorname{In}(M)$ such that $A \subset B \subset C$, then these three form a 3-cycle and $\operatorname{girth}(\operatorname{In}(M))=3$. From now on, assume such three vertices do not exist in $\operatorname{In}(M)$. Together with the assumption that $\operatorname{In}(M)$ has no isolated vertices, we obtain that the submodules in $\operatorname{In}(M)$ can be partitioned into two disjoint levels: Level 1 and Level 2. Each submodule in Level 1 is a subset of some submodule on Level 2 and each submodule on Level 2 contains some submodule on Level 1. It means that the graph $\operatorname{In}(M)$ is bipartite, so it has no odd cycles. Next we claim that for any two distinct $A, B \in \operatorname{In}(M)$ such that both $A$ and $B$ are on Level 1, the sum $A+B \neq M$. Assume the opposite, that $A+B=M$. Let $C \in \operatorname{In}(M)$ be on Level 2 such that $B \subset C$. As $A+B=M$, thus $A+C=M$, and hence $C$ and $A$ are incomparable. Therefore, $A \cap C=(0)$ and hence also $A \cap B=(0)$. Again, we obtain that (0), $A, B, C, M$ form a sublattice of the submodule lattice of $M$ isomorphic to the pentagon $N_{5}$, a contradiction. The contradiction proves the claim. By a dual argument, with reversing all inclusions, and transposing + and $\cap$, we prove the dual claim, that for any two distinct $A, B \in \operatorname{In}(M)$ such that both $A$ and $B$ are on Level 2 , the intersection $A \cap B \neq(0)$. Putting the two claims together, we know that for any two distinct $A, B \in \operatorname{In}(M)$, if $A$ and $B$ are both on Level 1 , then $A+B$ is on Level 2 , while if $A$ and $B$ are both on Level 2 , then $A \cap B$ is on Level 1 .

Now, assume that the graph $\operatorname{In}(M)$ has an $n$-cycle. We claim that it is impossible that $\operatorname{In}(M)$ has a 4 -cycle or 5 -cycle. To prove this claim, let $M_{1}, M_{2}, M_{3}, M_{4}, M_{5}$ be a 5 -cycle of $\operatorname{In}(M)$. We can see easily that there exists a chain $M_{i} \subset M_{j} \subset M_{k}$ in $M$, where $1 \leq i, j, k \leq 5$. Thus, $\operatorname{In}(M)$ contains a 3-cycle. Now, let $M_{1}, M_{2}, M_{3}, M_{4}$ is a 4-cycle of $\operatorname{In}(M)$. If $M_{1}$ and $M_{3}$ are incomparable, also $M_{2}$ and $M_{4}$ are incomparable, then $M_{1}+M_{3} \subseteq M_{2}, M_{4}$ or $M_{2}, M_{4} \subseteq M_{1} \cap M_{3}$. Hence, $M_{1} \cap M_{3} \neq M_{i}$ and $M_{1}+M_{3} \neq M_{i}$ for $1 \leq i \leq 4$. Therefore, $\left(M_{1}, M_{1}+M_{3}, M_{2}\right)$ or $\left(M_{1}, M_{1} \cap M_{3}, M_{2}\right)$ is a 3 -cycle. Consequently, $\operatorname{girth}(\operatorname{In}(M))=3$.

Now we are ready to complete the proof of this theorem. In the case when the graph $\operatorname{In}(M)$ contains an $n$-cycle of length greater than 6 , the case when $n$ is odd produces a 3 -cycle similarly as the case when $n=5$ which is mentioned in the argument above. Finally, suppose that $A_{1}, A_{2}, \ldots, A_{2 n}$ is a cycle in $\operatorname{In}(M)$. If $2 n=6$, we are finished, as the girth of $\operatorname{In}(M)$ must be 6 . So assume that $2 n>6$, and without loss of generality, we assume that $A_{1}$ is on Level

1. Thus $A_{3}$ and $A_{5}$ are also on Level 1 , while $A_{2}$ and $A_{4}$ are on Level 2. By the previous argument, $A_{1}+A_{5}$ is on Level 2. We claim that $A_{1}+A_{5} \neq A_{2}$. Assume the opposite, $A_{1}+A_{5}=A_{2}$, thus $A_{5} \subset A_{2}$ and $A_{3}, A_{4}, A_{5}, A_{2}$ form a 4 -cycle in $\operatorname{In}(M)$, which is a contradiction. An analogous argument proves that $A_{1}+A_{5} \neq A_{4}$. Therefore, $A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{1}+A_{5}$ form a 6 -cycle in $\operatorname{In}(M)$, and hence $\operatorname{girth}(\operatorname{In}(M))=6$.

