# An inverse free Broyden's method for solving equations 

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#### Abstract

Based on a center-Lipschitz-type condition and our idea of the restricted convergence domain, we present a new semi-local convergence analysis for an inverse free Broyden's method (BM) in order to approximate a locally unique solution of an equation in a Hilbert space setting. The operators involved have regularly continuous divided differences. This way we provide weaker sufficient semi-local convergence conditions, tighter error bounds, and a more precise information on the location of the solution. Hence, our approach extends the applicability of BM under the same hypotheses as before. Finally, we consider some special cases.


AMS Mathematics Subject Classification (2010):65J15; 47H17; 47J05; 49M15
Key words and phrases: Broyden's method; Hilbert space; semi-local convergence; regularly continuous divided differences

## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^{\star}$ of equation

$$
\begin{equation*}
F(x)=0, \tag{1.1}
\end{equation*}
$$

where $F$ is a continuous operator defined on a open convex subset $\Omega$ of a Hilbert space $\mathcal{B}_{1}$ with values in a Hilbert space $\mathcal{B}_{2}$.

Broyden's method BM is
(1.2) $x_{+}=x-A F(x), \quad y=F\left(x_{+}\right)-F(x), \quad A_{+}=A-\frac{A F\left(x_{+}\right)<y, \cdot>}{<y, y>}$,
where $\mathcal{L}\left(\mathcal{B}_{2}, \mathcal{B}_{1}\right):=\left\{A: \mathcal{B}_{2} \longrightarrow \mathcal{B}_{1}\right.$, bounded and linear $\}$, and $<\cdot, \cdot>$ stands for the inner product.

Numerous convergence results for this type of methods have appeared in the literature [1, 3, 5, 8, 9, 10, 11] (see also, e.g. [4], and the references therein). BM requires no inverse, so no linear subproblem needs to be solved at each iteration.

[^0]The convergence domain for such methods is small in general $12,13,14,15$. In the present study, we extend the convergence domain for BM. To achieve this goal, we first introduce the center-Lipschitz condition which determines a subset of the original domain for the operator containing the iterates. The scalar functions are then related to the subset instead of the original domain. This way, the scalar functions are more precise than if they were depending on the original domain. The new technique leads to : weaker sufficient convergence conditions, tighter error bounds on the distances involved, and an at least as precise information on the location of the solution. These advantages are obtained under the same computational cost as in earlier studies [8, 9, 10, 11, since in practice the new functions are special cases of the old functions. This idea can be used to study other iterative methods requiring inverses of linear mappings [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

The study is structured as follows. Section 2 contains some preliminary results for regularly continuous dd. In Section 3, we provide the semi-local convergence analysis of BM. Finally, in Section 4 , we provide special cases, as applications.

## 2. Preliminaries: regularly continuous dd

In order to make the paper as self-contained as possible, we reintroduce some definitions and some results on regularly continuous dd. The proofs are omitted, and can be found in [4, 11]. In this section, $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ are Banach spaces, equipped with the norm $\|$.$\| . We denote by U(z, R)=\left\{x \in \mathcal{B}_{1}: \|\right.$ $x-z \|<R$,$\} the open ball centered at z$ and of radius $R>0$, whereas $\bar{U}(z, R)$ denotes its closure. For $x \in \mathcal{B}_{1}$, denote by $\mathcal{K}_{x}$ the subspace of operators vanishing at $x \mathcal{K}_{x}=\left\{A \in \mathcal{L}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right): A x=0\right\}$. Let $\mathcal{N}$ be the class of increasing concave functions $v: \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$, with $v(0)=0$. Note that $\mathcal{N}$ contains the functions in the form $\varphi(t)=c t^{p},(c \geq 0$, and $p \in(0,1])$.

Definition 2.1. 11] An operator [., .; F] belonging in $\mathcal{L}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right)$ is called the first order divided difference (briefly dd) of $F$ at the points $x$ and $y$ in $\mathcal{B}_{1}$ $(x \neq y)$, if the following secant equation holds $[x, y ; F](y-x)=F(y)-F(x)$. If $F$ is Fréchet differentiable at $x$, then $[x, x ; F]=F^{\prime}(x)$. Otherwise, the following limit (if it exists) $\lim _{t \searrow 0}[x, x+t h ; F] h=\lim _{t \searrow 0} \frac{F(x+t h)-F(x)}{t}$ vary according to $h$, with $\|h\|=1$, and this limit is the Fréchet derivative (or the directional derivative) $F^{\prime}(x) h$ of $F$ in the direction $h$ (i.e., if we suppose that $F$ is Fréchet differentiable at $x$, then the Fréchet derivative is characterized as a limit of dd in the uniform topology of the space of continuous linear mappings of $\mathcal{B}_{1}$ into $\mathcal{B}_{2}$ ).

Remark 2.2. (a) Let $(x, y) \in \mathcal{B}_{1} \times \mathcal{B}_{2}$, the set $\left\{A \in \mathcal{L}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right): A x=y\right\}$ constitutes an affine manifold in $\mathcal{L}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right)$.
(b) Let $A$ and $A_{0}$ in $\mathcal{L}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right)$, and $(x, y) \in \mathcal{B}_{1} \times \mathcal{B}_{2}$, such that $A_{0} x=A x=y$. Then $\left(A-A_{0}\right) x=0$, and $A \in A_{0}+\mathcal{K}_{x}$.

The following result gives some properties of set-valued mapping $\Upsilon_{x, y}$ : $\mathcal{C}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right) \rightrightarrows \mathcal{L}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right)$ given by $\Upsilon_{x, y}(F)=[x, y ; F]$ for the pair $(x, y) \in \mathcal{B}_{1}^{2}$.

Proposition 2.3. (a) $\Upsilon_{x, y}(F)=F$ if and only if $F$ is linear.
(b) $\Upsilon_{x, y}$ is linear, i.e., for $F_{1}, F_{2}$ in $\mathcal{C}\left(\mathcal{B}_{1}, \mathcal{B}_{2}\right)$, and $(\alpha, \beta) \in \mathbb{K}^{2}(\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ ), we have

$$
\Upsilon_{x, y}\left(\alpha F_{1}+\beta F_{2}\right)=\alpha \Upsilon_{x, y}\left(F_{1}\right)+\beta \Upsilon_{x, y}\left(F_{2}\right)
$$

(c) If $F$ is a composition of operators $F_{1}$ and $F_{2}$ (i.e., $F=F_{1} \circ F_{2}$ ), then

$$
\Upsilon_{x, y}(F)=\Upsilon_{F_{2}(x), F_{2}(y)}\left(F_{1}\right) \Upsilon_{x, y}\left(F_{2}\right) .
$$

Definition 2.4. [11] The dd $[x, y ; F]$ is said to be $w_{1}$-regularly continuous on $\Omega \subseteq \mathcal{B}_{1}$ for $w_{1} \in \mathcal{N}$ (call it regularity modulus), if the following inequality holds for each $x, y, u, v \in \Omega$

$$
\begin{align*}
& w_{1}^{-1}(\min \{\|[x, y ; F]\|,\|[u, v ; F]\|\}+\|[x, y ; F]-[u, v ; F]\|) \\
& -w_{1}^{-1}(\min \{\|[x, y ; F]\|,\|[u, v ; F]\|\}) \leq\|x-u\|+\|y-v\| \tag{2.1}
\end{align*}
$$

The dd $[x, y ; F]$ is said to be regularly continuous on $\Omega$ if it has a regularity modulus there.

