# Value sharing of non linear differential polynomials with the glimpse of normal family 

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#### Abstract

In this paper we consider the situation when non-linear differential polynomials of two non-constant meromorphic functions share one value. Actually the results in this paper significantly improve and generalize the result due to Zhang [22].


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## 1. Introduction and preliminary results

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. In this paper, it is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna value distribution theory of meromorphic functions. For a meromorphic function $f(z)$ in the complex plane $\mathbb{C}$, we shall use the following standard notations of the value distribution theory: $T(r, f), m(r, \infty ; f), N(r, \infty ; f), \bar{N}(r, \infty ; f), \ldots$ (see, e.g., [6, 17]). We adopt the standard notation $S(r, f)$ for any quantity satisfying the relation $S(r, f)=o(T(r, f))$ as $r \rightarrow \infty$ except possibly a set of finite linear measure. A meromorphic function $a$ is said to be a small function of $f$ if $T(r, a)=S(r, f)$. We denote by $S(f)$ the set of all small functions of $f$.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a \in S(f) \cap$ $S(g)$. We say that $f$ and $g$ share $a$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities.

We use the symbol $\rho(f)$ to denote the order of $f$. Let $f$ be a meromorphic function in a domain $\Omega \subset \mathbb{C}$. Then the derivative of $f$ at $z_{0} \in \Omega$ in the spherical metric, called the spherical derivative, is denoted by $f^{\#}\left(z_{0}\right)$, where

$$
f^{\#}\left(z_{0}\right)= \begin{cases}\frac{\left|f^{\prime}\left(z_{0}\right)\right|}{1+\mid f\left(\left.z_{0}\right|^{2}\right.}, & \text { if } z_{0} \in \Omega \text { is not a pole } \\ \lim _{z \rightarrow z_{0}} \frac{\left|f^{\prime}(z)\right|}{1+|f(z)|^{2}}, & \text { if } z_{0} \in \Omega \text { is a pole. }\end{cases}
$$

[^0]A family $\mathcal{F}$ of functions meromorphic in the domain $\Omega \subset \mathbb{C}$ is said to be normal in $\Omega$ if every sequence $\left\{f_{n}\right\}_{n} \subseteq \mathcal{F}$ contains a subsequence which converges spherically uniformly on compact subsets of $\Omega$ (see [12]). It is assumed that the reader is familiar with the well known Marty Criterion which is one of the most widely used for determining the normality of a family of meromorphic functions (see [12]). The following well known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time ( $\mathbf{1}]-[4])$.

Theorem A. Let $f$ be a transcendental meromorphic function and $n \in \mathbb{N}$. Then $f^{n} f^{\prime}=1$ has infinitely many solutions.

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua [5, Yang and Hua [16] obtained the following result.

Theorem B. Let $f$ and $g$ be two non-constant entire (meromorphic) functions, $n \in \mathbb{N}$ such that $n \geq 6(n \geq 11)$. If $f^{n} f^{\prime}$ and $g^{n} g^{\prime}$ share $1 C M$, then either $f(z)=c_{1} e^{c z}, g(z)=c_{2} e^{-c z}$, where $c_{1}, c_{2}, c \in \mathbb{C}$ and $4\left(c_{1} c_{2}\right)^{n+1} c^{2}=-1$, or $f \equiv t g$ for a constant $t$ such that $t^{n+1}=1$.

Gradually the research work in the above directions gained pace and today it has become one of the most prominent branches of uniqueness theory.

We recall the following result by Xu et al. [13] or Zhang and Li [23], respectively.

Theorem C. Let $f$ be a transcendental meromorphic function, $n, k \in \mathbb{N}$ such that $n \geq 2$. Then $f^{n} f^{(k)}$ takes every finite non-zero value infinitely many times or has infinitely many fixed points.

Recently, Cao and Zhang [2] proved the following theorem.
Theorem D. Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n>$ $\max \left\{2 k-1, k+\frac{4}{k}+4\right\}$. If $f^{n} f^{(k)}$ and $g^{n} g^{(k)}$ share $1 C M$, $f$ and $g$ share $\infty$ IM, one of the following two conclusions holds:
(i) $f^{n} f^{(k)} \equiv g^{n} g^{(k)}$;
(ii) $f(z)=c_{3} e^{d z}, g(z)=c_{4} e^{-d z}$, where $c_{3}, c_{4}, d \in \mathbb{C}$ such that $(-1)^{k}\left(c_{3} c_{4}\right)^{n+1} d^{2 k}=1$.

In 2014, X. B. Zhang [22] proved that in Theorem D the condition " $f$ and $g$ share $\infty$ IM" can be removed and obtained the following result.

Theorem E. Let $f$ and $g$ be two non-constant meromorphic functions with $\rho(f)<+\infty$, whose zeros are of multiplicities at least $k$, where $k \in \mathbb{N}$ and let $n \in \mathbb{N}$ such that $n>\max \left\{2 k-1,2(\rho(f)-1) k-1, k+\frac{4}{k}+5\right\}$. Suppose $f^{n} f^{(k)}$ and $g^{n} g^{(k)}$ share 1 CM. Then the conclusion of Theorem $D$ holds.

We now explain the notation of weighted sharing as introduced in 7 .

Definition 1.1. 77 Let $k \in \mathbb{N} \cup\{0\} \cup\{\infty\}$. For $a \in \mathbb{C} \cup\{\infty\}$ we denote by $E_{k}(a ; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m>k$. If $E_{k}(a ; f)=E_{k}(a ; g)$, we say that $f, g$ share the value $a$ with weight $k$.
We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$ respectively.

Throughout this paper, we always use $L(f)$ to denote a differential polynomial in $f$ as follows:

$$
\begin{equation*}
L(f)=f^{(k)}+a_{k-1} f^{(k-1)}+\ldots+a_{1} f^{\prime}+a_{0} f \tag{1.1}
\end{equation*}
$$

where $a_{j} \in \mathbb{C}(j=0,1, \ldots, k-1)$.
Now observing the above results the following questions are inevitable.

Question 1.2. Can one remove the condition " $\rho(f)<+\infty$ " in Theorem E ?
Question 1.3. What happens if the differential monomials $f^{(k)}$ and $g^{(k)}$ are replaced by non linear differential polynomials in $f$ and $g$, namely of the form $(L(f))^{l}$ and $(L(g))^{l}$ in Theorem E ?

Question 1.4. Can "CM" sharing in Theorem E be reduced to finite weight sharing?

Question 1.5. Can the lower bound of $n$ be further reduced in Theorems E?

## 2. Main results and some definitions

In this paper, taking the possible answers to the above questions into background we obtain our main results as follows.

Theorem 2.1. Let $f$ and $g$ be two non-constant meromorphic functions such that either $f$ and $g$ have no zeros or zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $l, n \in \mathbb{N}$ be such that $n>\max \left\{2 l k-l, 3+l k+2 l+\frac{4}{k}\right\}$. If $f^{n}(L(f))^{l}$ and $g^{n}(L(g))^{l}$ share $(1,2)$, then one of the following two cases holds:
(i) $f^{n}(L(f))^{l} \equiv g^{n}(L(g))^{l}$;
(ii) $f(z)=c_{1} e^{c z}, g(z)=c_{2} e^{-c z}$, where $c_{1}, c_{1}, c \in \mathbb{C}$ such that $\left(c_{1} c_{2}\right)^{n+l}\left(c^{k}+a_{k-1} c^{k-1}+\ldots+a_{1} c+a_{0}\right)^{l}$
$\times\left\{(-c)^{k}+a_{k-1}(-c)^{k-1}+a_{1}(-c)+a_{0}\right\}^{l}=1$.
Corollary 2.2. Let $f$ and $g$ be two non-constant meromorphic functions such that zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $l, n \in \mathbb{N}$ be such that $n>\max \left\{2 l k-l, 3+l k+2 l+\frac{4}{k}\right\}$. If $f^{n}(L(f))^{l}$ and $g^{n}(L(g))^{l}$ share $(1,2)$, then $f^{n}(L(f))^{l} \equiv g^{n}(L(g))^{l}$.

We now explain some definitions and notations which are used in the paper.

Definition 2.3. 9] Let $p \in \mathbb{N}$ and $a \in \mathbb{C} \cup\{\infty\}$. Then
(i) $N(r, a ; f \mid \geq p)(\bar{N}(r, a ; f \mid \geq p))$ denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not less than $p$.
(ii) $N(r, a ; f \mid \leq p)(\bar{N}(r, a ; f \mid \leq p))$ denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not greater than $p$.