Example 3.7. Consider the $\mathbb{Z}$-modules $\mathbb{Z}, \mathbb{Z}_{20}, \mathbb{Z}_{30}, \mathbb{Z}_{36}$ and $\mathbb{Z}_{5} \oplus \mathbb{Z}_{5}$. By Theorem 2.1, we know that $\operatorname{In}(\mathbb{Z})$ and $\operatorname{In}\left(\mathbb{Z}_{36}\right)$ are two connected graphs and they contain 3 -cycles of the forms $(2 \mathbb{Z}, 4 \mathbb{Z}, 8 \mathbb{Z})$ and $\left(2 \mathbb{Z}_{36}, 4 \mathbb{Z}_{36}, 12 \mathbb{Z}_{36}\right)$, respectively. Also, by Examples 2.3 2.5 and 2.6. $\operatorname{In}\left(\mathbb{Z}_{20}\right) \cong P_{4}$ and $\operatorname{In}\left(\mathbb{Z}_{30}\right) \cong$ $C_{6}$ and also $\operatorname{In}\left(\mathbb{Z}_{5} \oplus \mathbb{Z}_{5}\right)$ is a null graph. Then by Theorem 3.6, $\operatorname{girth}(\operatorname{In}(\mathbb{Z}))=$ $\operatorname{girth}\left(\operatorname{In}\left(\mathbb{Z}_{36}\right)\right)=3$ and $\operatorname{girth}\left(\operatorname{In}\left(\mathbb{Z}_{30}\right)\right)=6$ and also

$$
\operatorname{girth}\left(\operatorname{In}\left(\mathbb{Z}_{20}\right)\right)=\operatorname{girth}\left(\operatorname{In}\left(\mathbb{Z}_{5} \oplus \mathbb{Z}_{5}\right)\right)=\infty
$$

## 4. Clique number and chromatic number of the inclusion submodule graphs

Let $M$ be an R-module. In this section, we obtain some results on the clique number and the chromatic number of $\operatorname{In}(M)$. We study the condition under which the chromatic number of $\operatorname{In}(M)$ is finite. It is proved that $\chi(\operatorname{In}(M))$ is finite, provided $\omega(\operatorname{In}(M))$ is finite and also if $\delta=\delta(\operatorname{In}(M)) \geq 1$ and $\Delta=$ $\Delta(\operatorname{In}(M))<\infty$, then both $\omega(\operatorname{In}(M))$ and $\chi(\operatorname{In}(M))$ are finite.

Lemma 4.1. Let $M$ be an $R$-module. Suppose that $S(M)$ and $E(M)$ are the set of all nontrivial small submodules and the set of all nontrivial essential submodules of $M$, respectively. Then the following statements hold:
(1) $\omega(\operatorname{In}(M))=1$ if and only if either $|\operatorname{In}(M)|=1$ or $|\operatorname{In}(M)| \geq 2$ and $M$ is a direct sum of two simple $R$-modules.
(2) If $\omega(\operatorname{In}(M))<\infty$, then $l_{R}(M)<\infty$.
(3) If $\operatorname{card}(S(M))<\infty$ or $\operatorname{card}(E(M))<\infty$, then for every nontrivial submodule $N$ of $M, \omega(\operatorname{In}(N))<\infty$ and $\omega(\operatorname{In}(M / N))<\infty$.