We introduce a special notion (see also [5, 6, 7]).
Definition 2.5. The $\mathrm{dd}[x, y ; F]$ is said to be $w_{0}$ - center regularly continuous on $\Omega \subset X$ for $w_{0} \in \mathcal{N}$ (call it center regularity modulus), if for fixed $x_{-1}, x_{0} \in \Omega$ the following inequality holds for each $x, y$ in $\Omega$

$$
\begin{align*}
& w_{0}^{-1}\left(\min \left\{\|[x, y ; F]\|,\left\|\left[x_{0}, x_{-1} ; F\right]\right\|\right\}+\left\|[x, y ; F]-\left[x_{0}, x_{-1} ; F\right]\right\|\right)  \tag{2.2}\\
& -w_{0}^{-1}\left(\min \left\{\|[x, y ; F]\|,\left\|\left[x_{0}, x_{-1} ; F\right]\right\|\right\}\right) \leq\left\|x-x_{0}\right\|+\left\|y-x_{-1}\right\|
\end{align*}
$$

Clearly, we have that Definition 2.5 is a special case of Definition 2.4 ,

$$
\begin{equation*}
w_{0}(t) \leq w_{1}(t) \quad \text { for each } \quad t \in[0, \infty) \tag{2.3}
\end{equation*}
$$

holds in general, and $\frac{w_{1}}{w_{0}}$ can be arbitrarily large [2, 4]. If $w_{0}, w_{1}$ are linear functions $\left(w_{1}(t)=c_{1} t\right.$ and $\left.w_{0}(t)=c_{0} t\right)$, then 2.2), and 2.3) become Lipschitz, and center-Lipschitz continuous conditions, respectively, i.e., the following hold respectively for each $(x, y, u, v) \in \Omega^{4}$ :

$$
\begin{equation*}
\|[x, y ; F]-[u, v ; F]\| \leq c_{1}(\|x-u\|+\|y-v\|) \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|[x, y ; F]-\left[x_{0}, x_{-1} ; F\right]\right\| \leq c_{0}\left(\left\|x-x_{0}\right\|+\left\|y-x_{-1}\right\|\right) \tag{2.5}
\end{equation*}
$$

Then, estimate 2.3) gives

$$
\begin{equation*}
c_{0} \leq c_{1} . \tag{2.6}
\end{equation*}
$$

We need the following auxiliary result.
Lemma 2.6. [9] If $d d[x, y ; F]$ is $w$-regularly continuous on $\Omega$, then we have

$$
\left|w_{1}^{-1}(\|[x, y ; F]\|)-w_{1}^{-1}(\|[u, v ; F]\|)\right| \leq\|x-u\|+\|y-v\|,
$$

for each $(x, y, u, v) \in \Omega^{4}$.
Then, the following holds for all $(x, y, u, v) \in \Omega^{4}$ :

$$
\begin{equation*}
w_{1}^{-1}(\|[x, y ; F]\|) \geq\left(w_{1}^{-1}(\|[u, v ; F]\|)-\|x-u\|-\|y-v\|\right)^{+} \tag{2.7}
\end{equation*}
$$

where $\rho^{+}(\rho \in \mathbb{R})$ denotes the nonnegative part of $\rho: \rho^{+}=\max \{\rho, 0\}$.
In particular, if $d d[x, y ; F]$ is $w_{0}$-regularly continuous on $\Omega$ (i.e., condition (2.2) holds), then, (2.7) holds, with $w_{0}, x_{0}$, and $x_{-1}$ replacing $w$, $u$, and $v$, respectively.

Suppose that the equation

$$
\begin{equation*}
w_{0}(t)=1 \tag{2.8}
\end{equation*}
$$

has at least one positive solution. Denote by $r_{0}$ the smallest such solution. Moreover, define

$$
\begin{equation*}
\Omega_{0}=\Omega \cap U\left(x_{0}, r_{0}\right) \tag{2.9}
\end{equation*}
$$

Notice also that we have a similar estimate for the function $w$ on $\Omega^{4}$.
Definition 2.7. The dd $[x, y ; F]$ is said to be restricted $w$-regularly continuous on $\Omega_{0} \subset \Omega$ for $w \in \mathcal{N}$, if the following inequality holds for each $x, y, u, v \in \Omega_{0}$

$$
\begin{align*}
& w^{-1}(\min \{\|[x, y ; F]\|,\|[u, v ; F]\|\}+\|[x, y ; F]-[u, v ; F]\|) \\
& -w^{-1}(\min \{\|[x, y ; F]\|,\|[u, v ; F]\|\}) \leq\|x-u\|+\|y-v\| \tag{2.10}
\end{align*}
$$

Notice that

$$
\begin{equation*}
w(t) \leq w_{1}(t) \text { for each } t \in\left[0, r_{0}\right) \tag{2.11}
\end{equation*}
$$

holds, since $\Omega_{0} \subseteq \Omega$. The function $w$ depends on the function $w_{0}$. Construction of function $w$ was not possible in the earlier studies using only the function $w_{1}$ [11. Clearly, in those studies $w$ can simply replace $w_{1}$, since the iterates lie in $\Omega_{0}$ related to $w$, which is a more precise location than $\Omega$ used in [11] related to $w_{1}$. This modification leads to the already stated advantages, if strict inequality holds in 2.3 or 2.11. We suppose from now on until Remark 4.5 (b) that

$$
\begin{equation*}
w_{0}(t) \leq w(t) \text { for each } t \in\left[0, r_{0}\right) \tag{2.12}
\end{equation*}
$$

## 3. Semi-local convergence analysis of BM

We present a semi-local convergence result for BM . The proofs are the proper modifications of the ones in [11], where, we use the more precise 2.2), (2.10) instead of (2.1). First, we denote

$$
\begin{equation*}
A_{0}=\left[x_{0}, x_{-1} ; F\right]^{-1} . \tag{3.1}
\end{equation*}
$$

As in [11, for the selected dd $[x, y ; F]$, such that (2.1) holds with $w$ modulus, we associate the current iteration $(x, A)$, and we consider $q=(\bar{t}, \bar{\gamma}, \bar{\delta})$, where

$$
\bar{t}=\left\|x-x_{0}\right\|, \quad \bar{\gamma}=\left\|x-x_{-}\right\|, \quad \bar{\delta}=\left\|x_{+}-x\right\|=\|A F(x)\| .
$$

Finally, let $q_{+}$, and $\psi_{w}: \mathbb{R}^{+2} \longrightarrow \mathbb{R}^{+}$be given by

$$
q_{+}=\left(\bar{t}_{+}, \bar{\gamma}_{+}, \bar{\delta}_{+}\right)
$$

and

$$
\psi_{w}(u, t)=w\left((u-t)^{+}+t\right)-w\left((u-t)^{+}\right), \quad \text { for each }(u, t) \in \mathbb{R}^{+^{2}},
$$

respectively. Note that $\psi_{w}$ is not increasing in the first argument, and not decreasing in the second, since $w$ is concave, and increasing.