Definition 2.4. 18] For $a \in \mathbb{C} \cup\{\infty\}$ and a positive integer $p$ we denote by $N_{p}(r, a ; f)$ the sum $\bar{N}(r, a ; f)+\bar{N}(r, a ; f \mid \geq 2)+\ldots+\bar{N}(r, a ; f \mid \geq p)$. Clearly $N_{1}(r, a ; f)=\bar{N}(r, a ; f)$.

Definition 2.5. [7] Let $f, g$ share a value $a$ IM. We denote by $\bar{N}_{*}(r, a ; f, g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$. Clearly $\bar{N}_{*}(r, a ; f, g) \equiv$ $\bar{N}_{*}(r, a ; g, f)$.

## 3. Lemmas

Let $F, G$ be two non-constant meromorphic functions. Henceforth we shall denote by $H$ the following function

$$
\begin{equation*}
H=\left(\frac{F^{\prime \prime}}{F^{\prime}}-\frac{2 F^{\prime}}{F-1}\right)-\left(\frac{G^{\prime \prime}}{G^{\prime}}-\frac{2 G^{\prime}}{G-1}\right) \tag{3.1}
\end{equation*}
$$

Lemma 3.1. [15] Let $f$ be a non-constant meromorphic function and $P(f)=$ $a_{0}+a_{1} f+a_{2} f^{2}+\ldots+a_{n} f^{n}$, where $a_{0}, a_{1}, a_{2} \ldots, a_{n} \in \mathbb{C}$ and $a_{n} \neq 0$. Then $T(r, P(f))=n T(r, f)+O(1)$.

Lemma 3.2. ([10], Corollary 2.3.4.) Let $f$ be a transcendental meromorphic function of finite order growth and $k \in \mathbb{N}$. Then $m\left(r, \frac{f^{(k)}}{f}\right)=O(\log r)$.

Lemma 3.3. Let $f$ be a transcendental meromorphic function of finite order growth. $L_{1}(f)$ is a differential polynomial defined as follows:

$$
\begin{equation*}
L_{1}(f)=f^{(k)}+b_{k-1} f^{(k-1)}+b_{k-2} f^{(k-2)}+\ldots+b_{1} f^{\prime}+b_{0} f \tag{3.2}
\end{equation*}
$$

where $k \in \mathbb{N}, b_{j} \in \mathbb{C}(j=0,1, \ldots, k-1)$. If $L_{1}(f) \not \equiv 0$, we have

$$
N\left(r, 0, L_{1}\right) \leq k \bar{N}(r, f)+N(r, 0, f)+O(\log r)
$$

as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.
Proof. By the first fundamental theorem and Lemma 3.2, we have

$$
m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{L_{1}}{f}\right)+m\left(r, \frac{1}{L_{1}}\right)+O(1)
$$

$$
\begin{aligned}
& \text { i.e., } \begin{aligned}
T(r, f)-N(r, 0 ; f) & \leq m\left(r, \frac{1}{L_{1}}\right)+O(\log r) \\
& =T\left(r, L_{1}\right)-N\left(r, 0, L_{1}\right)+O(\log r)
\end{aligned} \\
& \text { i.e., } \quad \begin{aligned}
N\left(r, 0 ; L_{1}\right) \leq & T\left(r, L_{1}\right)-T(r, f)+N(r, 0 ; f)+O(\log r) \\
\leq & m\left(r, L_{1}\right)+N\left(r, L_{1}\right)-T(r, f)+N(r, 0 ; f)+O(\log r) \\
\leq & m\left(r, \frac{L_{1}}{f}\right)+m(r, f)+N(r, f)+k \bar{N}(r, f)-T(r, f) \\
& +N(r, 0 ; f)+O(\log r) \\
\leq & k \bar{N}(r, f)+N(r, 0, f)+O(\log r) .
\end{aligned}
\end{aligned}
$$

This completes the proof.

Lemma 3.4. [21] Let $f$ be a non-constant meromorphic function. $L_{1}(f)$ is a differential polynomial defined as in (3.2). If $L_{1}(f) \not \equiv 0$ and $p \in \mathbb{N}$, we have

$$
\begin{gathered}
N_{p}\left(r, 0 ; L_{1}\right) \leq T\left(r, L_{1}\right)-T(r, f)+N_{p+k}(r, 0 ; f)+S(r, f), \\
\quad N_{p}\left(r, 0 ; L_{1}\right) \leq k \bar{N}(r, \infty ; f)+N_{p+k}(r, 0 ; f)+S(r, f)
\end{gathered}
$$

Lemma 3.5. [8] If $N\left(r, 0 ; f^{(k)} \mid f \neq 0\right)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then
$N\left(r, 0 ; f^{(k)} \mid f \neq 0\right) \leq k \bar{N}(r, \infty ; f)+N(r, 0 ; f \mid<k)+k \bar{N}(r, 0 ; f \mid \geq k)+S(r, f)$.

Lemma 3.6. Let $f$ be a non-constant meromorphic function and $F=f^{n}(L(f))^{l}$, where $l, k, n \in \mathbb{N}$. Then

$$
(n-l) T(r, f) \leq T(r, F)-l N(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right)+S(r, f)
$$

Proof. Note that

$$
\begin{aligned}
N(r, \infty ; F) & =N\left(r, \infty ; f^{n}\right)+N\left(r, \infty ;(L(f))^{l}\right) \\
& =N\left(r, \infty ; f^{n}\right)+l N(r, \infty ; f)+l k \bar{N}(r, \infty ; f),
\end{aligned}
$$

i.e.,

$$
N\left(r, \infty ; f^{n}\right)=N(r, \infty, F)-l N(r, \infty ; f)-l k \bar{N}(r, \infty, f)+S(r, f)
$$

Also

$$
\begin{aligned}
m\left(r, \infty ; f^{n}\right)= & m\left(r, \infty ; \frac{F}{(L(f))^{l}}\right) \\
\leq & m(r, \infty ; F)+m\left(r, \infty ; \frac{1}{(L(f))^{l}}\right)+S(r, f) \\
= & m(r, \infty ; F)+T\left(r,(L(f))^{l}\right)-N\left(r, 0 ;(L(f))^{l}\right)+S(r, f) \\
= & m(r, \infty ; F)+N\left(r, \infty ;(L(f))^{l}\right)+m\left(r, \infty ;(L(f))^{l}\right) \\
& -N\left(r, 0 ;(L(f))^{l}\right)+S(r, f) \\
\leq & m(r, \infty ; F)+l N(r, \infty ; f)+l k \bar{N}(r, \infty ; f) \\
& +m\left(r, \infty ; \frac{(L(f))^{l}}{f^{l}}\right)+m\left(r, \infty ; f^{l}\right)-N\left(r, 0 ;(L(f))^{l}\right) \\
& +S(r, f) \\
= & m(r, \infty ; F)+l T(r, f)+l k \bar{N}(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right) \\
& +S(r, f) .
\end{aligned}
$$

Now by Lemma 3.1, we have

$$
\begin{aligned}
n T(r, f) & =N\left(r, \infty ; f^{n}\right)+m\left(r, \infty ; f^{n}\right) \\
& \leq T(r, F)+l T(r, f)-l N(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right)+S(r, f)
\end{aligned}
$$

i.e., $(n-l) T(r, f) \leq T(r, F)-l N(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right)+S(r, f)$. This completes the proof.

Lemma 3.7. Let $f$ and $g$ be two transcendental meromorphic functions of finite order. Let $n, l, k \in \mathbb{N}$ be such that $n>l+1$. If $f^{n}(L(f))^{l}$ and $g^{n}(L(g))^{l}$ share $\alpha I M$, where $\alpha(z)(\not \equiv 0, \infty) \in S(f) \cap S(g)$, then $\rho(f)=\rho(g)$.