Proof. (1) Suppose that $\omega(\operatorname{In}(M))=1$ and $|\operatorname{In}(M)| \geq 2$. This implies that $\operatorname{In}(M)$ is not connected. Hence, by Theorem 2.1, $M$ is a direct sum of two simple $R$-modules. The converse is straightforward.
(2) Clearly, $l_{R}(M) \leq \omega(\operatorname{In}(M))+1$ and since $\omega(\operatorname{In}(M))<\infty, l_{R}(M)<\infty$.
(3) In order to establish this part, first we claim that $\omega(\operatorname{In}(M))<\infty$. To see this, let $\omega(\operatorname{In}(M))=\infty$. Then $\operatorname{In}(M)$ has an infinite maximal clique $H$. Hence by Proposition 2.14, every vertex of the maximal clique $H$ is a small and essential submodule in $M$. Then $\operatorname{card}(S(M))=\operatorname{card}(E(M))=\infty$, a contradiction. Hence, $\omega(\operatorname{In}(M))<\infty$. Now, suppose that $N$ is a nontrivial submodule of $M$. Since $\omega(\operatorname{In}(N)) \leq \omega(\operatorname{In}(M))$ and $\omega(\operatorname{In}(M / N)) \leq \omega(\operatorname{In}(M))$, thus $\omega(\operatorname{In}(N))<\infty$ and $\omega(\operatorname{In}(M / N))<\infty$.

Lemma 4.2. Let $M$ be an $R$-module. If $N$ is a vertex of the graph $\operatorname{In}(M)$ such that $\operatorname{deg}(N)<\infty$, then $l_{R}(M)<\infty$.

Proof. Suppose that $N$ contains an infinite strictly increasing sequence of submodules $N_{0} \subset N_{1} \subset N_{2} \subset \ldots$ Then $N_{i} \subset N$, for all $i \in I$, which contradicts $\operatorname{deg}(N)<\infty$. Similarly, the assumption that $N$ contains an infinite strictly decreasing sequence of submodules, again yields a contradiction. Also assume that $M / N$ contains an infinite strictly increasing sequence of submodules $M_{0} / N \subset M_{1} / N \subset M_{2} / N \subset \ldots$. Since $N \subset M_{0} \subset M_{1} \subset M_{2} \subset \ldots$. Then $M_{i} \supset N$, for all $i \in I$, a contradiction. Similarly, if $M / N$ contains an infinite strictly decreasing sequence of submodules, again yields a contradiction. Hence, $N$ and $M / N$ cannot contain an infinite strictly increasing or decreasing sequence of submodules. Thus, they are Noetherian $R$-module as well as Artinian $R$-module. Hence, $M$ is Noetherian $R$-module as well as Artinian $R$-module. Therefore, $l_{R}(M)<\infty$.

Proposition 4.3. Let $M$ be an $R$-module and $\operatorname{In}(M)$ be a connected graph. If $M$ has at least one minimal and one maximal submodule, both of finite degrees, then $\operatorname{card}(\operatorname{Min}(M))<\infty$ and $\operatorname{card}(\operatorname{Max}(M))<\infty$.

Proof. Let $N \in \operatorname{Min}(M)$ and $K \in \operatorname{Max}(M)$ be such that $\operatorname{deg}(N)<\infty$ and $\operatorname{deg}(K)<\infty$. We consider four cases.
Case 1. If $N \cap K=(0)$ and $N+K \neq M$, since $K \subseteq N+K \neq M$, the maximality of $K$ implies that $K=N+K$, then $N \subset K$ and so $N=(0)$, a contradiction. Case 2. If $N \cap K \neq(0)$ and $N+K=M$, since $N \cap K \subseteq N$, the minimality of $N$ implies that $N \cap K=N$, thus $N \subset K$ and so $K=M$, a contradiction. Case 3. Let $N \cap K=(0)$ and $N+K=M$, then $M=N \oplus K$. If $N$ and $K$ are comparable, then $M=K$ and $N=(0)$, which is a contradiction. However, if $N$ and $K$ are not comparable, since $\operatorname{In}(M)$ is a connected graph, there exists a path between them. So with no loss of generality we can suppose that there exists a vertex $X$ of $\operatorname{In}(M)$ such that $N-X-K$ is an $N, K$-path. Since $N$ is a minimal submodule, $N \subset X$ and since $K$ is a maximal submodule, $X \subset K$. Then $N \subset K$, which is again a contradiction. Case 4. If $N \cap K \neq(0)$ and $N+K \neq M$, the above argument shows that $N \subset K$. Therefore, $N$ is adjacent to any $K_{j} \in \operatorname{Max}(M), j \in J$ and $K$ is adjacent to any $N_{i} \in \operatorname{Min}(M)$, $i \in I$ and since $\operatorname{deg}(N)<\infty$ and $\operatorname{deg}(K)<\infty$, we have $\operatorname{card}(\operatorname{Min}(M))<\infty$ and $\operatorname{card}(\operatorname{Max}(M))<\infty$.

