

On generalized weakly (Ricci) ϕ -symmetric Lorentzian Para Sasakian manifold

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Abstract. The present paper attempts to introduce the notion of generalized weakly ϕ -symmetric and generalized weakly Ricci ϕ -symmetric Lorentzian Para Sasakian manifold. Furthermore, we study generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetimes. In addition, the existence of a generalized weakly ϕ -symmetric Lorentzian Para Sasakian manifold is ensured by a suitable example.

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1. Introduction

Throughout the paper, we shall denote the Levi-Civita connection, Riemannian curvature tensor, Ricci tensor and Ricci operator by the symbols ∇ , R (or \bar{R}), and Q , respectively. As a mild version of local symmetry[5], Takahashi [28] started the studies on locally ϕ -symmetric manifold. Further research has been conducted to weaken such notion by many authors. For details, we refer the reader to ([10], [11], [16], [12], [7], [23], [24], [21], [27], [8], [17], [18], [19] and the references therein).

Recently, in [2], the author has introduced the concept of a generalized weakly symmetric manifold which is defined as follows:

A Riemannian (or semi-Riemannian) manifold of dimension n is said to be generalized weakly symmetric if it admits the equation

$$\begin{aligned} (\nabla_X \bar{R})(Y, U, V, Z) &= A(X)\bar{R}(Y, U, V, Z) + B(Y)\bar{R}(X, U, V, Z) \\ &+ B(U)\bar{R}(Y, X, V, Z) + D(V)\bar{R}(Y, U, X, Z) + D(Z)\bar{R}(Y, U, V, X) \\ &+ \alpha(X)\bar{G}(Y, U, V, Z) + \beta(Y)\bar{G}(X, U, V, Z) + \beta(U)\bar{G}(Y, X, V, Z) \\ (1.1) \quad &+ \gamma(V)\bar{G}(Y, U, X, Z) + \gamma(Z)\bar{G}(Y, U, V, X), \end{aligned}$$

where

$$(1.2) \quad \bar{G}(Y, U, V, W) = g(U, V)g(Y, W) - g(Y, V)g(U, W)$$

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for any vector fields X, Y and U and the 1-forms $A(X) = g(X, \pi_1)$, $B(X) = g(X, \pi_2)$, $D(X) = g(X, \varrho)$, $\alpha(X) = g(X, \delta_1)$, $\beta(X) = g(X, \delta_2)$ and $\gamma(X) = g(X, \sigma)$. The relation (1.1) can also be written as

$$\begin{aligned} & (\nabla_X R)(Y, U)V \\ &= A(X)R(Y, U)V + B(Y)R(X, U)V + B(U)R(Y, X)V + D(V)R(Y, U)X \\ & \quad + g(R(Y, U)V, X)\varrho + \alpha(X)G(Y, U)V + \beta(Y)G(X, U)V \\ (1.3) \quad & + \beta(U)G(Y, X)V + \gamma(V)G(Y, U)X + g(G(Y, U)V, X)\sigma. \end{aligned}$$

Analogously, a semi-Riemannian (or Riemannian) manifold (M^n, g) is said to be generalized weakly Ricci-symmetric, if it satisfies the condition

$$\begin{aligned} (\nabla_X S)(U, V) &= A^*(X)S(U, V) + B^*(U)S(V, X) + D^*(V)S(U, X) \\ (1.4) \quad & + \alpha^*(X)g(U, V) + \beta^*(U)g(V, X) + \gamma^*(V)g(U, X) \end{aligned}$$

which can also be expressed as

$$\begin{aligned} (\nabla_X Q)(U) &= A^*(X)QU + B^*(U)QX + S(U, X)\varrho^* \\ (1.5) \quad & + \alpha^*(X)U + \beta^*(U)X + g(U, X)\sigma^* \end{aligned}$$

for any vector fields X, U and V and the 1-forms $A^*(X) = g(X, \pi_1^*)$, $B^*(X) = g(X, \pi_2^*)$, $D^*(X) = g(X, \varrho^*)$, $\alpha^*(X) = g(X, \delta_1^*)$, $\beta^*(X) = g(X, \delta_2^*)$ and $\gamma(X) = g(X, \sigma^*)$.