Proof. Let $F=f^{n}(L(f))^{l}$ and $G=g^{n}(L(g))^{l}$. Note that

$$
\begin{aligned}
N(r, \infty ; F) & =N\left(r, \infty ; f^{n}\right)+N\left(r, \infty ;(L(f))^{l}\right) \\
& =N\left(r, \infty ; f^{n}\right)+l N(r, \infty ; f)+l k \bar{N}(r, \infty ; f)
\end{aligned}
$$

Also

$$
\begin{aligned}
m(r, \infty ; F) & \leq m\left(r, \infty ; f^{n}\right)+m\left(r, \infty ; \frac{(L(f))^{l}}{f^{l}}\right)+m\left(r, \infty ; f^{l}\right)+S(r, f) \\
& =m\left(r, \infty ; f^{n}\right)+m\left(r, \infty ; f^{l}\right)+S(r, f)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T(r, F) & \leq T\left(r, f^{n}\right)+T\left(r, f^{l}\right)+l k \bar{N}(r, \infty ; f)+S(r, f) \\
& =(n+l+l k) T(r, f)+S(r, f)
\end{aligned}
$$

By the second fundamental theorem for small functions (see [14]), we have

$$
\begin{aligned}
T(r, F) \leq & \bar{N}(r, \infty ; F)+\bar{N}(r, 0 ; F)+\bar{N}(r, \alpha ; F) \\
& +\left(\frac{\varepsilon}{n+l+l k}+o(1)\right) T(r, F) \\
\leq & \bar{N}(r, \infty ; f)+\bar{N}(r, 0 ; F)+\bar{N}(r, \alpha ; F)+(\varepsilon+o(1)) T(r, f) \\
& +S(r, f)
\end{aligned}
$$

for all $\varepsilon>0$. Now in view of Lemma 3.6 and using the above, we get

$$
\begin{aligned}
& (n-l) T(r, f) \\
\leq & T(r, F)-l N(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right)+S(r, f) \\
\leq & \bar{N}(r, 0 ; F)+\bar{N}(r, \infty ; f)+\bar{N}(r, \alpha ; F)-l N(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right) \\
& +(\varepsilon+o(1)) T(r, f)+S(r, f) \\
\leq & \bar{N}\left(r, 0 ; f^{n}\right)+\bar{N}(r, \alpha ; G)+(\varepsilon+o(1)) T(r, f)+S(r, f) \\
\leq & T(r, f)+(n+l(k+1)) T(r, g)+(\varepsilon+o(1)) T(r, f)+S(r, f), \\
\text { i.e., } \quad & (n-l-1) T(r, f) \leq(n+l(k+1)) T(r, g)+(\varepsilon+o(1)) T(r, f) .
\end{aligned}
$$

Since $n>l+1$, take $0<\varepsilon<1$. Then it follows by Lemma 1.1.2 [10] that $\rho(f) \leq \rho(g)$. Similarly we have $\rho(g) \leq \rho(f)$. Therefore $\rho(f)=\rho(g)$. This completes the proof.

Lemma 3.8. Let $f$ and $g$ be two non-constant rational functions such that either $f$ and $g$ have no zeros or zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $n>2 l k-l$. Then there are no solutions of the functional differential equation of the form

$$
f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv 1
$$

Proof. Suppose

$$
\begin{equation*}
f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv 1 \tag{3.3}
\end{equation*}
$$

Now two cases may arise.
Case 1. Suppose zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $z_{0}$ be a zero of $f$ with multiplicity $q_{0}(\geq k)$. Then $z_{0}$ is a zero of $f^{n}(L(f))^{l}$ with multiplicity $(n+l) q_{0}-l k$. Clearly $z_{0}$ will be a pole of $g$ with multiplicity $p_{0}$, say. Note that $z_{0}$ will be pole of $g^{n}(L(g))^{l}$ with multiplicity $(n+l) p_{0}+l k$. Obviously $(n+l) q_{0}-l k=(n+l) p_{0}+l k$. Now $(n+l) q_{0}-l k=(n+l) p_{0}+l k$ implies that $(n+l)\left(q_{0}-p_{0}\right)=2 l k$. Since $n>2 l k-l$, we arrive at a contradiction.
Case 2. Suppose $f$ and $g$ have no zeros. Let $f(z)=\frac{1}{R(z)}$ and $g(z)=\frac{1}{K(z)}$, where $R$ and $K$ are non-constant polynomials. Then $f^{(i)}=\frac{R_{1 i}}{R_{2 i}}$ and $g^{(i)}=$ $\frac{K_{1 i}}{K_{2 i}}$, where $R_{1 i}, R_{2 i} K_{1 i}$ and $K_{2 i}$ are polynomials such that $\operatorname{deg}\left(R_{2 i}\right)>\operatorname{deg}\left(R_{1 i}\right)$
and $\operatorname{deg}\left(K_{2 i}\right)>\operatorname{deg}\left(K_{1 i}\right)$, for $i=1,2, \ldots, k-1$.
Therefore $L(f)=\frac{R_{1}}{R_{2}}$ and $L(g)=\frac{K_{1}}{K_{2}}$, where $R_{1}, R_{2}, K_{1}$ and $K_{2}$ are nonconstant polynomials such that $\operatorname{deg}\left(R_{2}\right)>\operatorname{deg}\left(R_{1}\right)$ and $\operatorname{deg}\left(K_{2}\right)>\operatorname{deg}\left(K_{1}\right)$. Combining this with (3.3) leads to a contradiction. This completes the proof.

Lemma 3.9. [[6], Lemma 3.5] Suppose that $F$ is meromorphic in a domain $D$ and set $f=\frac{F^{\prime}}{F}$. Then for $n \geq 1$,

$$
\frac{F^{(n)}}{F}=f^{n}+\frac{n(n-1)}{2} f^{n-2} f^{\prime}+a_{n} f^{n-3} f^{\prime \prime}+b_{n} f^{n-4}\left(f^{\prime}\right)^{2}+P_{n-3}(f)
$$

where $a_{n}=\frac{1}{6} n(n-1)(n-2), b_{n}=\frac{1}{8} n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n>3$.

Lemma 3.10. [[3]], Lemma 1] Let $f$ be a meromorphic function on $\mathbb{C}$. If $f$ has a bounded spherical derivative on $\mathbb{C}, f$ is of order at most 2 . If in addition $f$ is entire, then the order of $f$ is at most 1 .

Lemma 3.11. [20] Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc $\Delta$ such that all zeros of functions in $\mathcal{F}$ have multiplicity greater than or equal to $l$ and all poles of functions in $F$ have multiplicity greater than or equal to $j$ and $\alpha$ be a real number satisfying $-l<\alpha<j$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_{0} \in \Delta$, if and only if there exist
(i) points $z_{n} \in \Delta, z_{n} \rightarrow z_{0}$,
(ii) positive numbers $\rho_{n}, \rho_{n} \rightarrow 0^{+}$and
(iii) functions $f_{n} \in \mathcal{F}$,
such that $\rho_{n}^{\alpha} f_{n}\left(z_{n}+\rho_{n} \zeta\right) \rightarrow g(\zeta)$ spherically uniformly on compact subsets of $\mathbb{C}$, where $g$ is a non-constant meromorphic function. The function $g$ may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0)=1(\zeta \in \mathbb{C})$.

Remark 3.12. Clearly with no special restrictions on the zeros and poles of functions in $\mathcal{F}$, Lemma 3.11 holds for $-1<\alpha<1$, on the other hand if all functions in $\mathcal{F}$ are holomorphic (so that the condition on the poles is satisfied vacuously for arbitrary $j$ ), we may take $-1<\alpha<\infty$. Similarly for families of meromorphic functions which do not vanish, one may choose $-\infty<\alpha<1$.

Lemma 3.13. Let $f$ and $g$ be two non-constant meromorphic functions such that either $f$ and $g$ have no zeros or zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $l, k$ and $n$ be three positive integers such that $n>2 l k-l$. Suppose $f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv 1$. Then $f(z)=c_{1} e^{c z}$ and $g(z)=c_{2} e^{-c z}$, where $c_{1}, c_{2}$ and $c$ are constants such that $\left(c_{1} c_{2}\right)^{n+l}\left(c^{k}+a_{k-1} c^{k-1}+\ldots+a_{1} c+a_{0}\right)^{l}$ $\times\left((-c)^{k}+a_{k-1}(-c)^{k-1}+a_{1}(-c)+a_{0}\right)^{l}=1$.

Proof. Suppose

$$
\begin{equation*}
f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv 1 \tag{3.4}
\end{equation*}
$$

Now two cases may arise.
Case 1. Suppose that zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$.
Let $z_{1}$ be a zero of $f$ with multiplicity $q_{1}(\geq k)$. Then $z_{1}$ is a zero of $f^{n}(L(f))^{l}$ with multiplicity $(n+l) q_{1}-l k$. Clearly $z_{1}$ will be a pole of $g$ with multiplicity $p_{1}$, say. Note that $z_{1}$ will be pole of $g^{n}(L(g))^{l}$ with multiplicity $(n+l) p_{1}+l k$. Obviously $(n+l) q_{1}-l k=(n+l) p_{1}+l k$. Now $(n+l) q_{1}-l k=(n+l) p_{1}+l k$ implies that $(n+l)\left(q_{1}-p_{1}\right)=2 l k$. Since $n>2 l k-l$, we arrive at a contradiction.
Case 2. Suppose that $f$ and $g$ have no zeros. By Lemma 3.8, we have $f$ and $g$ are transcendental meromorphic functions. Let $\mathcal{F}=\left\{f_{\omega}\right\}$ and $\mathcal{G}=\left\{g_{\omega}\right\}$, where $f_{\omega}(z)=f(z+\omega)$ and $G_{\omega}(z)=g(z+\omega), z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.
Sub-case 2.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $f^{\#}(\omega)=f_{\omega}^{\#}(0) \leq M$ for some $M>0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 3.10, we have $\rho(f) \leq 2$. Using Lemma 3.6, we have

$$
\begin{aligned}
(n-l) T(r, g) \leq T\left(r, g^{n}(L(g))^{l}\right)+S(r, g) & \leq T\left(r, \frac{1}{f^{n}(L(f))^{l}}\right)+S(r, g) \\
& =T\left(r, f^{n}(L(f))^{l}\right)+S(r, g) \\
& \leq T\left(r, f^{n}\right)+l T(r, L(f))+S(r, g) \\
& \leq(n+(k+1) l) T(r, f)+S(r, g)
\end{aligned}
$$