Corollary 4.4. Let $M$ be an $R$-module and $\operatorname{In}(M)$ be a connected graph. If $M$ has at least a minimal and maximal submodule $N$ such that $\operatorname{deg}(N)<\infty$, then the following statements hold:
(1) $\operatorname{card}(\operatorname{Min}(M))=\operatorname{card}(\operatorname{Max}(M))=1$.
(2) $\alpha(\operatorname{In}(M)) \geq 1$.
(3) $\chi(\operatorname{In}(M)) \leq 1+\operatorname{deg}(N)$.

Proof. (1) Suppose that $\operatorname{Min}(M)=\left\{L \mid L \leq^{\min } M\right\}$. Clearly, $\operatorname{Min}(M) \neq \emptyset$. Since $\operatorname{In}(M)$ is connected, by Proposition 2.7 Part $2, L+N \neq M$ for all $L \in \operatorname{Min}(M)$. As $N$ is a maximal submodule of $M$ and $N \subseteq L+N \neq M$, then the maximality of $N$ implies that $N=L+N$, so $L \subseteq N$ and since $N$ is a minimal submodule of $M$, the minimality of $N$ implies that $N=L$. Hence $\operatorname{Min}(M)=\{N\}$. Now, assume that $\operatorname{Max}(M)=\left\{U \mid U \leq^{\max } M\right\}$.

Clearly, $\operatorname{Max}(M) \neq \emptyset$. Since $\operatorname{In}(M)$ is connected, by Proposition 2.7 Part 1, $U \cap N \neq(0)$ for all $U \in \operatorname{Max}(M)$. As $N$ is a minimal submodule of $M$ and $(0) \neq U \cap N \subseteq N$, then the minimality of $N$ implies that $N=U \cap N$, so $N \subseteq U$ and since $N$ is a maximal submodule of $M$, the maximality of $N$ implies that $N=U$. Hence $\operatorname{Max}(M)=\{N\}$.
(2) This is an immediate consequence of the first part and Corollary 2.8.
(3) In order to establish this part, let $\left\{X_{i}\right\}_{i \in I}$ be a family of nontrivial submodules which are not adjacent to $N$. Since $N$ is both a minimal and a maximal submodule of $M$ and since $N \subset X_{i}+N \subseteq M$ and ( 0$) \subseteq N \cap X_{i} \subset N$, the maximality and the minimality of $N$ implies that $X_{i}+N=M$ and $X_{i} \cap N=(0)$, for all $i \in I$, respectively. Hence by [[4], Lemma 3.7] and [[12], Lemma 3.5], $X_{i}$ is both a minimal and a maximal submodule of $M$, for all $i \in I$. Thus $X_{i}+X_{j}=M$ and $X_{i} \cap X_{j}=(0)$, for $i \neq j$. So $X_{i} \oplus X_{j}=M$ and also $X_{i}$ and $X_{j}$ are two simple $R$-modules. Then by Theorem 2.1, $\operatorname{In}(M)$ is not connected, a contradiction. Hence, the family $\left\{X_{i}\right\}_{i \in I}$ is empty. Now, we use a color for $N$ and new colors for the other vertices which are a finite numbers of adjacent vertices to $N$. We can color these vertices by at most $1+\operatorname{deg}(N)$ new colors to obtain a proper vertex coloring of $\operatorname{In}(M)$. Therefore, $\chi(\operatorname{In}(M)) \leq 1+\operatorname{deg}(N)$.