We provide now a result on $q_{+}$using $w$ and $w_{0}-$ regularity.
Lemma 3.1. Under the hypotheses (2.2), and 2.10, the following estimates hold:

$$
\begin{gather*}
\bar{t}_{+}:=\left\|x_{+}-x_{0}\right\| \leq \bar{t}+\bar{\delta},  \tag{3.2}\\
\bar{\gamma}_{+}:=\left\|x_{+}-x\right\|=\bar{\delta} \tag{3.3}
\end{gather*}
$$

and

$$
\begin{equation*}
\bar{\delta}_{+} \leq \bar{\delta} e_{w_{0}, w}(q) \tag{3.4}
\end{equation*}
$$

where

$$
e_{w_{0}, w}(q)=\frac{\psi_{w}\left(w^{-1}\left(\left\|A_{0}^{-1}\right\|-\bar{\gamma}_{0}-2 \bar{t}_{-}-\bar{\gamma}\right)^{+}, \bar{\gamma}+\bar{\delta}\right)}{w_{0}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|-\bar{\gamma}_{0}-2 \bar{t}-\bar{\delta}\right)\right)} .
$$

Proof. Estimates (3.2), and (3.3) follow from $\left\|x_{+}-x_{0}\right\| \leq\left\|x_{+}-x\right\|+\|$ $x-x_{0} \|=\bar{t}+\bar{\delta}$, and the expression of $\bar{\delta}$, respectively.

We must show (3.4). We have

$$
\bar{\delta}_{+} \leq\left\|A_{+}\right\|\left\|F\left(x_{+}\right)\right\| .
$$

By the Banach lemma on invertible operators [4, and (3.1), we obtain

$$
\begin{equation*}
\left\|A_{+}^{-1}\right\| \geq\left\|A_{0}^{-1}\right\|-\left\|A_{+}^{-1}-A_{0}^{-1}\right\|=\left\|A_{0}^{-1}\right\|^{-1}-\left\|\left[x_{+}, x ; F\right]-\left[x_{0}, x_{-1} ; F\right]\right\| . \tag{3.5}
\end{equation*}
$$

Using 2.1 , we get that
(3.6)

$$
\begin{aligned}
& \|[x, y ; F]-[u, v ; F]\| \leq \\
& w\left(w^{-1}(\min \{\|[x, y ; F]\|,\|[u, v ; F]\|\})+\|[x, y ; F]-[u, v ; F]\|\right) \\
& -\min \{\|[x, y ; F]\|,\|[u, v ; F]\|\}= \\
& w\left(\min \left\{w^{-1}(\|[x, y ; F]\|), w^{-1}(\|[u, v ; F]\|)\right\}+\|[x, y ; F]-[u, v ; F]\|\right) \\
& -w\left(\min \left\{w^{-1}(\|[x, y ; F]\|), w^{-1}(\|[u, v ; F]\|)\right\}\right) .
\end{aligned}
$$

By Lemma 2.6, we have

$$
w^{-1}(\|[u, v ; F]\|) \geq\left(w^{-1}(\|[x, y ; F]\|)-\|x-u\|-\|y-v\|\right)^{+}
$$

By (3.6), and the concavity of $w$, we get

$$
\begin{align*}
& \|[x, y ; F]-[u, v ; F]\| \leq  \tag{3.7}\\
& w\left(w^{-1}(\|[x, y ; F]\|-\|x-u\|-\|y-v\|)^{+}+\|x-u\|+\|y-v\|\right) \\
& -w\left(w^{-1}(\|[x, y ; F]\|-\|x-u\|-\|y-v\|)^{+}\right)= \\
& \psi_{w}\left(w^{-1}(\|[x, y ; F]\|,\|x-u\|+\|y-v\|)\right)
\end{align*}
$$

Clearly, estimate (3.7) holds with $w_{0}, x_{+}, x_{0}$, and $x_{-1}$ replacing $w, y, u$, and $v$, respectively. Consequently,

$$
\begin{align*}
\left\|\left[x_{+}, x ; F\right]-\left[x_{0}, x_{-1} ; F\right]\right\| & \leq \psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|\left[x_{0}, x_{-1} ; F\right]\right\|\right),\left\|x_{+}-x_{0}\right\|+\left\|x-x_{-1}\right\|\right)  \tag{3.8}\\
& \leq \psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|,\left\|x_{+}-x_{0}\right\|+\left\|x-x_{0}\right\|+\left\|x_{0}-x_{-1}\right\|\right)\right) \\
& =\psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|, \bar{t}_{+}+\bar{t}+\bar{\gamma}_{0}\right)\right),
\end{align*}
$$

so,
$\left\|A_{+}\right\| \leq\left(\left\|A_{0}\right\|^{-1}-\psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|\right), \bar{\gamma}_{0}+\bar{t}_{+}+\bar{t}\right)\right)^{-1} \Longrightarrow \bar{\gamma}_{0}+\bar{t}_{+}+\bar{t}<w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|\right)$
since, otherwise

$$
\psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|\right), \bar{t}_{+}+\bar{t}+\bar{\gamma}_{0}\right)=w_{0}\left(\bar{\gamma}_{0}+\bar{t}_{+}+\bar{t}\right) \geq\left\|A_{0}^{-1}\right\| \geq\left\|A_{0}\right\|^{-1}
$$

and

$$
\left\|A_{0}\right\|^{-1}-\psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|\right), \bar{t}_{+}+\bar{t}+\bar{\gamma}_{0}\right) \leq 0
$$

We also have

$$
\begin{equation*}
\left\|A_{+}\right\| \leq \frac{1}{w_{0}\left(w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\delta}-2 \bar{t}-\bar{\gamma}_{0}\right)\right)} \tag{3.10}
\end{equation*}
$$

Using (3.10), and since $w_{0}$ is concave and increasing, we deduce

$$
\begin{aligned}
\psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|, \bar{t}_{+}+\bar{t}+\bar{\gamma}_{0}\right)\right) & =\left\|A_{0}\right\|^{-1}-w_{0}\left(w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{t}_{+}-\bar{t}-\bar{\gamma}_{0}\right)\right) \\
& \leq\left\|A_{0}\right\|^{-1}-w_{0}\left(w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\delta}-2 \bar{t}-\bar{\gamma}_{0}\right)\right) .
\end{aligned}
$$