Since $n>2 l k-l$, it follows by Lemma 1.1.2 [10] that $\rho(g) \leq \rho(f)$. Hence $g$ is of finite order. Also from (3.4, we see that $f^{n}(L(f))^{l}$ and $g^{n}(L(g))^{l}$ share 1 IM and so by Lemma 3.7, we have $\rho(f)=\rho(g) \leq 2$. Now from 3.4) and Lemma 3.3, we have

$$
\begin{aligned}
(n+l) N(r, \infty ; f)+l k \bar{N}(r, \infty ; f) & =N\left(r, \infty ; f^{n}(L(f))^{l}\right) \\
& =N\left(r, \infty ; \frac{1}{g^{n}(L(g))^{l}}\right) \\
& =N\left(r, 0 ; g^{n}(L(g))^{l}\right) \\
& \leq l N(r, 0 ; g)+l k \bar{N}(r, \infty ; g)+O(\log r) \\
& =l k \bar{N}(r, \infty ; g)+O(\log r),
\end{aligned}
$$

as $r \rightarrow \infty$. Similarly

$$
(n+l) N(r, \infty ; g)+l k \bar{N}(r, \infty ; g) \leq l k \bar{N}(r, \infty ; f)+O(\log r)
$$

as $r \rightarrow \infty$. Therefore, we have

$$
N(r, \infty ; f)+N(r, \infty ; g) \leq O(\log r)
$$

as $r \rightarrow \infty$. This shows that $f$ and $g$ have at most finitely many poles. Since $\rho(f)=\rho(g) \leq 2$, so let us assume that

$$
\begin{equation*}
f=\frac{1}{P_{1}} e^{\alpha} \text { and } g=\frac{1}{P_{2}} e^{\beta}, \tag{3.5}
\end{equation*}
$$

where $P_{1}$ and $P_{2}$ are non-zero polynomials and $\alpha$ and $\beta$ are non-constant polynomials such that $\operatorname{deg}(\alpha) \leq 2$ and $\operatorname{deg}(\beta) \leq 2$.
From (3.4), one can easily conclude that either both $P_{1}$ and $P_{2}$ are non-zero constants or both $P_{1}$ and $P_{2}$ are non-constant polynomials. Here we claim that both $P_{1}$ and $P_{2}$ are non-zero constants. If not, suppose that both $P_{1}$ and $P_{2}$ are non-constant polynomials.
Now from (3.5) and Lemma 3.9, we have

$$
\begin{aligned}
f^{(i)} & =\left[\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)^{i}+P_{i-1}^{*}\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)\right] \frac{e^{\alpha}}{P_{1}} \\
\text { and } g^{(i)} & =\left[\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)^{i}+P_{i-1}^{*}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)\right] \frac{e^{\beta}}{P_{2}}
\end{aligned}
$$

for $i=1,2, \ldots, k$; where $P_{i-1}^{*}\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)\left(P_{i-1}^{*}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)\right)$ is a differential polynomial of degree at most $i-1$ in $\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)$. Therefore

$$
\begin{equation*}
L(f)=\left[\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)\right] \frac{e^{\alpha}}{P_{1}} \tag{3.6}
\end{equation*}
$$

$$
\begin{equation*}
\text { and } L(g)=\left[\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)\right] \frac{e^{\beta}}{P_{2}} \tag{3.7}
\end{equation*}
$$

where $\mathcal{P}_{k-1}^{*}\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)\left(\mathcal{P}_{k-1}^{*}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)\right)$ is a differential polynomial of degree at most $k-1$ in $\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)$. Now from 3.4, 3.6, and 3.7, we have

$$
\begin{array}{r}
{\left[\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)\right]^{l}}  \tag{3.8}\\
\times\left[\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(\beta^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)\right]^{l} e^{(n+l)(\alpha+\beta)}=\left(P_{1} P_{2}\right)^{n+l} .
\end{array}
$$

Since $\alpha$ and $\beta$ are non-constant polynomials, from 3.8 we have $\alpha+\beta=d \in \mathbb{C}$. Therefore $\alpha^{\prime}+\beta^{\prime}=0$. Now from 3.8, we have

$$
\begin{array}{r}
{\left[\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(\alpha^{\prime}-\frac{P_{1}^{\prime}}{P_{1}}\right)\right]^{l}}  \tag{3.9}\\
\times\left[\left(-\alpha^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(-\alpha^{\prime}-\frac{P_{2}^{\prime}}{P_{2}}\right)\right]^{l} e^{(n+l) d}=\left(P_{1} P_{2}\right)^{n+l}
\end{array}
$$

Letting $|z| \rightarrow \infty$, we see that

$$
\begin{equation*}
2 l k \operatorname{deg}\left(\alpha^{\prime}\right)=(n+l) \operatorname{deg}\left(P_{1} P_{2}\right) \tag{3.10}
\end{equation*}
$$

Since $\operatorname{deg}\left(\alpha^{\prime}\right) \leq 1$ and $n>2 l k-l$, from (3.10) we arrive at a contradiction.
Therefore both $P_{1}$ and $P_{2}$ are non-zero constants and so $f$ and $g$ are transcendental entire functions. Without loss of generality we may assume that $f(z)=e^{\alpha(z)}$ and $g(z)=e^{\beta(z)}$. Now from (3.6) and 3.7), we have

$$
\begin{equation*}
L(f)=\left(\left(\alpha^{\prime}\right)^{k}+\mathcal{P}_{k-1}^{*}\left(\alpha^{\prime}\right)\right) e^{\alpha} \text { and } L(g)=\left(\left(\beta^{\prime}\right)^{k}+\mathcal{P}_{k-1}^{*}(\beta)\right) e^{\beta} \tag{3.11}
\end{equation*}
$$

Also from (3.4), we see that $L(f) \neq 0$ and $L(g) \neq 0$. Therefore from (3.11) we can conclude that $\alpha^{\prime}$ and $\beta^{\prime}$ must be constants and so $\operatorname{deg}(\alpha)=\operatorname{deg}(\beta)=1$. Again since $\alpha+\beta=d \in \mathbb{C}$, we can take $f(z)=c_{1} e^{c z}$ and $g(z)=c_{2} e^{-c z}$, where $c, c_{1}$ and $c_{2} \in \mathbb{C} \backslash\{0\}$ such that $\left(c_{1} c_{2}\right)^{n+l}\left(c^{k}+a_{k-1} c^{k-1}+\ldots+a_{1} c+a_{0}\right)^{l}$ $\times\left((-c)^{k}+a_{k-1}(-c)^{k-1}+a_{1}(-c)+a_{0}\right)^{l}=1$.

Sub-case 2.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is not normal on $\mathbb{C}$. Then there exists at least one $z_{0}$ such that $\mathcal{F}$ is not normal at $z_{0}$, i.e., $\mathcal{F}$ is not normal in any neighbourhood of $z_{0}$. For the sake of simplicity we assume that $z_{0}=0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\left\{f\left(z+\omega_{j}\right)\right\} \subset \mathcal{F}$, where $z \in \Delta$ and $\left\{\omega_{j}\right\} \subset \mathbb{C}$ is some sequence such that $f_{j}^{\#}(0)=f^{\#}\left(\omega_{j}\right) \rightarrow \infty$, as $\left|\omega_{j}\right| \rightarrow \infty$. Then by Lemma 3.11 there exist
(i) points $z_{j} \in \Delta, z_{j} \rightarrow 0$,
(ii) positive numbers $\rho_{j}, \rho_{j} \rightarrow 0$,
(iii) a subsequence of functions $\left\{f_{j}\left(z_{j}+\rho_{j} \zeta\right)=f\left(\omega_{j}+z_{j}+\rho_{j} \zeta\right)\right\}$ of $\left\{f\left(\omega_{j}+z\right)\right\}$ such that