Corollary 4.5. Let $M$ be an $R$-module with the graph $\operatorname{In}(M)$. Then the following statements hold:
(1) If $M$ has no maximal or no minimal submodule, then $\operatorname{In}(M)$ is infinite.
(2) If $M$ contains at least two distinct minimal and maximal submodules of $M$ which every minimal and maximal submodule of $M$ has finite degree, then In $(M)$ is either null or finite.

Proof. (1) If $M$ has no maximal submodule, since (0) $\subset M$ and (0) is not maximal, there exists a submodule $M_{0}$ of $M$ such that $(0) \subset M_{0} \subset M$. Since $M_{0}$ is not maximal, then there exists a submodule $M_{1}$ of $M$ such that $(0) \subset M_{0} \subset M_{1} \subset M$. Consequently, there exists $(0) \subset M_{0} \subset M_{1} \subset \cdots \subset M$ and for $i<j, M_{i} \subset M_{j}$. Thus $M$ contains an infinite strictly increasing sequence of submodules. Therefore, $\operatorname{In}(M)$ is infinite. Also, if $M$ has no minimal submodule, since $M \supset(0)$ and $M$ is not minimal, there exists a submodule $N_{0}$ such that $M \supset N_{0} \supset(0)$. since $N_{0}$ is not minimal, then there exists a submodule $N_{1}$ such that $M \supset N_{0} \supset N_{1} \supset(0)$. Consequently, there exists $M \supset N_{0} \supset N_{1} \supset \cdots \supset(0)$ and for $i<j, N_{i} \supset N_{j}$. Thus $M$ contains an infinite strictly decreasing sequence of submodules. Therefore, $\operatorname{In}(M)$ is infinite.
(2) Suppose that $\operatorname{In}(M)$ is not null and by contrary assume that $\operatorname{In}(M)$ is infinite. Since $\operatorname{In}(M)$ is not null, by Corollary $2.2, \operatorname{In}(M)$ is connected and also by Lemma 4.2, $l_{R}(M)<\infty$. So $M$ is both Artinian and Noetherian $R$-module. However, by Proposition 4.3, the number of minimal and maximal submodules is finite. Since $\operatorname{In}(M)$ is infinite, there exist two distinct submodules, a minimal submodule $N$ and a maximal $N^{\star}$ such that $N$ is contained in an infinite number of submodules and $N^{\star}$ contains an infinite number of submodules. This contradicts with $\operatorname{deg}(N)<\infty$ and $\operatorname{deg}\left(N^{\star}\right)<\infty$. Hence, $\operatorname{In}(M)$ is a finite graph.

In the following theorem we consider the condition under which the chromatic number of $\operatorname{In}(M)$ is finite but the independence number of $\operatorname{In}(M)$ is infinity.

Theorem 4.6. Let $M$ be an $R$-module. If $\operatorname{In}(M)$ is an infinite graph and $\omega(\operatorname{In}(M))<\infty$, then the following statements hold:
(1) The number of minimal and maximal submodules of $M$ is infinite.
(2) The number of non-minimal and non-maximal submodules of $M$ is finite.
(3) $\chi(\operatorname{In}(M))<\infty$.
(4) $\alpha(\operatorname{In}(M))=\infty$.