By BM, we can have the identity

$$
\begin{align*}
F\left(x_{+}\right) & =F\left(x_{+}\right)-F(x)+F(x) \\
& =\left[x_{+}, x ; F\right]\left(x_{+}-x\right)-A^{-1}\left(x_{+}-x\right)  \tag{3.11}\\
& =\left(\left[x_{+}, x ; F\right]-\left[x, x_{-} ; F\right]\right)\left(x_{+}-x\right)
\end{align*}
$$

Using (3.7), and (3.11), we obtain

$$
\begin{aligned}
\left\|F\left(x_{+}\right)\right\| & \leq \bar{\delta}\left\|\left[x_{+}, x ; F\right]-\left[x, x_{-} ; F\right]\right\| \\
& \leq \psi_{w}\left(w^{-1}\left(\left\|A^{-1}\right\|\right), \bar{\gamma}+\bar{\delta}\right)
\end{aligned}
$$

By 2.10, we get

$$
\begin{aligned}
w^{-1}\left(\left\|A^{-1}\right\|\right) & =w^{-1}\left(\left[x, x_{-} ; F\right]\right) \\
& \geq\left(w^{-1}\left(\left[x_{0}, x_{-1} ; F\right]-\left\|x-x_{0}\right\|-\left\|x_{-}-x_{-1}\right\|\right)^{+}\right) \\
& \geq\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}-\left\|x-x_{0}\right\|-\left\|x_{-}-x_{0}\right\|-\left\|x_{0}-x_{-1}\right\|\right)^{+}\right) \\
& \geq\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{t}-\bar{t}_{-}-\bar{\gamma}_{0}\right)^{+}\right) \\
& \geq\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\gamma}-2 \bar{t}_{-}-\bar{\gamma}_{0}\right)^{+}\right) .
\end{aligned}
$$

Consequently,

$$
\begin{align*}
& \psi_{w}\left(w^{-1}\left(\|A\|^{-1}\right), \bar{\gamma}+\bar{\delta}\right) \leq \psi_{w}\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\gamma}-2 \bar{t}_{-}-\bar{\gamma}_{0}\right)^{+}, \bar{\gamma}+\bar{\delta}\right), \\
& 12) \quad\left\|F\left(x_{+}\right)\right\| \leq \bar{\delta} \psi_{w}\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\gamma}-2 \bar{t}_{-}-\bar{\gamma}_{0}\right)^{+}, \bar{\gamma}+\bar{\delta}\right), \tag{3.12}
\end{align*}
$$

and

$$
\bar{\delta}_{+} \leq \bar{\delta} e_{w, w_{0}}(q)
$$

We define the function $\chi_{w, w_{0}}$ for all $q=(t, \gamma, \delta)$ by $\chi_{w, w_{0}}(q)=q_{+}=$ $\left(t_{+}, \gamma_{+}, \delta_{+}\right)$,

$$
\begin{equation*}
t_{+}=t+\delta, \quad \gamma_{+}=\delta, \quad \delta_{+}=\delta \frac{\psi_{w}\left((a-2 t+\gamma)^{+}, \gamma+\delta\right)}{w_{0}\left(a_{0}-2 t-\delta\right)} \tag{3.13}
\end{equation*}
$$

where

$$
a \leq w^{-1}\left(\left\|A_{0}\right\|^{-1}\right)-\left\|x_{0}-x_{-1}\right\| \quad \text { and } \quad a_{0} \leq w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}\right)-\left\|x_{0}-x_{-1}\right\| .
$$

In view of the definitions of $a_{0}$ and $a$, we can certainly assume $a \leq a_{0}$.
Remark 3.2. (a) Since $2 t-\gamma \leq 2 t+\delta<a$, then

$$
\psi_{w}\left((a-2 t+\gamma)^{+}, \gamma+\delta\right)=\psi_{w}(a-2 t+\gamma, \gamma+\delta)=w(a-2 t+\gamma)-w(a-2 t-\delta)
$$

Consequently, we can simplify the third component $\delta_{+}$in the expression of $\chi_{w, w_{0}}$ by:

$$
\delta_{+}=\delta \frac{w(a-2 t+\gamma)-w(a-2 t-\delta)}{w_{0}\left(a_{0}-2 t-\delta\right)} .
$$

(b) As $\bar{t}_{0}=0$, we can take $t_{0}=0$.

Consider the relation order " $\prec$ " for $q=(t, \gamma, \delta)$ and $q^{\prime}=\left(t^{\prime}, \gamma^{\prime}, \delta^{\prime}\right)$. We say that $q^{\prime}$ majorizes $q$, if

$$
q \prec q^{\prime} \Longleftrightarrow t \leq t^{\prime}, \quad \gamma \leq \gamma^{\prime} \text { and } \quad \delta \leq \delta^{\prime} .
$$

Lemma 3.3. Let $q=(t, \gamma, \delta)$ and $q^{\prime}=\left(t^{\prime}, \gamma^{\prime}, \delta^{\prime}\right)$. Then

$$
0 \prec q \prec q^{\prime} \Longrightarrow 0 \prec \chi_{w, w_{0}}(q) \prec \chi_{w, w_{0}}\left(q^{\prime}\right)
$$

Proof. We suppose $q \prec q^{\prime}$. Then, we obtain

$$
t \leq t^{\prime} \text { and } \delta \leq \delta^{\prime} \Longrightarrow t_{+}:=t+\delta \leq t^{\prime}+\delta^{\prime}=: t_{+}^{\prime} \text { and } \gamma_{+}:=\delta \leq \delta^{\prime}=: \gamma_{+}^{\prime}
$$

We show now $\delta_{+} \leq \delta_{+}^{\prime}$. Functions $w, w_{0}$ are concave and increasing, and by using Remark 3.2, we have

$$
\begin{aligned}
\delta_{+} & =\delta \frac{w(a-2 t+\gamma)-w(a-2 t-\delta)}{w_{0}\left(a_{0}-2 t-\delta\right)} \\
& \leq \delta^{\prime} \frac{w(a-2 t-\delta+(\delta+\gamma))-w(a-2 t-\delta)}{w_{0}\left(a_{0}-2 t^{\prime}-\delta^{\prime}\right)} \\
& \leq \delta^{\prime} \frac{w\left(a-2 t^{\prime}-\delta^{\prime}+(\delta+\gamma)\right)-w\left(a-2 t^{\prime}-\delta^{\prime}\right)}{w_{0}\left(a_{0}-2 t^{\prime}-\delta^{\prime}\right)} \\
& \leq \delta^{\prime} \frac{w\left(a-2 t^{\prime}-\delta^{\prime}+\left(\delta^{\prime}+\gamma^{\prime}\right)\right)-w\left(a-2 t^{\prime}-\delta^{\prime}\right)}{w_{0}\left(a_{0}-2 t^{\prime}-\delta^{\prime}\right)}=\delta_{+}^{\prime}
\end{aligned}
$$

Consider the sequence $q_{n}$ with the initial iterate $q_{0}=\left(t_{0}, \gamma_{0}, \delta_{0}\right)$ by

$$
\begin{equation*}
q_{n+1}=\chi_{w, w_{0}}\left(q_{n}\right) . \tag{3.14}
\end{equation*}
$$

Then, $q_{n}$ is a majorizing sequence, if $\bar{q}_{n} \prec q_{n}$ for each $n \geq 0$, where $\bar{q}_{n}$ is produced by the $n$-th iteration $\left(x_{n}, A_{n}\right)$ of BM .