$$
\begin{equation*}
h_{j}(\zeta)=\rho_{j}^{-\frac{l k}{n+l}} f_{j}\left(z_{j}+\rho_{j} \zeta\right) \rightarrow h(\zeta) \tag{3.12}
\end{equation*}
$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $h(\zeta)$ is a non-constant meromorphic function such that $h^{\#}(\zeta) \leq h^{\#}(0)=1(\zeta \in \mathbb{C})$. Now using Lemma 3.10, we have $\rho(h) \leq 2$. Since $f(z) \neq 0$, by Hurwitz's theorem we conclude that $h$ has no zeros. In the proof of Zalcman's lemma (see [11, 19] ), we see that $\rho_{j}=\frac{1}{f_{j}^{\#}\left(z_{j}\right)}$. Now from 3.12, we have

$$
\left\{\begin{array}{c}
h_{j}^{n}(\zeta)=\rho_{j}^{-\frac{l k n}{n+l}} f_{j}^{n}\left(z_{j}+\rho_{j} \zeta\right) \rightarrow h^{n}(\zeta)  \tag{3.13}\\
h_{j}^{(s)}(\zeta)=\rho_{j}^{\frac{l(s-k)-n s}{n+l}} f_{j}^{(s)}\left(z_{j}+\rho_{j} \zeta\right) \rightarrow h^{(s)}(\zeta)
\end{array}\right.
$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $s \in \mathbb{N}$. Then from 3.12) and (3.13), we have

$$
\begin{align*}
& h_{j}^{n}(\zeta)\left(h_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j}^{i} h_{j}^{(k-i)}(\zeta)\right)^{l}  \tag{3.14}\\
= & \left(f_{j}\left(z_{j}+\rho_{j} \zeta\right)\right)^{n}\left(L\left(f_{j}\left(z_{j}+\rho_{j} \zeta\right)\right)\right)^{l} \rightarrow h^{n}(\zeta)\left(h^{(k)}(\zeta)\right)^{l}
\end{align*}
$$

spherically uniformly on compact subsets of $\mathbb{C} \backslash h^{-1}\{\infty\}$. Let

$$
\begin{equation*}
\widehat{h}_{j}(\zeta)=\rho_{j}^{-\frac{l k}{n+l}} g_{j}\left(z_{j}+\rho_{j} \zeta\right) \tag{3.15}
\end{equation*}
$$

Therefore from 3.15, we have

$$
\begin{align*}
& \widehat{h}_{j}^{n}(\zeta)\left(\widehat{h}_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j}^{i} \widehat{h}_{j}^{(k-i)}(\zeta)\right)^{l}  \tag{3.16}\\
& \quad=\left(g_{j}\left(z_{j}+\rho_{j} \zeta\right)\right)^{n}\left(L\left(g_{j}\left(z_{j}+\rho_{j} \zeta\right)\right)\right)^{l}
\end{align*}
$$

Consequently from (3.4), (3.14) and (3.16), we have

$$
\begin{array}{r}
h_{j}^{n}(\zeta)\left(h_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j}^{i} h_{j}^{(k-i)}(\zeta)\right)^{l}  \tag{3.17}\\
\times \widehat{h}_{j}^{n}(\zeta)\left(\widehat{h}_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j}^{i} \widehat{h}_{j}^{(k-i)}(\zeta)\right)^{l} \equiv 1
\end{array}
$$

Now taking $\rho_{j} \rightarrow 0$ as $j \rightarrow \infty$, we get from (3.14) and 3.17) that

$$
\begin{equation*}
h^{n}(\zeta)\left(h^{(k)}(\zeta)\right)^{l} \widehat{H}(\zeta) \equiv 1 \tag{3.18}
\end{equation*}
$$

for all $\zeta \in \mathbb{C} \backslash\left\{h^{-1}(\infty) \cup \widehat{H}^{-1}(\infty)\right\}$, where $\widehat{H}$ is a non-constant meromorphic function in $\mathbb{C}$ such that

$$
\begin{equation*}
\widehat{h}_{j}^{n}(\zeta)\left(\widehat{h}_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j}^{i} \widehat{h}_{j}^{(k-i)}(\zeta)\right)^{l} \rightarrow \widehat{H}(\zeta) \tag{3.19}
\end{equation*}
$$

spherically uniformly on compact subsets of $\mathbb{C} \backslash\left\{\widehat{H}^{-1}(\infty)\right\}$. On the other hand we see that for fixed $r$ with $r<1$, the functions $\widehat{h}_{j}(\zeta)=\rho_{j}^{-\frac{l k}{n+l}} g_{j}\left(z_{j}+\rho_{j} \zeta\right)$ are defined on $|\zeta|<R_{j}=\frac{r-\left|z_{j}\right|}{\rho_{j}}$ and $R_{j} \rightarrow \infty$ as $j \rightarrow \infty$. Moreover taking a fixed $R$ with $|\zeta| \leq R<R_{j}$, we have $\left|z_{j}+\rho_{j} \zeta\right|<r$. Clearly $\left\{\widehat{h}_{j}(\zeta)\right\}$ is a family of meromorphic functions defined on $|\zeta|<R$. Now we consider the following two sub-cases.
Sub-case 2.2.1. Suppose that $\left\{\widehat{h}_{j}(\zeta)\right\}$ is not normal in $|\zeta|<R$. Then there exists $\zeta_{0}$ such that $\left\{\widehat{h}_{j}(\zeta)\right\}$ is not normal at $\zeta_{0}$. For the sake of simplicity we assume that $\zeta_{0}=0$. Now proceeding in the same way as above and using Lemma 3.11, we see that there exist
(i) points $\zeta_{j} \in \Delta, \zeta_{j} \rightarrow 0$,
(ii) positive numbers $\eta_{j}, \eta_{j} \rightarrow 0$,
(iii) a subsequence of functions $\left\{\widehat{h}_{j}\left(\zeta_{j}+\eta_{j} \zeta\right)\right\}$ of $\left\{\widehat{h}_{j}(\zeta)\right\}$ such that

$$
\begin{equation*}
\widehat{H}_{j}(\zeta):=\eta_{j}^{-\frac{l k}{n+l}} \widehat{h}_{j}\left(\zeta_{j}+\eta_{j} \zeta\right) \rightarrow \widehat{h}_{1}(\zeta) \tag{3.20}
\end{equation*}
$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $\widehat{h}_{1}(\zeta)$ is a non-constant meromorphic function such that $\widehat{h}_{1}^{\#}(\zeta) \leq \widehat{h}_{1}^{\#}(0)=1(\zeta \in \mathbb{C})$. Using Lemma 3.10 we get $\rho\left(\widehat{h}_{1}\right) \leq 2$. Now from 3.15 and 3.20 , we have

$$
\begin{equation*}
\widehat{H}_{j}(\zeta):=\eta_{j}^{-\frac{l k}{n+l}} \widehat{h}_{j}\left(\zeta_{j}+\eta_{j} \zeta\right)=\xi_{j}^{-\frac{l k}{n+l}} g_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right) \rightarrow \widehat{h}_{1}(\zeta) \tag{3.21}
\end{equation*}
$$

where $\xi_{j}=\rho_{j} \eta_{j}$. Since $g(z) \neq 0$, by Hurwitz's theorem we can see that $\widehat{h}_{1}$ has no zeros. Also from Zalcman's Lemma (see [11, 19] ), we see that

$$
\begin{equation*}
\xi_{j}=\frac{1}{g_{j}^{\#}\left(z_{j}+\rho_{j} \zeta_{j}\right)} \tag{3.22}
\end{equation*}
$$

Now from 3.21, we have

$$
\left\{\begin{array}{c}
\widehat{H}_{j}^{n}(\zeta)=\xi_{j}^{-\frac{l k n}{n+l}} g_{j}^{n}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right) \rightarrow \widehat{h}_{1}^{n}(\zeta)  \tag{3.23}\\
\widehat{H}_{j}^{(s)}(\zeta)=\xi_{j}^{\frac{l(s-k)+n s}{n+l}} g_{j}^{(s)}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right) \rightarrow \widehat{h}_{1}^{(s)}(\zeta)
\end{array}\right.
$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $s \in \mathbb{N}$ and so

$$
\begin{align*}
& \widehat{H}_{j}^{n}(\zeta)\left(\widehat{H}_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \xi_{j}^{i} \widehat{H}_{j}^{(k-i)}(\zeta)\right)^{l}  \tag{3.24}\\
= & \left(g_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right)\right)^{n}\left(L\left(g_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right)\right)\right)^{l} \rightarrow \widehat{h}_{1}^{n}(\zeta)\left(\widehat{h}_{1}^{(k)}(\zeta)\right)^{l}
\end{align*}
$$

spherically uniformly on compact subset of $\mathbb{C} \backslash h^{-1}\{\infty\}$. Again from 3.12, we have