Proof. (1) On the contrary, assume that the number of minimal and maximal submodules of $M$ is finite. Since $\operatorname{In}(M)$ is infinite, $\operatorname{In}(M)$ has an infinite clique which contradicts the finiteness of $\omega(\operatorname{In}(M))$.
(2) Suppose that $\omega(\operatorname{In}(M))<\infty$, then by Part 2 of Lemma 4.1, $l_{R}(M)<\infty$. Also for each $U \leq M, l_{R}(M / U) \leq l_{R}(M), l_{R}(M / U)<\infty$. We claim that the number of non-minimal and non-maximal submodules of $M$ is finite. To see this, assume that $S_{m}=\left\{X \subset M \mid l_{R}(X)=m\right\}$ and $T_{n}=\left\{Y \subset M \mid l_{R}(M / Y)=\right.$ $n\}$ such that $m_{0}=\max \left\{m \mid \operatorname{card}\left(S_{m}\right)=\infty\right\}$ and $n_{0}=\max \left\{n \mid \operatorname{card}\left(T_{n}\right)=\infty\right\}$. Since $S_{1}=\left\{X \subset M \mid l_{R}(X)=1\right\}$ and $T_{1}=\left\{Y \subset M \mid l_{R}(M / Y)=1\right\}, X$ and $M / Y$ are simple $R$-modules, thus $X$ is a minimal submodule and $Y$ is a maximal submodule of $M$. Hence, $S_{1}=\left\{X \mid X \leq^{\min } M\right\}$ and $T_{1}=\left\{Y \mid Y \leq^{\max } M\right\}$. By Part $1, S_{1}$ and $T_{1}$ are infinite, then there exist $m_{0}$ and $n_{0}$, where $m_{0}, n_{0} \geq 1$. Since $l_{R}(X)<l_{R}(M)$ and $l_{R}(M / Y)<l_{R}(M)$ and also $l_{R}(M) \leq \omega(\operatorname{In}(M))+1$, clearly $1 \leq m_{0}, n_{0} \leq \omega(\operatorname{In}(M))$. However, since $l_{R}(M)<\infty$, Theorem 5 of [15], p. 19] implies that every proper submodule of length $m_{0}$ is contained in a submodule of length $m_{0}+1$ and $n_{0}$ is also so. Moreover, by the definition of $m_{0}$ and $n_{0}$, we have that the numbers of submodules of length $m_{0}+1$ and $n_{0}+1$ are finite. Hence there exists a submodule $Z$ of $M$ such that $l_{R}(Z)=m_{0}+1$ and $Z$ contains an infinite number of submodules $\left\{X_{i}\right\}_{i \in I}$ of $M$, where $l_{R}\left(X_{i}\right)=m_{0}$, for all $i \in I$. Since $\omega(\operatorname{In}(M))<\infty$, there exist submodules $K_{1}$ and $L_{1}$ of $M$ with $K_{1}, L_{1} \subseteq Z$ and $l_{R}\left(K_{1}\right)=l_{R}\left(L_{1}\right)=m_{0}$ such that $K_{1} \cap L_{1}=(0)$. Since $K_{1} \cap L_{1} \subseteq Z, m_{0}+1=l_{R}(Z) \geq l_{R}\left(K_{1}+L_{1}\right) \geq$ $\left.l_{R}\left(K_{1} \oplus L_{1}\right)\right)=l_{R}\left(K_{1}\right)+l_{R}\left(L_{1}\right)=2 m_{0}$. Then $m_{0}=1$. Also, there exists a submodule $N$ of $M$ such that $l_{R}(M / N)=n_{0}+1$ and $N$ contains an infinite number of submodules $\left\{N_{i}\right\}_{i \in I}$ of $M$, where $\left.l_{R}\left(M / N_{i}\right)\right)=n_{0}$, for all $i \in I$. Now, $\omega(\operatorname{In}(M))<\infty$ implies that there exist submodules $K$ and $L$ of $M$ with $K, L \subseteq N$ and $l_{R}(M / K)=l_{R}(M / L)=n_{0}$ such that $K+L=M$. Since $K \bigcap L \subseteq N$ and $M /(K \bigcap L) \cong M / K \bigoplus M / L, n_{0}+1=l_{R}(M / N) \geq$ $l_{R}(M /(K \bigcap L))=l_{R}(M / K \bigoplus M / L)=l_{R}(M / K)+l_{R}(M / L)=2 n_{0}$. Then $n_{0}=1$. Therefore, only $S_{1}$ and $T_{1}$ are infinite and thus the number of nonminimal and non-maximal submodules of $M$ is finite.
(3) In order to establish this part, if $\omega(\operatorname{In}(M))=1$, then there is nothing to prove. Let $\omega(\operatorname{In}(M))>1$. By Proposition 2.7, each two distinct maximal submodules and each two distinct minimal submodules are not two adjacent vertices of $\operatorname{In}(M)$. Now, by Part 1, the number of minimal and maximal submodules of $M$ is infinite. Hence, we can color all maximal submodules by a
color and all the minimal submodules by another color and also other vertices, which are a finite number, by a new color but different from them, to obtain a proper vertex coloring of $\operatorname{In}(M)$. Therefore, $\chi(\operatorname{In}(M))<\infty$.
(4) This is an immediate consequence of the first part and Corollary 2.8.