Lemma 3.4. Let $q_{0}$ be an initial iterate for sequence $q_{n}=\left(t_{n}, \gamma_{n}, \delta_{n}\right)$ given by (3.14), such that $\bar{q}_{0} \prec q_{0}$, and $2 t_{n}+\delta_{n}<a$ for each $n \geq 0$. Then, the following hold for each $n \geq 0$ :
(a)

$$
\bar{q}_{n} \prec q_{n} ;
$$

(b)

$$
\gamma_{\infty}=\delta_{\infty}=0 \quad \text { and } \quad t_{n}=\sum_{k=0}^{k=n-1} \delta_{k} \leq 0.5\left(a-\delta_{n}\right)
$$

where

$$
\gamma_{\infty}=\lim _{n \longrightarrow \infty} \gamma_{n}, \quad \delta_{\infty}=\lim _{n \longrightarrow \infty} \delta_{n} \text { and } t_{\infty}=\lim _{n \longrightarrow \infty} t_{n}
$$

(c) The sequence $\left(x_{n}, A_{n}\right)$ generated by BM from the initial iterate $\left(x_{0}, A_{0}\right)$ converges to a solution $\left(x^{\star}, A_{\infty}\right)$ of the system

$$
F(x)=0 \quad \text { and } \quad A[x, x ; F]=I
$$

(d) The solution $x^{\star}$ is unique in $U\left(x_{0}, a_{0}-t_{\infty}\right)$;
(e)

$$
\begin{gathered}
\left\|F\left(x_{n+1}\right)\right\| \leq \delta_{n}\left(w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-2 t_{n}-\delta_{n}\right)\right), \\
\left\|x_{n}-x_{0}\right\| \leq t_{n}<t_{\infty} \leq 0.5 a, \\
\Delta_{n}:=\left\|x^{\star}-x_{n}\right\| \leq t_{\infty}-t_{n}, \\
\left\|I-A_{n}\left[x^{\star}, x^{\star} ; F\right]\right\| \leq \frac{w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-2 t_{\infty}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)},
\end{gathered}
$$

and

$$
\frac{\Delta_{n+1}}{\Delta_{n}} \leq \frac{w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-t_{n}-t_{\infty}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)} .
$$

## Proof.

(a) We show (a) using induction on $n$. We suppose that $\bar{q} \prec q$. By Lemma 3.1. we have

$$
\begin{equation*}
q_{+}=\left(\bar{t}_{+}, \bar{\gamma}_{+}, \bar{\delta}_{+}\right) \prec\left(\bar{t}+\bar{\delta}, \bar{\delta}, \bar{\delta} e_{w_{0}, w}(q)\right), \tag{3.15}
\end{equation*}
$$

where

$$
w\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}\right)-\bar{\gamma}_{0}-2 \bar{t}-\bar{\delta}\right) \geq w(a-2 t-\delta)
$$

and

$$
\begin{aligned}
\psi_{w}\left(\left(w^{-1}\left(\left\|A_{0}^{-1}\right\|-\bar{\gamma}_{0}-2 \bar{t}_{-}-\bar{\gamma}\right)^{+}, \bar{\gamma}+\bar{\delta}\right)\right) & \leq \psi_{w}\left(\left(a-2 \bar{t}_{-}-\bar{\gamma}\right)^{+}, \bar{\gamma}+\bar{\delta}\right) \\
& \leq \psi_{w}\left((a-2 t--\gamma)^{+}, \gamma+\delta\right) \\
& =\psi_{w}(a-2 t+\gamma, \gamma+\delta) \\
& =w(a-2 t+\gamma)-w(a-2 t-\delta) .
\end{aligned}
$$

Then, we get the estimate

$$
\begin{equation*}
\bar{\delta} e_{w_{0}, w}(q) \leq \delta \frac{w(a-2 t+\gamma)-w(a-2 t-\delta)}{w_{0}\left(a_{0}-2 t-\delta\right)} . \tag{3.16}
\end{equation*}
$$

By (3.15), and (3.16), we deduce

$$
\bar{q}_{+} \prec \chi_{w, w_{0}}(q)=q_{+} .
$$

The induction is completed.
(b) By hypotheses of Lemma $2,2 t_{n}+\delta_{n}<a$ for all $n \geq 0$, so we deduce that $t_{n}<0.5\left(a-\delta_{n}\right)$ for each $n \geq 0$. Then, $\left\{t_{n}\right\}$ is increasing and bounded. Thus, $\left\{t_{n}\right\}$ converges to a finite limit $t_{\infty}$. Since $t_{n+1}=t_{n}+\delta_{n}$ and $\gamma_{n+1}=\delta_{n}$, for each $n \geq 0$, we obtain $\gamma_{\infty}=\delta_{\infty}=0$, and $t_{\infty} \leq 0.5 a$.
(c) to (e) First, by (a), we have for $n, m \geq 0$

$$
\begin{align*}
\left\|x_{n+m}-x_{n}\right\| & \leq \sum_{\substack{k=n \\
k=n+m-1} x_{k+1}-x_{k} \|}^{k=n+m-1} \bar{\delta}_{k} \leq \sum_{k=n}^{k=n+m-1} \delta_{k}<\sum_{k=n}^{\infty} \delta_{k}=t_{\infty}-t_{n} \tag{3.17}
\end{align*}
$$

Hence, $\left\{x_{n}\right\}$ is a complete sequence in a Banach space, and as such it converges to $x^{\star}$. By letting $m \longrightarrow \infty$ in (3.17), we deduce for $n \geq 0$ the following estimate $\Delta_{n}=\left\|x^{\star}-x_{n}\right\| \leq t_{\infty}-t_{n}$. Moreover, by (3.12), we get

$$
\begin{aligned}
\left\|F\left(x_{n+1}\right)\right\| & \leq \overline{\delta_{n}} \psi_{w}\left(\left(w^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\gamma}_{0}-\bar{t}_{n}-\bar{t}_{n-1}\right)^{+}, \bar{\gamma}_{n}+\bar{\delta}_{n}\right)\right. \\
& \leq \delta_{n} \psi_{w}\left(a-2 t_{n}+\gamma_{n}, \gamma_{n}+\delta_{n}\right) \\
& =\delta_{n}\left(w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-2 t_{n}-\delta_{n}\right)\right),
\end{aligned}
$$

where $\delta_{n} \longrightarrow \infty$, so, $F\left(x^{\star}\right)=0$. By letting $n \longrightarrow \infty$ in equality $A_{n}\left[x_{n}, x_{n-1} ; F\right]=I$, we get $A_{\infty}\left[x^{\star}, x^{\star} ; F\right]=0$, and (c) is completed.
Substituting $A_{+}$by $A_{n}$ in 3.10, we have

$$
\begin{align*}
\left\|A_{n}\right\| & \leq \frac{1}{w_{0}\left(w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\gamma}_{0}-\bar{t}_{n}-\bar{t}_{n-1}\right)^{+}\right)}  \tag{3.18}\\
& \leq \frac{1}{w_{0}\left(a_{0}-t_{n}-t_{n-1}\right)}=\frac{1}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)}
\end{align*}
$$