$$
\begin{equation*}
H_{j}(\zeta):=\eta_{j}^{-\frac{l k}{n+l}} h_{j}\left(\zeta_{j}+\eta_{j} \zeta\right)=\xi_{j}^{-\frac{l k}{n+l}} f_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right) \rightarrow h_{1}(\zeta) \tag{3.25}
\end{equation*}
$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $h_{1}(\zeta)$ is a non-constant meromorphic function such that $h_{1}^{\#}(\zeta) \leq h_{1}^{\#}(0)=1(\zeta \in \mathbb{C})$. Using Lemma 3.10 , we get $\rho\left(h_{1}\right) \leq 2$. Also since $f(z) \neq 0$, by Hurwitz's theorem we can see that $h_{1}$ has no zeros. Therefore from (3.25), we have

$$
\left\{\begin{array}{c}
H_{j}^{n}(\zeta)=\xi_{j}^{-\frac{l k n}{n+l}} f_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right) \rightarrow h_{1}^{n}(\zeta)  \tag{3.26}\\
H_{j}^{(s)}(\zeta)=\xi_{j}^{\frac{l(s-k)+n s}{n+l}} f_{j}^{(s)}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right) \rightarrow h_{1}^{(s)}(\zeta)
\end{array}\right.
$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $s \in \mathbb{N}$ and so

$$
\begin{align*}
& H_{j}^{n}(\zeta)\left(H_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \xi_{j}^{i} H_{j}^{(k-i)}(\zeta)\right)^{l}  \tag{3.27}\\
= & \left(f_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right)\right)^{n}\left(L\left(f_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right)\right)^{l} \rightarrow h_{1}^{n}(\zeta)\left(h_{1}^{(k)}(\zeta)\right)^{l}\right.
\end{align*}
$$

spherically uniformly on compact subset of $\mathbb{C} \backslash h^{-1}\{\infty\}$. Again from (3.17), we have

$$
\begin{array}{r}
H_{j}^{n}(\zeta)\left(H_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \xi_{j}^{i} H_{j}^{(k-i)}(\zeta)\right)^{l}  \tag{3.28}\\
\times \widehat{H}_{j}^{n}(\zeta)\left(\widehat{H}_{j}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \xi_{j}^{i} \widehat{H}_{j}^{(k-i)}(\zeta)\right)^{l} \equiv 1
\end{array}
$$

Taking $\xi_{j} \rightarrow 0$ as $j \rightarrow \infty$, we get from (3.24, (3.27) and 3.28) that

$$
\begin{equation*}
h_{1}^{n}(\zeta)\left(h_{1}^{(k)}(\zeta)\right)^{l} \widehat{h}_{1}^{n}(\zeta)\left(\widehat{h}_{1}^{(k)}(\zeta)\right)^{l} \equiv 1 \tag{3.29}
\end{equation*}
$$

for all $\zeta \in \mathbb{C} \backslash\left\{h_{1}^{-1}(\infty) \cup \widehat{h}_{1}^{-1}(\infty)\right\}$. Also from 3.29 , we see that $h_{1}^{n}\left(h_{1}^{(k)}\right)^{l}$ and $\widehat{h}_{1}^{n}\left(\widehat{h}_{1}^{(k)}\right)^{l}$ share 1 IM. Then by Lemma 3.7. we have $\rho\left(\widehat{h}_{1}\right)=\rho\left(h_{1}\right) \leq 2$. Also by Lemma 3.8, we conclude that both $h_{1}$ and $\widehat{h}_{1}$ are transcendental meromorphic functions. Now using Lemma 3.3, we get from (3.29) that

$$
\begin{aligned}
(n+l) N\left(r, \infty ; h_{1}\right)+l k \bar{N}\left(r, \infty ; h_{1}\right) & =N\left(r, \infty ; h_{1}^{n}\left(h_{1}^{(k)}\right)^{l}\right) \\
& =N\left(r, \infty ; \frac{1}{\widehat{h}_{1}^{n}\left(\widehat{h}_{1}^{(k)}\right)^{l}}\right) \\
& =N\left(r, 0 ; \widehat{h}_{1}^{n}\left(\widehat{h}_{1}^{(k)}\right)^{l}\right) \\
& \leq l N\left(r, 0 ; \widehat{h}_{1}\right)+l k \bar{N}\left(r, \infty ; \widehat{h}_{1}\right)+O(\log r) \\
& =l k \bar{N}\left(r, \infty ; \widehat{h}_{1}\right)+O(\log r),
\end{aligned}
$$

as $r \rightarrow \infty$. Similarly

$$
(n+l) N\left(r, \infty ; \widehat{h}_{1}\right)+l k \bar{N}\left(r, \infty ; \widehat{h}_{1}\right) \leq l k \bar{N}\left(r, \infty ; h_{1}\right)+O(\log r)
$$

as $r \rightarrow \infty$. Therefore, we have $N\left(r, \infty ; h_{1}\right)+N\left(r, \infty ; \widehat{h}_{1}\right) \leq O(\log r)$ as $r \rightarrow \infty$. This shows that $h_{1}$ and $\widehat{h}_{1}$ have at most finitely many poles. Since $\rho\left(h_{1}\right)=$ $\rho\left(\widehat{h}_{1}\right) \leq 2$, so we let

$$
h_{1}=\frac{1}{P_{3}} e^{\alpha_{1}} \text { and } \widehat{h}_{1}=\frac{1}{P_{4}} e^{\beta_{1}}
$$

where $P_{3}$ and $P_{4}$ are non-zero polynomials and $\alpha_{1}$ and $\beta_{1}$ are non-constant polynomials with degree at most 2. Next proceeding in the same manner as done in Sub-case 2.1, we get

$$
\begin{equation*}
2 l k \operatorname{deg}\left(\alpha_{1}^{\prime}\right)=(n+l) \operatorname{deg}\left(P_{3} P_{4}\right) \tag{3.30}
\end{equation*}
$$

Since $\operatorname{deg}\left(\alpha_{1}^{\prime}\right) \leq 1$ and $n>2 l k-l$, we can deduce from (3.30 that $P_{3}$ and $P_{4}$ are both non-zero constants. Now observing Sub-case 2.1, we can take

$$
\begin{equation*}
h_{1}(z)=d_{1} e^{d z} \text { and } \widehat{h}_{1}(z)=d_{2} e^{-d z} \tag{3.31}
\end{equation*}
$$

where $d_{1}, d_{2}$ and $d \in \mathbb{C} \backslash\{0\}$ satisfying $(-1)^{l k}\left(d_{1} d_{2}\right)^{n+l} d^{2 l k}=1$. From 3.21) and (3.31), we have

$$
\begin{equation*}
\frac{\widehat{H}_{j}^{\prime}(\zeta)}{\widehat{H}_{j}(\zeta)}=\xi_{j} \frac{g_{j}^{\prime}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right)}{g_{j}\left(z_{j}+\rho_{j} \zeta_{j}+\xi_{j} \zeta\right)} \rightarrow \frac{\widehat{h}_{1}^{\prime}(\zeta)}{\widehat{h}_{1}(\zeta)}=-d \tag{3.32}
\end{equation*}
$$

spherically uniformly on compact subsets of $\mathbb{C}$. Now from 3.22 and (3.32), we get

$$
\begin{aligned}
\xi_{j}\left|\frac{g_{j}^{\prime}\left(z_{j}+\rho_{j} \zeta_{j}\right)}{g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right)}\right| & =\frac{1+\left|g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right)\right|^{2}}{\left|g_{j}^{\prime}\left(z_{j}+\rho_{j} \zeta_{j}\right)\right|} \frac{\left|g_{j}^{\prime}\left(z_{j}+\rho_{j} \zeta_{j}\right)\right|}{\left|g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right)\right|} \\
& =\frac{1+\left|g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right)\right|^{2}}{\left|g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right)\right|} \rightarrow\left|\frac{\widehat{h}_{1}^{\prime}(0)}{\widehat{h}_{1}(0)}\right|=|-d|=|d|
\end{aligned}
$$

which implies that $\lim _{j \rightarrow \infty} g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right) \neq 0, \infty$ and so from 3.21), we have $\widehat{H}_{j}(0)=\xi_{j}^{-\frac{l k}{n+l}} g_{j}\left(z_{j}+\rho_{j} \zeta_{j}\right) \rightarrow \infty$. Again from 3.21 and 3.31), we have $\widehat{H}_{j}(0) \rightarrow \widehat{h}_{1}(0)=d_{2}$. Therefore we arrive at a contradiction.
Sub-case 2.2.2. Suppose that $\left\{\widehat{h}_{j}(\zeta)\right\}$ is normal in $|\zeta|<R$. Then there exists a subsequence $\left\{\widehat{h}_{j_{m}}(\zeta)\right\}$ of $\left\{\widehat{h}_{j}(\zeta)\right\}$ such that