Recently, Hui [10] studied ϕ -pseudo symmetric and ϕ -pseudo Ricci symmetric Kenmotsu manifolds. In tune with [10], in this paper we would like to introduce the notion of a generalized weakly ϕ -symmetric manifold and a generalized weakly Ricci ϕ -symmetric manifold.

A Lorentzian Para-Sasakian manifold is said to be generalized weakly ϕ -symmetric if R admits the equation

$$\begin{aligned} & \phi^2((\nabla_X R)(Y, U)V) \\ &= A(X)R(Y, U)V + B(Y)R(X, U)V + B(U)\bar{R}(Y, X)V + D(V)\bar{R}(Y, U)X \\ & \quad + g(R(Y, U)V, X)\varrho + \alpha(X)G(Y, U)V + \beta(Y)G(X, U)V \\ (1.6) \quad & + \beta(U)G(Y, X)V + \gamma(V)G(Y, U)X + g(G(Y, U)V, X)\sigma. \end{aligned}$$

The beauty of such generalized weakly ϕ -symmetric manifold is that it has the flavour of a

- (i) locally ϕ -symmetric space [28] (for $A = B = D = 0 = \alpha = \beta = \gamma$),
- (ii) locally ϕ -recurrent space [8] (for $A \neq 0, B = D = \alpha = \beta = \gamma = 0$),
- (iii) generalized ϕ -recurrent space in the sense of [9] (for $A \neq 0, \alpha \neq 0, B = D = \beta = \gamma = 0$),
- (iv) quasi ϕ -recurrent space in the sense of [20] ($A \neq 0, B = D = 0, \alpha \neq 0, \beta = \gamma = (\beta^* - \gamma^*)\alpha$),
- (v) pseudo ϕ -symmetric space in the sense of [10] (for $\frac{A}{2} = B = D = H \neq 0, \alpha = \beta = \gamma = 0$),

(vi) generalized pseudo ϕ -symmetric space in the sense of [1] (for $\frac{A}{2} = B = D = H_1 \neq 0, \frac{\alpha}{2} = \beta = \gamma = H_2 \neq 0$),

(vii) semi-pseudo ϕ -symmetric space in the sense of [30] ($A = \alpha = \beta = \gamma = 0, B = D \neq 0$),

(viii) generalized semi-pseudo ϕ -symmetric space in the sense of [3] ($A = 0 = \alpha, B = D \neq 0, \beta = \gamma \neq 0$),

(ix) almost pseudo ϕ -symmetric space in the sense of [6] (for $A = H_1 + K_1, B = D = H_1 \neq 0$ and $\alpha = \beta = \gamma = 0$),

(x) almost generalized pseudo ϕ -symmetric space in the sense of [3] ($A = H_1 + K_1, B = D = H_1 \neq 0, \alpha = H_2 + K_2, \beta = \gamma = H_2 \neq 0$),

(xi) weakly ϕ -symmetric space in the sense of [29] (for $A, B, D \neq 0, \alpha = \beta = \gamma = 0$).

A Lorentzian Para-Sasakian manifold is said to be generalized weakly Ricci ϕ -symmetric if Q admits the following

$$\begin{aligned}
 & \phi^2((\nabla_X Q)(U)) \\
 = & A^*(X)QU + B^*(U)QX + S(U, X)\rho^* \\
 (1.7) \quad & + \alpha^*(X)U + \beta^*(U)X + g(U, X)\sigma^*.
 \end{aligned}$$

In 1989, Matsumoto [13] started the studies of Lorentzian Para-Sasakian manifolds which were also independently defined by Mihai and Rosca [15]. Matsumoto, Mihai and Rosca [14] gave a five dimensional example of Lorentzian Para-Sasakian manifold.