Using (3.7), we have

$$
\begin{align*}
& \left\|I-A_{n}\left[x^{\star}, x^{\star} ; F\right]\right\|  \tag{3.19}\\
& \leq\left\|A_{n}\right\|\left\|\left[x_{n}, x_{n-1} ; F\right]-\left[x^{\star}, x^{\star} ; F\right]\right\| \\
& \leq\left\|A_{n}\right\| \psi_{w}\left(w^{-1}\left(\left\|\left[x_{n}, x_{n-1} ; F\right]\right\|\right),\left\|x_{n}-x^{\star}\right\|+\left\|x_{n-1}-x^{\star}\right\|\right) .
\end{align*}
$$

By Lemma 2.6, we get

$$
\begin{align*}
w^{-1}\left(\left\|\left[x_{n}, x_{n-1} ; F\right]\right\|\right) & \geq\left(w^{-1}\left(\left\|\left[x_{0}, x_{-1} ; F\right]\right\|-\left\|x_{n}-x_{0}\right\|-\left\|x_{n-1}-x_{-1}\right\|\right)^{+}\right)  \tag{3.20}\\
& \geq\left(w^{-1}\left(\left\|A_{0}\right\|-\bar{t}_{n}-\bar{t}_{n-1}-\bar{\gamma}_{0}\right)^{+}\right) \\
& \geq a-2 t_{n}+\gamma_{n}
\end{align*}
$$

Using (3.18-3.20), we obtain

$$
\begin{equation*}
\left\|I-A_{n}\left[x^{\star}, x^{\star} ; F\right]\right\| \leq \frac{w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-2 t_{\infty}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)} \tag{3.21}
\end{equation*}
$$

By the identity

$$
\begin{aligned}
x_{n+1}-x^{\star} & =x_{n}-x^{\star}-A_{n} F\left(x_{n}\right) \\
& =A_{n}\left(\left[x_{n}, x_{n-1} ; F\right]-\left[x_{n}, x^{\star} ; F\right]\right)\left(x_{n}-x^{\star}\right),
\end{aligned}
$$

we get, similarly as in (3.21),

$$
\begin{align*}
\Delta_{n+1} & \leq \Delta_{n}\left\|A_{n}\right\|\left\|\left[x_{n}, x_{n-1} ; F\right]-\left[x_{n}, x^{\star} ; F\right]\right\| \\
& \leq \Delta_{n} \frac{\psi_{w}\left(a-2 t_{n}+\gamma_{n}, \Delta_{n-1}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)} \\
& \leq \Delta_{n} \frac{\psi_{w}\left(a-2 t_{n}+\gamma_{n}, t_{\infty}-t_{n}+\gamma_{n}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)}  \tag{3.22}\\
& =\Delta_{n} \frac{w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-t_{n}-t_{\infty}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)}
\end{align*}
$$

The proof of (e) is completed.
Now we prove (d). Let $y^{\star}$ be a solution of $F(x)=0$. Then, $F_{0}\left(y^{\star}\right)=0$, where $F_{0}=A_{0} F$, and $F_{0}\left(y^{\star}\right)-F_{0}\left(x^{\star}\right)=0=\left[y^{\star}, x^{\star} ; F_{0}\right]\left(y^{\star}-x^{\star}\right)$. Then, we deduce by the Banach lemma on invertible operators [2, 4, and Proposition 2.3 that $\left[y^{\star}, x^{\star} ; F_{0}\right]=A_{0}\left[y^{\star}, x^{\star} ; F\right]$ is not invertible, and $\left\|I-A_{0}\left[y^{\star}, x^{\star} ; F\right]\right\| \geq 1$.
Using (3.19), we have

$$
\begin{aligned}
1 & \leq\left\|I-A_{0}\left[x^{\star}, x^{\star} ; F\right]\right\| \\
& \leq\left\|A_{0}\right\|\left\|\left[x_{0}, x_{-1} ; F\right]-\left[y^{\star}, x^{\star} ; F\right]\right\| \\
& \leq\left\|A_{0}\right\| \psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|\left[x_{0}, x_{-1} ; F\right]\right\|\right),\left\|y^{\star}-x_{0}\right\|+\left\|x_{-1}-x^{\star}\right\|\right) \\
& \leq\left\|A_{0}\right\| \psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}^{-1}\right\|\right),\left\|y^{\star}-x_{0}\right\|+\left\|x^{\star}-x_{0}\right\|+\left\|x_{0}-x_{-1}\right\|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|A_{0}^{-1}\right\| & \leq \psi_{w_{0}}\left(w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}, \Lambda\right)\right) \\
& = \begin{cases}\left\|A_{0}\right\|^{-1}-w_{0}\left(w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}, \Lambda\right)\right), & \text { if } \Lambda \leq w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}\right) \\
w_{0}(\Lambda), & \text { if } \Lambda \geq w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}\right)\end{cases}
\end{aligned}
$$

where

$$
\Lambda=\bar{\gamma}_{0}+t_{\infty}+\left\|y^{\star}-x_{0}\right\| .
$$

It follows that

$$
\left\|y^{\star}-x_{0}\right\| \geq w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}-\bar{\gamma}_{0}-t_{\infty}\right) \geq a_{0}-t_{\infty} \geq 0.5 a .
$$

## 4. Applications

In the special cases 1 and 2 that follow, we provide more precise estimates than [11, Section 4].

Case 1: Semi-local convergence under (2.4) and (2.5)
Let $x_{0}, x_{-1}, A_{0}, a, a_{0}$, and $q_{0}$ given, such that

$$
\begin{equation*}
\bar{q}_{0} \prec q_{0} \quad \text { and } \quad 2 t_{n}+\delta_{n}<a \quad \text { for each } n \geq 0 . \tag{4.1}
\end{equation*}
$$

Condition (4.1) guarantee the convergence of sequence ( $x_{n}, A_{n}$ ). We denote by $\mathcal{Q}_{c}$ the set

$$
\mathcal{Q}_{c}=\left\{\left(q_{0}, t_{n}, \delta_{n}\right): 4.1 \text { holds }\right\} .
$$