$$
\begin{equation*}
\widehat{h}_{j_{m}}(\zeta) \rightarrow \widehat{h}(\zeta) \text { as } m \rightarrow \infty \tag{3.33}
\end{equation*}
$$

spherically uniformly on compact subsets of $\mathbb{C}$. Since $g(z) \neq 0$, by Hurwitz's theorem we can see that $\widehat{h}$ has no zeros. Now from 3.33, we have

$$
\left\{\begin{array}{c}
\widehat{h}_{j_{m}}^{n}(\zeta)=\rho_{j_{m}}^{-\frac{l k n}{n t}} g_{j_{m}}\left(z_{j_{m}}+\rho_{j_{m}} \zeta\right) \rightarrow \widehat{h}^{n}(\zeta)  \tag{3.34}\\
\widehat{h}_{j_{m}}^{(s)}(\zeta)=\rho_{j_{m}}^{\frac{l(s-k)+n s}{n+l}} g_{j_{m}}^{(s)}\left(z_{j_{m}}+\rho_{j_{m}} \zeta\right) \rightarrow \widehat{h}^{(s)}(\zeta)
\end{array}\right.
$$

spherically uniformly on compact subset of $\mathbb{C}$ and so from (3.34), we have

$$
\begin{align*}
& \widehat{h}_{j_{m}}^{n}(\zeta)\left(\widehat{h}_{j_{m}}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j_{m}}^{i} \widehat{h}_{j_{m}}^{(k-i)}(\zeta)\right)^{l}  \tag{3.35}\\
= & \left(g_{j_{m}}\left(z_{j_{m}}+\rho_{j_{m}} \zeta\right)\right)^{n}\left(L\left(g_{j_{m}}\left(z_{j_{m}}+\rho_{j_{m}} \zeta\right)\right)\right)^{l} \rightarrow \widehat{h}^{n}(\zeta)\left(\widehat{h}^{(k)}(\zeta)\right)^{l}
\end{align*}
$$

spherically uniformly on compact subset of $\mathbb{C}$. On the other hand from (3.19), we have

$$
\begin{equation*}
\widehat{h}_{j_{m}}^{n}(\zeta)\left(\widehat{h}_{j_{m}}^{(k)}(\zeta)+\sum_{i=1}^{k} a_{k-i} \rho_{j_{m}}^{i} \widehat{h}_{j_{m}}^{(k-i)}(\zeta)\right)^{l} \rightarrow \widehat{H}(\zeta) \tag{3.36}
\end{equation*}
$$

spherically uniformly on compact subset of $\mathbb{C} \backslash \widehat{H}^{-1}\{\infty\}$. First we suppose that $\widehat{H} \not \equiv \widehat{h}^{n}\left(\widehat{h}^{(k)}\right)^{l}$. Then from 3.35 and 3.36$)$, we get a contradiction. Next we suppose that $\widehat{H} \equiv \widehat{h}^{n}\left(\widehat{h}^{(k)}\right)^{l}$. Then from 3.18, we have

$$
h^{n}(\zeta)\left(h^{(k)}(\zeta)\right)^{l} \widehat{h}^{n}(\zeta)\left(\widehat{h}^{(k)}(\zeta)\right)^{l} \equiv 1
$$

for all $\zeta \in \mathbb{C} \backslash\left\{h^{-1}(\infty) \cup \widehat{h}^{-1}(\infty)\right\}$. Now proceeding in the same way as done in Sub-case 2.2.1, one can easily arrive at a contradiction. This completes the proof.

Lemma 3.14. Let $f$ and $g$ be two non-constant meromorphic functions such that either $f$ and $g$ have no zeros or zeros of $f$ and $g$ are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $F=\frac{f^{n}(L(f))^{l}}{\alpha}$ and $G=\frac{g^{n}(L(g))^{l}}{2^{\alpha}}$, where $\alpha(z)(\not \equiv$ $0, \infty) \in S(f) \cap S(g)$ and $l, n \in \mathbb{N}$ such that $n>l+k+\frac{2}{k}+1$. If $H \equiv 0$, then one of the following two cases holds:
(1) $f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv \alpha^{2}$,
(2) $f^{n}(L(f))^{l} \equiv g^{n}(L(g))^{l}$.

Proof. Since $H \equiv 0$, on integration we get

$$
\begin{equation*}
\frac{1}{F-1} \equiv \frac{b G+a-b}{G-1} \tag{3.37}
\end{equation*}
$$

where $a, b \in \mathbb{C}$ such that $a \neq 0$. We now consider the following cases.
Case 1. Suppose that $b \neq 0$. Then the following sub-cases are immediate consequences.
Sub-case 1.1. Suppose $a \neq b$. If $b=-1$, then from (3.37), we have $F \equiv$ $\frac{-a}{G-a-1}$. Therefore $\bar{N}(r, a+1 ; G)=\bar{N}(r, \infty ; F)=\bar{N}(r, \infty ; f)+S(r, f)$. So in view of Lemma 3.6 and the second fundamental theorem, we get

$$
\begin{aligned}
& (n-l) T(r, g) \\
\leq & T(r, G)-l N(r, \infty ; g)-N\left(r, 0 ;(L(g))^{l}\right)+S(r, g) \\
\leq & \bar{N}(r, \infty ; G)+\bar{N}(r, 0 ; G)+\bar{N}(r, a+1 ; G)-l N(r, \infty ; g) \\
& -N\left(r, 0 ;(L(g))^{l}\right)+S(r, g) \\
\leq & \bar{N}(r, 0 ; g)+\bar{N}\left(r, 0 ;(L(g))^{l}\right)+\bar{N}(r, \infty ; f)-N\left(r, 0 ;(L(g))^{l}\right)+S(r, g) \\
\leq & \frac{1}{k} N(r, 0 ; g)+N(r, \infty ; f)+S(r, g) \leq \frac{1}{k} T(r, g)+T(r, f)+S(r, g)
\end{aligned}
$$

Without loss of generality, we suppose that there exists a set $I$ with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$. So for $r \in I$, we have $(n-$ l) $T(r, g) \leq\left(1+\frac{1}{k}\right) T(r, g)+\bar{S}(r, g)$, which is contradiction, since $n>1+l+\frac{1}{k}$. If $b \neq-1$, from 3.37 we obtain that $F-\left(1+\frac{1}{b}\right) \equiv \frac{-a}{b^{2}\left[G+\frac{a-b}{b}\right]}$. So $\bar{N}\left(r, \frac{b-a}{b} ; G\right)=$ $\bar{N}(r, \infty ; F)=\bar{N}(r, \infty ; f)+S(r, f)$. Using Lemma 3.6 and the same argument as used in the case when $b \neq-1$, we can get a contradiction.
Sub-case 1.2. Suppose $a=b$. If $b=-1$, then (3.37) yields $F G \equiv 1$, i.e., $f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv \alpha^{2}$. If $b \neq-1$, from 3.37 we have $\frac{1}{F} \equiv \frac{b G}{(1+b) G-1}$. Therefore $\bar{N}\left(r, \frac{1}{1+b} ; G\right)=\bar{N}(r, 0 ; F)$. So in view of Lemmas 3.5, 3.6 and the
second fundamental theorem we get

$$
\begin{aligned}
& (n-l) T(r, g) \\
\leq & T(r, G)-l N(r, \infty ; g)-N\left(r, 0 ;(L(g))^{l}\right)+S(r, g) \\
\leq & \bar{N}(r, \infty ; G)+\bar{N}(r, 0 ; G)+\bar{N}\left(r, \frac{1}{1+b} ; G\right)-l N(r, \infty ; g) \\
& -N\left(r, 0 ;(L(g))^{l}\right)+S(r, g) \\
\leq & \bar{N}(r, 0 ; g)+\bar{N}\left(r, 0 ;(L(g))^{l}\right)+\bar{N}(r, 0 ; F)-N\left(r, 0 ;(L(g))^{l}\right)+S(r, g) \\
\leq & \bar{N}(r, 0 ; g)+\bar{N}(r, 0 ; f)+\bar{N}(r, 0 ; L(f))+S(r, g) \\
\leq & \frac{1}{k} N(r, 0 ; g)+\frac{1}{k} N(r, 0 ; f)+N_{k+1}(r, 0 ; f)+k \bar{N}(r, \infty ; f)+S(r, g) \\
\leq & \left(k+1+\frac{1}{k}\right) T(r, f)+\frac{1}{k} T(r, g)+S(r, f)+S(r, g) .
\end{aligned}
$$

So for $r \in I$ we have $(n-l) T(r, g) \leq\left(k+1+\frac{2}{k}\right) T(r, g)+S(r, g)$, which is a contradiction since $n>l+k+\frac{2}{k}+1$.
Case 2. Suppose that $b=0$. Then (3.37) yields $F \equiv \frac{G+a-1}{a}$. If $a \neq 1$ then from above we have $\bar{N}(r, 1-a ; G)=\bar{N}(r, 0 ; F)$. We can similarly deduce a contradiction as in Sub-case 1.2. Therefore $a=1$ and so we get $F \equiv G$, i.e., $f^{n}(L(f))^{l} \equiv g^{n}(L(g))^{l}$. Hence the proof.