We represent our paper as follows: Section 2, is concerned with some known results on Lorentzian Para Sasakian manifolds. In Section 3, we study generalized weakly ϕ -symmetric Lorentzian Para Sasakian manifolds. We observe that a generalized weakly ϕ -symmetric Lorentzian Para Sasakian manifold is η -Einstein whereas Lorentzian Para Sasakian manifold can not be η -Einstein for each of the curvature restrictions: (i) ϕ -symmetric, (ii) ϕ -recurrent, (iii) generalized ϕ -recurrent, (iv) pseudo ϕ -symmetric, (v) almost pseudo ϕ -symmetric, (vi) generalized pseudo ϕ -symmetric and (vii) generalized almost pseudo ϕ -symmetric. It is also found that there does not exist a Lorentzian Para-Sasakian manifold which is (i) ϕ -recurrent, (ii) generalized ϕ -recurrent provided the 1-forms are collinear, (iii) pseudo ϕ -symmetric, (iv) generalized semi-pseudo ϕ -symmetric provided the 1-forms are collinear, (v) generalized almost pseudo ϕ -symmetric provided the 1-forms are collinear. Section 4 deals with generalized weakly Ricci ϕ -symmetric Lorentzian Para Sasakian manifolds. Keeping in tune with Shaikh, Yoon and Hui [25], in Section 5 we study a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime and obtain some interesting results. Finally, we have constructed an example of a generalized weakly ϕ -symmetric Lorentzian Para Sasakian manifold.

2. Properties of a Lorentzian Para Sasakian manifold

Let M be an n -dimensional differential manifold endowed with a $(1, 1)$ tensor field ϕ , a vector field ξ , a 1-form η , and a Lorentzian metric g of type $(0, 2)$

such that for each point $a \in M$, the tensor $g_a : T_aM \times T_aM \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, \dots, +)$, where T_aM denotes the tangent space of M at a and \mathbb{R} is the real number space which satisfies

$$(2.1) \quad \phi^2 = I + \eta \otimes \xi,$$

$$(2.2) \quad \eta(\xi) = -1,$$

$$(2.3) \quad g(X, \xi) = \eta(X),$$

$$(2.4) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for all vector fields X, Y on M^n . Then, the structure (ϕ, ξ, η, g) is called Lorentzian almost para contact structure and the manifold with the structure (ϕ, ξ, η, g) is called a Lorentzian almost para contact manifold. In a Lorentzian almost para contact manifold M , the following relations hold [13]:

$$(2.5) \quad \phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$(2.6) \quad g(\phi X, Y) = g(X, \phi Y).$$

If we put

$$(2.7) \quad \Omega(X, Y) = g(\phi X, Y) = g(X, \phi Y)$$

for any vector fields X and Y then the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field.

A Lorentzian almost para contact manifold M endowed with the structure (ϕ, ξ, η, g) is called a Lorentzian Para-Sasakian manifold if

$$(2.8) \quad (\nabla_X \phi)Y - g(\phi X, \phi Y)\xi = \eta(Y)\phi^2 X,$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g . In a Lorentzian Para-Sasakian manifold M with the structure (ϕ, ξ, η, g) , it is easily seen that ([13], [4], [22])

$$(2.9) \quad \nabla_X \xi = \phi X,$$

$$(2.10) \quad (\nabla_X \eta)Y = g(X, \phi Y) = \Omega(X, Y) = (\nabla_Y \eta)X,$$

$$(2.11) \quad S(X, \xi) = (n - 1)\eta(X), \quad Q\xi = (n - 1)\xi,$$

$$(2.12) \quad R(\xi, X)Y + \eta(Y)X = g(X, Y)\xi,$$

$$(2.13) \quad R(Y, U)\xi + \eta(Y)U = \eta(U)Y,$$

$$(2.14) \quad \begin{aligned} R(Y, U)\phi X &= \phi R(Y, U)X + g(U, X)\phi Y - g(Y, X)\phi U + g(\phi Y, X)U \\ &\quad - g(\phi U, X)Y + 2[g(\phi Y, X)\eta(U) - g(\phi U, X)\eta(Y)]\xi \\ &\quad + 2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X), \end{aligned}$$

$$(2.15) \quad \begin{aligned} R(X, Y)Z &= \phi R(X, Y)\phi Z + g(X, Z)Y - g(Y, Z)X \\ &\quad + \Omega(Y, Z)\phi X - \Omega(X, Z)\phi Y + 2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi \\ &\quad + 2[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned}$$

$$(2.16) \quad \begin{aligned} (\nabla_X R)(Y, U)\xi &= 2[g(\phi U, X)Y - g(\phi Y, X)U] - \phi R(Y, U)X \\ &\quad + g(Y, X)\phi U - g(U, X)\phi Y - 2[g(\phi Y, X)\eta(U) \\ &\quad - g(\phi U, X)\eta(Y)]\xi - 2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X). \end{aligned}$$

for all vector fields X, Y, U and Z on M^n .