Theorem 4.7. Let $M$ be an $R$-module. If $\delta=\delta(\operatorname{In}(M)) \geq 1$ and $\Delta=$ $\Delta(\operatorname{In}(M))<\infty$, then the following statements hold:
(1) $\omega(\operatorname{In}(M))<\infty$ and $\chi(\operatorname{In}(M))<\infty$.
(2) Every nontrivial submodule of $M$ contains finitely many submodules of $M$ and is contained in finitely many submodules of $M$.
Proof. (1) In order to establish this part, first we discuss the connectivity of the graph $\operatorname{In}(M)$. Let $S$ and $T$ be two non-adjacent vertices of the graph $\operatorname{In}(M)$. Since $\delta \geq 1$, there exists at least a nontrivial submodule $X$ of $M$ such that $S-X-T$ is a path in $\operatorname{In}(M)$. However, if there exist two distinct nontrivial submodules $S^{\star}$ and $T^{\star}$ of $M$ such that $S \subset S^{\star}$ or $S^{\star} \subset S$ and $T \subset T^{\star}$ or $T^{\star} \subset T$, then we consider four cases.
Case 1. Let $S \subset S^{\star}$ and $T \subset T^{\star}$. If $S^{\star} \cap T^{\star} \neq(0)$, then there exists a path of the form $S-S^{\star}-S^{\star} \cap T^{\star}-T^{\star}-T$. But, if $S^{\star} \cap T^{\star}=(0)$, clearly $S \cap T=(0), S^{\star} \cap T=(0)$ and $S \cap T^{\star}=(0)$, but $S+T \neq M$, otherwise by the modularity condition, $S^{\star}=S^{\star} \cap M=S^{\star} \cap(S+T)=S+S^{\star} \cap T=S$, a contradiction. Similarly $S+T^{\star} \neq M$ and $S^{\star}+T \neq M$. Hence we obtain three paths of the forms $S-S+T-T, S-S+T^{\star}-T$ and $S-S^{\star}+T-T$. Case 2. Let $S \subset S^{\star}$ and $T^{\star} \subset T$. If $S^{\star} \cap T \neq(0)$, then there exists a path of the form $S-S^{\star}-S^{\star} \cap T-T$, otherwise there exist three paths of the forms $S-S+T-T, S-S+T^{\star}-T^{\star}-T$ and $S-S^{\star}-S^{\star}+T^{\star}-T^{\star}-T$. Case 3. Let $S^{\star} \subset S$ and $T \subset T^{\star}$. If $S \cap T^{\star} \neq(0)$, then there exist a path of the form $S-S \cap T^{\star}-T^{\star}-T$, otherwise there exist three paths of the forms $S-S+T-T, S-S^{\star}-S^{\star}+T-T$ and $S-S^{\star}-S^{\star}+T^{\star}-T^{\star}-T$. Case 4. Let $S^{\star} \subset S$ and $T^{\star} \subset T$. If $S \cap T \neq(0)$, then there exists a path of the form $S-S \cap T-T$, otherwise there exist three paths of the forms $S-S+T^{\star}-T^{\star}-T$, $S-S^{\star}-S^{\star}+T-T$ and $S-S^{\star}-S^{\star}+T^{\star}-T^{\star}-T$. Consequently, $\operatorname{In}(M)$ is a connected graph. Hence, by Part 1 of Theorem 10.3 of [[10], p. 289], $\omega(\operatorname{In}(M)) \leq \chi(\operatorname{In}(M)) \leq \Delta+1$ and since $\Delta<\infty$, we have $\omega(\operatorname{In}(M))<\infty$ and $\chi(\operatorname{In}(M))<\infty$.
(2) It is clear.

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