## 4. Proof of the main theorem

Proof of Theorem 2.1. Let $F=f^{n}(L(f))^{l}$ and $G=g^{n}(L(g))^{l}$. Clearly $F$ and $G$ share $(1,2)$. We now consider the following two cases:
Case 1. Let $H \not \equiv 0$. From (3.1) it can be easily calculated that the possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) those 1 points of $F$ and $G$ whose multiplicities are different, (iii) poles of $F$ and $G$, (iv) zeros of $F^{\prime}\left(G^{\prime}\right)$ which are not the zeros of $F(F-1)(G(G-1))$.
Since $H$ has only simple poles we get

$$
\leq \begin{align*}
& N(r, \infty ; H)  \tag{4.1}\\
& \leq \bar{N}(r, \infty ; f)+\bar{N}(r, \infty ; g)+\bar{N}_{*}(r, 1 ; F, G)+\bar{N}(r, 0 ; F \mid \geq 2) \\
& +\bar{N}(r, 0 ; G \mid \geq 2)+\bar{N}_{0}\left(r, 0 ; F^{\prime}\right)+\bar{N}_{0}\left(r, 0 ; G^{\prime}\right)+S(r, f)+S(r, g),
\end{align*}
$$

where $\bar{N}_{0}\left(r, 0 ; F^{\prime}\right)$ is the reduced counting function of those zeros of $F^{\prime}$ which are not the zeros of $F(F-1)$ and $\bar{N}_{0}\left(r, 0 ; G^{\prime}\right)$ is similarly defined.
Let $z_{0}$ be a simple zero of $F-1$. Then $z_{0}$ is a simple zero of $G-1$ and a zero of $H$. So

$$
\begin{equation*}
N(r, 1 ; F \mid=1) \leq N(r, 0 ; H) \leq N(r, \infty ; H)+S(r, f)+S(r, g) \tag{4.2}
\end{equation*}
$$

Using (4.1) and 4.2), we get

$$
\begin{align*}
& \bar{N}(r, 1 ; F)  \tag{4.3}\\
\leq & N(r, 1 ; F \mid=1)+\bar{N}(r, 1 ; F \mid \geq 2) \\
\leq & \bar{N}(r, \infty ; f)+\bar{N}(r, \infty ; g)+\bar{N}(r, 0 ; F \mid \geq 2)+\bar{N}(r, 0 ; G \mid \geq 2) \\
& +\bar{N}_{*}(r, 1 ; F, G)+\bar{N}(r, 1 ; F \mid \geq 2)+\bar{N}_{0}\left(r, 0 ; F^{\prime}\right)+\bar{N}_{0}\left(r, 0 ; G^{\prime}\right) \\
& +S(r, f)+S(r, g)
\end{align*}
$$

Now in view of Lemma 3.5, we get

$$
\begin{align*}
& \bar{N}_{0}\left(r, 0 ; G^{\prime}\right)+\bar{N}(r, 1 ; F \mid \geq 2)+\bar{N}_{*}(r, 1 ; F, G)  \tag{4.4}\\
\leq & \bar{N}_{0}\left(r, 0 ; G^{\prime}\right)+\bar{N}(r, 1 ; F \mid \geq 2)+\bar{N}(r, 1 ; F \mid \geq 3) \\
= & \bar{N}_{0}\left(r, 0 ; G^{\prime}\right)+\bar{N}(r, 1 ; G \mid \geq 2)+\bar{N}(r, 1 ; G \mid \geq 3) \\
\leq & N\left(r, 0 ; G^{\prime} \mid G \neq 0\right) \leq \bar{N}(r, 0 ; G)+\bar{N}(r, \infty ; g)+S(r, g) .
\end{align*}
$$

Hence using (4.3), (4.4) and Lemma 3.4, we get from second fundamental theorem that

$$
\begin{align*}
& T(r, F)  \tag{4.5}\\
\leq & \bar{N}(r, 0 ; F)+\bar{N}(r, \infty ; F)+\bar{N}(r, 1 ; F)-N_{0}\left(r, 0 ; F^{\prime}\right)+S(r, f) \\
\leq \quad & 2 \bar{N}(r, \infty, f)+\bar{N}(r, \infty ; g)+N_{2}(r, 0 ; F)+\bar{N}(r, 0 ; G \mid \geq 2) \\
& +\bar{N}(r, 1 ; F \mid \geq 2)+\bar{N}_{*}(r, 1 ; F, G)+\bar{N}_{0}\left(r, 0 ; G^{\prime}\right)+S(r, f)+S(r, g) \\
\leq \quad & 2 \bar{N}(r, \infty ; f)+2 \bar{N}(r, \infty ; g)+N_{2}(r, 0 ; F)+N_{2}(r, 0 ; G) \\
& +S(r, f)+S(r, g) \\
\leq \quad & 2 \bar{N}(r, \infty ; f)+2 \bar{N}(r, \infty ; g)+2 \bar{N}(r, 0 ; f)+N_{2}\left(r, 0 ;(L(f))^{l}\right) \\
& +2 \bar{N}(r, 0 ; g)+N_{2}\left(r, 0 ;(L(f))^{l}\right)+S(r, f)+S(r, g) \\
\leq \quad & 2 \bar{N}(r, \infty ; f)+2 \bar{N}(r, \infty ; g)+2 \bar{N}(r, 0 ; f)+N\left(r, 0 ;(L(f))^{l}\right) \\
& +2 \bar{N}(r, 0 ; g)+l N_{k+2}(r, 0 ; g)+l k \bar{N}(r, \infty ; g)+S(r, f)+S(r, g) \\
\leq & 2 \bar{N}(r, \infty ; f)+(2+l k) \bar{N}(r, \infty ; g)+2 \bar{N}(r, 0 ; f)+2 \bar{N}(r, 0 ; g) \\
& +l N(r, 0 ; g)+N\left(r, 0 ;(L(f))^{l}\right)+S(r, f)+S(r, g) .
\end{align*}
$$

Now usingLemma 3.6, we get from (4.5) that

$$
\begin{align*}
(n-l) T(r, f) \leq & T(r, F)-l N(r, \infty ; f)-N\left(r, 0 ;(L(f))^{l}\right)+S(r, f)  \tag{4.6}\\
\leq & \bar{N}(r, \infty ; f)+(2+l k) \bar{N}(r, \infty ; g)+2 \bar{N}(r, 0 ; f) \\
& +2 \bar{N}(r, 0 ; g)+l N(r, 0 ; g)+S(r, f)+S(r, g) \\
\leq & \bar{N}(r, \infty ; f)+(2+l k) \bar{N}(r, \infty ; g)+\frac{2}{k} N(r, 0 ; f) \\
& +\frac{2}{k} N(r, 0 ; g)+l N(r, 0 ; g)+S(r, f)+S(r, g) \\
\leq & \left(1+\frac{2}{k}\right) T(r, f)+\left(2+l k+l+\frac{2}{k}\right) T(r, g) \\
& +S(r, f)+S(r, g) .
\end{align*}
$$

In a similar way we can obtain

$$
\begin{align*}
(n-l) T(r, g) \leq & \left(2+l k+l+\frac{2}{k}\right) T(r, f)+\left(1+\frac{2}{k}\right) T(r, g)  \tag{4.7}\\
& +S(r, f)+S(r, g)
\end{align*}
$$

Adding (4.6) and 4.7, we see that

$$
\begin{align*}
(n-l)\{T(r, f)+T(r, g)\} \leq & \left(3+l k+l+\frac{4}{k}\right)\{T(r, f)+T(r, g)\}  \tag{4.8}\\
& +S(r, f)+S(r, g)
\end{align*}
$$

Since $n>3+l k+2 l+\frac{4}{k}$, 4.8 leads to a contradiction.
Case 2. Let $H \equiv 0$. Then from Lemma 3.14. we get either $f^{n}(L(f))^{l} \equiv$ $g^{n}(L(g))^{l}$ or

$$
\begin{equation*}
f^{n}(L(f))^{l} g^{n}(L(g))^{l} \equiv 1 \tag{4.9}
\end{equation*}
$$

The remaining part of the theorem follows from Lemma 3.13 and 4.9 .

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