3. Generalized weakly ϕ -symmetric Lorentzian Para-Sasakian manifolds

In this section we consider a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian manifold. Then by the virtue of (2.1), the equation (1.6) takes the following form

$$(3.1) \quad \begin{aligned} &(\nabla_X R)(Y, U)V + \eta((\nabla_X R)(Y, U)V)\xi \\ &= A(X)R(Y, U)V + B(Y)R(X, U)V \\ &\quad + B(U)R(Y, X)V + D(V)R(Y, U)X \\ &\quad + g(R(Y, U)V, X)\rho + \alpha(X)G(Y, U)V \\ &\quad + \beta(Y)G(X, U)V + \beta(U)G(Y, X)V \\ &\quad + \gamma(V)G(Y, U)X + g(G(Y, U)V, X)\sigma. \end{aligned}$$

From which we get

$$(3.2) \quad \begin{aligned} &g((\nabla_X R)(Y, U)V, W) + \eta((\nabla_X R)(Y, U)V)\eta(W) \\ &= A(X)g(R(Y, U)V, W) + B(Y)g(R(X, U)V, W) \\ &\quad + B(U)g(R(Y, X)V, W) + D(V)g(R(Y, U)X, W) \\ &\quad + D(W)g(R(Y, U)V, X) + \alpha(X)g(G(Y, U)V, W) \\ &\quad + \beta(Y)g(G(X, U)V, W) + \beta(U)g(G(Y, X)V, W) \\ &\quad + \gamma(V)g(G(Y, U)X, W) + \gamma(W)g(G(Y, U)V, X). \end{aligned}$$

Now, taking an orthonormal frame field and then contracting (3.2) over Y and W , we obtain

$$\begin{aligned}
 & (\nabla_X S)(U, V) + (\nabla_X \bar{R})(\xi, U, V, \xi) \\
 &= A(X)S(U, V) + B(R(X, U)V) + B(U)S(X, V) + D(V)S(U, X) \\
 &\quad + D(R(X, V)U) + (n-1)\alpha(X)g(U, V) + \beta(G(X, U)V) \\
 (3.3) \quad & + (n-1)\beta(U)g(X, V) + (n-1)\gamma(V)g(U, X) + \gamma(G(X, V)U).
 \end{aligned}$$

Using (2.12), (2.5) and (2.9), we obtain

$$(3.4) \quad (\nabla_X R)(\xi, U, V, \xi) = 0.$$

In view of (3.3) and (3.4), we have

$$\begin{aligned}
 & (\nabla_X S)(U, V) \\
 &= A(X)S(U, V) + B(U)S(X, V) + D(V)S(U, X) \\
 &\quad + (n-1)\{\alpha(X)g(U, V) + \beta(U)g(X, V) + \gamma(V)g(U, X)\} \\
 (3.5) \quad & + B(R(X, U)V) + D(R(X, V)U) + \beta(G(X, U)V) + \gamma(G(X, V)U).
 \end{aligned}$$

Theorem 3.1. *Every generalized weakly ϕ -symmetric Lorentzian Para Sasakian manifold is generalized weakly Ricci-symmetric, provided that $D(\xi) + \gamma(\xi) = 0$.*

Proof. From, (3.5) it is obvious that a generalized weakly ϕ -symmetric Lorentzian Para Sasakian manifold is generalized weakly Ricci-symmetric if

$$(3.6) \quad B(R(X, U)V) + D(R(X, V)U) + \beta(G(X, U)V) + \gamma(G(X, V)U) = 0.$$

From, (3.6), one can easily deduce that $D(\xi) + \gamma(\xi) = 0$. □

Now, setting $V = \xi$ in (3.5) we get

$$\begin{aligned}
 & (\nabla_X S)(U, \xi) \\
 &= (n-1)A(X)\eta(U) + (n-2)B(U)\eta(X) + D(\xi)S(U, X) \\
 &\quad + (n-1)\alpha(X)\eta(U) + (n-2)\beta(U)\eta(X) + (n-1)\gamma(\xi)g(U, X) \\
 &\quad + B(X)\eta(U) + D(X)\eta(U) - D(\xi)g(U, X) \\
 (3.7) \quad & + \beta(X)\eta(U) + \gamma(X)\eta(U) - \gamma(\xi)g(U, X).
 \end{aligned}$$