In this subsection $w(t)=c t, w_{0}(t)=c_{0} t$ with $c_{0} \leq c$ and $w_{1}(t)=c_{1} t$. The function $\chi_{w, w_{0}}$ defined in (3.13) by $\chi_{w, w_{0}}(q)=q_{+}=\left(t_{+}, \gamma_{+}, \delta_{+}\right)$for all $q=(t, \gamma, \delta)$, is simplified in the following form

$$
\begin{equation*}
t_{+}=t+\delta, \quad \gamma_{+}=\delta, \quad \delta_{+}=\delta \frac{\bar{c}(\gamma+\delta)}{1-\bar{c}_{0}\left(\left\|x_{0}-x_{-1}\right\|-2 t-\delta\right)} \tag{4.2}
\end{equation*}
$$

where

$$
\bar{c}=c\left\|A_{0}\right\|, \quad \bar{c}_{0}=c_{0}\left\|A_{0}\right\| \quad \text { and } \bar{c}_{1}=c_{1}\left\|A_{0}\right\| .
$$

If in the denominator of $\delta_{+}$, the function $w$ and $a$ replace $w_{0}$ and $a_{0}$, respectively, then 4.2 becomes

$$
\begin{equation*}
t_{+}=t+\delta, \quad \gamma_{+}=\delta, \quad \delta_{+}=\delta \frac{\gamma+\delta}{a-2 t-\delta} \tag{4.3}
\end{equation*}
$$

Define function $\Gamma$ for $q=(t, \gamma, \delta)$ by

$$
\begin{equation*}
\Gamma(q)=(0.5 a-t)^{2}-\delta(a-2 t+\gamma) \tag{4.4}
\end{equation*}
$$

We present now two results on simplified generator $\chi_{w, w_{0}}$ given by 4.3.
Lemma 4.1. [3, 9, 11]
(a) The function $\Gamma$ given by 4.4) is an invariant of the generator 4.3):

$$
\Gamma(q)=\Gamma\left(q_{+}\right)
$$

(b) For all $n \geq 0$,

$$
\begin{aligned}
2 t_{n}+\delta_{n}<a & \Longleftrightarrow \Gamma\left(0, \gamma_{0}, \delta_{0}\right) \geq 0 \Longleftrightarrow 4 \delta_{0}\left(a+\gamma_{0}\right) \leq a^{2} \\
& \Longleftrightarrow t_{n}=0.5 a-\delta_{n}-\sqrt{\delta_{n}\left(\gamma_{n}+\delta_{n}\right)+\Gamma\left(0, \gamma_{0}, \delta_{0}\right)} .
\end{aligned}
$$

Theorem 4.2. Suppose that (2.4), and (2.5) hold. Let $x_{-1}, x_{0}, A_{0}, a_{0}, a, \gamma_{0}$, $\delta_{0}$, be such that

$$
\begin{gathered}
\frac{\left\|A_{0}\right\|^{-1}}{c_{0}}-\left\|x_{0}-x_{-1}\right\| \geq a_{0}, \quad \frac{\left\|A_{0}\right\|^{-1}}{c}-\left\|x_{0}-x_{-1}\right\| \geq a \\
\left\|x_{0}-x_{-1}\right\| \leq \gamma_{0}, \quad\left\|A_{0} F\left(x_{0}\right)\right\| \leq \delta_{0}
\end{gathered}
$$

and

$$
\begin{equation*}
4 \delta_{0}\left(a+\gamma_{0}\right) \leq a^{2} \tag{4.5}
\end{equation*}
$$

Then, the following hold
(a) $\left(t_{n}, \gamma_{n}, \delta_{n}\right)$ generated by (4.3) started at $\left(0, \gamma_{0}, \delta_{0}\right)$ is well defined, and converges to $\left(t_{\infty}, 0,0\right)$, where,

$$
t_{\infty}=0.5\left(a-\sqrt{a^{2}-4 \delta_{0}\left(a+\gamma_{0}\right)}\right)
$$

(b) The sequence $\left(x_{n}, A_{n}\right)$ generated by $B M$ started at $\left(x_{-1}, x_{0}, A_{0}\right)$ converges to a solution $\left(x^{\star}, A_{\infty}\right)$ of the system

$$
F(x)=0 \quad \text { and } \quad A[x, x ; F]=I
$$

(c) $x^{\star}$ is the unique solution of (1.1) in $U\left(x_{0}, r\right)$, where,

$$
r=0.5\left(a+\sqrt{a^{2}-4 \delta_{0}\left(a+\gamma_{0}\right)}\right) ;
$$

(d) For each $n \geq 1$,

$$
\begin{gathered}
\left\|F\left(x_{n+1}\right)\right\| \leq c \delta_{n}\left(\gamma_{n}+\delta_{n}\right), \quad \Delta_{n} \leq \tilde{t}_{\infty}-\tilde{t}_{n} \\
\left\|x_{n}-x_{0}\right\| \leq \tilde{t}_{n}-\tilde{t}_{0} \\
\left\|I-A_{n}\left[x^{\star}, x^{\star} ; F\right]\right\| \leq p_{n}
\end{gathered}
$$

and

$$
\frac{\Delta_{n+1}}{\Delta_{n}} \leq q_{n}
$$

where

$$
\begin{gathered}
\tilde{t}_{-1}=0, \quad \tilde{t}_{0}=\left\|x_{0}-x_{-1}\right\|, \quad \tilde{t}_{1}=\tilde{t}_{0}+\left\|A_{0} F\left(x_{0}\right)\right\|, \\
\tilde{t}_{n}=\tilde{t}_{n-1}+\frac{\bar{c}\left(\tilde{t}_{n-1}-\tilde{t}_{n-3}\right)\left(\tilde{t}_{n-1}-\tilde{t}_{n-2}\right)}{1-\bar{c}_{0}\left(-\tilde{t}_{n-1}-\tilde{t}_{n-2}+\tilde{t}_{0}\right)}, \quad(n \geq 2), \\
p_{n}=\frac{\bar{c}\left(\gamma_{n}+2\left(\tilde{t}_{\infty}-\tilde{t}_{n}\right)\right)}{\bar{c}_{0}\left(a_{0}-2 \tilde{t}_{n}+\gamma_{n}\right)},
\end{gathered}
$$

and

$$
q_{n}=\frac{\bar{c}\left(\gamma_{n}+\tilde{t}_{\infty}-\tilde{t}_{n}\right)}{\bar{c}_{0}\left(a_{0}-2 \tilde{t}_{n}+\gamma_{n}\right)},
$$

with

$$
\tilde{t}_{\infty}=\lim _{n \longrightarrow \infty} \tilde{t}_{n} \leq t_{\infty}
$$

The estimates in Theorem 4.2 reduce to the corresponding ones in [11] for $c_{0}=c=c_{1}$. Otherwise, the new estimates are more precise.

## Case 2: Semi-local convergence under regular continuity of dd

Suppose that $w$ is nonlinear, and let $q_{0}=\left(t_{0}, \gamma_{0}, \delta_{0}\right)$. Define the scalar function $F\left(\cdot / q_{0}\right)$ on the sequence $\left\{t_{n}\right\}(n \geq 0)$ as follows

$$
\begin{equation*}
F\left(t_{n} / q_{0}\right)=\delta_{n} w\left(a-2 t_{n}+\gamma_{n}\right) \tag{4.6}
\end{equation*}
$$

The sequence $F\left(t_{n}\right)$ is decreasing, and consequently, the function $F$ is invertible on $\left\{t_{n}\right\}$, i.e., for each $n \geq 0$ :

$$
\begin{equation*}
t_{n}=F^{-1}\left(\delta_{n} w\left(a-2 t_{n}+\gamma_{n}\right) / q_{0}\right) \tag{4.7}
\end{equation*}
$$

We present now two results on the generator $\chi_{w, w_{0}}$ given by 3.13.
Lemma 4.3. [11]
(a) The function $F^{-1}(0 / q)$ with initial iterate $q=(t, \gamma, \delta)$ is an invariant of the generator 3.13.
(b) For all $n \geq 0$, and $q_{0}=\left(t_{0}, \gamma_{0}, \delta_{0}\right)$, we have the following equivalence

$$
2 t_{n}+\delta_{n}<a \Longleftrightarrow F^{-1}\left(0 / q_{0}\right) \leq 0.5 a
$$

Theorem 4.4. Suppose that (2.1), and (2.2) hold. Let $x_{-1}, x_{0}, A_{0}, a_{0}, a, \gamma_{0}$, $\delta_{0}$, such that

$$
\begin{gathered}
w_{0}^{-1}\left(\left\|A_{0}\right\|^{-1}\right)-\left\|x_{0}-x_{-1}\right\| \geq a_{0}, \quad w^{-1}\left(\left\|A_{0}\right\|^{-1}\right)-\left\|x_{0}-x_{-1}\right\| \geq a \\
\left\|x_{0}-x_{-1}\right\| \leq \gamma_{0}, \quad\left\|A_{0} F\left(x_{0}\right)\right\| \leq \delta_{0}
\end{gathered}
$$

and

$$
F^{-1}\left(0 /\left(0, \gamma_{0}, \delta_{0}\right)\right) \leq 0.5 a
$$

Then, the following hold
(a) $\left(t_{n}, \gamma_{n}, \delta_{n}\right)$ generated by (3.13) started at $\left(0, \gamma_{0}, \delta_{0}\right)$ is well defined, and converges to $\left(t_{\infty}, 0,0\right)$, where

$$
t_{\infty}=F^{-1}\left(0 /\left(0, \gamma_{0}, \delta_{0}\right)\right)
$$

(b) The sequence $\left(x_{n}, A_{n}\right)$ generated by $B M$ started at $\left(x_{-1}, x_{0}, A_{0}\right)$ converges to a solution $\left(x^{\star}, A_{\infty}\right)$ of the system

$$
F(x)=0 \quad \text { and } \quad A[x, x ; F]=I
$$

(c) $x^{\star}$ is the unique solution of (1.1) in $U\left(x_{0}, a_{0}-t_{\infty}\right)$;
(d) For each $n \geq 1$,

$$
\begin{aligned}
& \left\|F\left(x_{n+1}\right)\right\| \leq \delta_{n}\left(w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-2 t_{n}-\delta_{n}\right)\right) \\
& \left\|I-A_{n}\left[x^{\star}, x^{\star} ; F\right]\right\| \leq \frac{w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-2 t_{\infty}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)}
\end{aligned}
$$

and

$$
\frac{\Delta_{n+1}}{\Delta_{n}} \leq \frac{w\left(a-2 t_{n}+\gamma_{n}\right)-w\left(a-t_{n}-t_{\infty}\right)}{w_{0}\left(a_{0}-2 t_{n}+\gamma_{n}\right)}
$$

Remark 4.5. (a) The results obtained in this study reduce to the corresponding ones in [11], if equality holds in 2.3) and 2.12 . Otherwise, our results provide weaker sufficient convergence conditions, error bounds than in [11] (see also the definition of $a$ and $a_{0}$ ). Moreover, the information on the uniqueness of the solution $x^{\star}$ is more precise, since $a_{0}-t_{\infty}>a-t_{\infty}$ (see also Lemma 3.4 (b)).

As an example instead of studying the iteration in [11] corresponding to 4.3) and defined by

$$
\begin{aligned}
& s_{-1}=0, \quad s_{0}=\left\|x_{0}-x_{-1}\right\|, \quad s_{1}=s_{0}+\left\|A_{0} F\left(x_{0}\right)\right\| \\
& s_{n+1}=s_{n}+\frac{\bar{c}_{1}\left(s_{n}-s_{n-1}\right)\left(s_{n}-s_{n-2}\right)}{1-\bar{c}_{1}\left(s_{0}-s_{n}-s_{n-1}\right)}
\end{aligned}
$$

we study the more precise sequence defined by

$$
\begin{aligned}
& \alpha_{-1}=0, \quad \alpha_{0}=\left\|x_{0}-x_{-1}\right\|, \quad \alpha_{1}=\alpha_{0}+\left\|A_{0} F\left(x_{0}\right)\right\| \\
& \alpha_{n+1}=\alpha_{n}+\frac{\bar{c}\left(\alpha_{n}-\alpha_{n-1}\right)\left(\alpha_{n}-\alpha_{n-2}\right)}{1-c_{2}\left(\alpha_{0}-\alpha_{n}-\alpha_{n-1}\right)}
\end{aligned}
$$

where $c_{2}=\bar{c}$ or $c_{2}=\bar{c}_{0}$. Using (2.1), (2.2) and our idea of recurrent functions but not (2.10), we have already obtained weaker sufficient convergence conditions for many iterative methods such as Newton's, Secant, and Newton-type methods (under very general conditions [1, 2, 3, 3, 4, 5, 6, 7]). In particular, our work using regularly continuous divided differences can be found in [5].
(b) If $w(t) \leq w_{0}(t)$ for all $t \in\left[0, r_{0}\right)$ holds instead of 2.12$]$, then clearly the function $w_{0}$ (still at least as small as the function $w_{1}$ ) can replace $w$ in the preceding results.
(c) If $\Omega_{0}$ is replaced by $\Omega_{0}^{*}=\bigcup\left(x_{1}, r-\left\|A_{0} F\left(x_{0}\right)\right\|\right)$ then in Definition 2.7 a function even tighter than $w$ can be used, so, the results can be weakened even further, since $\Omega_{0}^{*} \subseteq \Omega_{0}$, and $x_{1}$ still depends on the initial data.

## Conclusion

We presented the convergence analysis of BM in order to approximate a locally unique solution of a nonlinear equation in a Hilbert space setting. Using a combination of $w$-regular continuous and $w_{0}$-center-regular continuous conditions and our new idea of restricted convergence domains, we provided a tighter semi-local convergence analysis than before [5, 8, 9, 10, 11]. Special cases are also given in this study. It is worth noticing that the new advantages are obtained under the same computational effort as before, since in practice the computation of the old function $w_{1}$ requires the computation of new functions $w_{0}$ and $w$ as special cases.

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Received by the editors March 27, 2018
First published online June 17, 2021


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