Again, from the relation

$$(\nabla_X S)(U, V) = \nabla_X S(U, V) - S(\nabla_X U, V) - S(U, \nabla_X V)$$

we get

$$(3.8) \quad (\nabla_X S)(U, \xi) = (n-1)g(\phi X, U) - S(\phi X, U).$$

Using (3.8) in (3.7) we have

$$\begin{aligned}
 & (n-1)g(\phi X, U) - S(\phi X, U) \\
 &= (n-1)A(X)\eta(U) + (n-2)B(U)\eta(X) + D(\xi)S(U, X) \\
 & \quad + (n-1)\alpha(X)\eta(U) + (n-2)\beta(U)\eta(X) + (n-1)\gamma(\xi)g(U, X) \\
 & \quad + B(X)\eta(U) + D(X)\eta(U) - D(\xi)g(U, X) \\
 (3.9) \quad & + \beta(X)\eta(U) + \gamma(X)\eta(U) - \gamma(\xi)g(U, X).
 \end{aligned}$$

Plugging $U = \xi$, $X = \xi$ and $U = X = \xi$ in succession, we obtain from (3.9) that

$$\begin{aligned}
 (3.10) \quad & (n-1)\{A(X) + \alpha(X)\} + B(X) + D(X) + \beta(X) + \gamma(X) \\
 &= (n-2)\eta(X)[B(\xi) + D(\xi) + \beta(\xi) + \gamma(\xi)],
 \end{aligned}$$

$$\begin{aligned}
 & (n-2)[B(U) + \beta(U)] \\
 &= \eta(U)[(n-1)\{A(\xi) + \alpha(\xi) + D(\xi) + \gamma(\xi)\} \\
 (3.11) \quad & + B(\xi) + \beta(\xi)],
 \end{aligned}$$

and

$$(3.12) \quad A(\xi) + B(\xi) + D(\xi) + \alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0,$$

respectively. From, (3.12) we infer that

Theorem 3.2. *In a generalized pseudo ϕ -symmetric Lorentzian Para-Sasakian manifold the vector field associate to the 1-form is perpendicular to the characteristic vector field ξ .*

Replacing U by X in (3.11), we get

$$\begin{aligned}
 (3.13) \quad & (n-2)[B(X) + \beta(X)] \\
 &= \eta(X)[(n-1)\{A(\xi) + \alpha(\xi) + D(\xi) + \gamma(\xi)\} + B(\xi) + \beta(\xi)].
 \end{aligned}$$

Now, by the virtue of (3.10), (3.13) and (3.12) we obtain

$$\begin{aligned}
 (3.14) \quad & (n-1)\{A(X) + B(X) + \alpha(X) + \beta(X)\} + D(X) + \gamma(X) \\
 &= (n-2)[D(\xi) + \gamma(\xi)]\eta(X).
 \end{aligned}$$

Now using (3.10), (3.11) and (3.12) in (3.9) we get

$$\begin{aligned}
 (3.15) \quad & (n-1)g(\phi X, U) - S(\phi X, U) \\
 &= D(\xi)S(U, X) + \{(n-2) - D(\xi)\}g(U, X) \\
 & + (n-2)[D(\xi) + \gamma(\xi)]\eta(U)\eta(X).
 \end{aligned}$$

Replacing X by ϕX in (3.15) we have

$$\begin{aligned}
 (3.16) \quad & (n-1)g(X, U) - S(X, U) \\
 &= D(\xi)S(\phi X, U) + \{(n-2) - D(\xi)\}g(\phi X, U).
 \end{aligned}$$

Using (3.16) in (3.15) we obtain

$$\begin{aligned}
 & [(n-1) + D(\xi)\{(n-2) - D(\xi)\}]g(X, U) \\
 & = [1 - D(\xi)D(\xi)]S(X, U) + \{(n-2) + (n-2)D(\xi)\}g(\phi X, U) \\
 (3.17) \quad & -(n-2)D(\xi)[D(\xi) + \gamma(\xi)]\eta(U)\eta(X).
 \end{aligned}$$

This leads to the following:

Theorem 3.3. *In a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian manifold the Ricci tensor S satisfies the relation (3.17).*

Again, setting $U = \xi$ in (3.5) we get

$$\begin{aligned}
 & (\nabla_X S)(V, \xi) \\
 & = (n-1)A(X)\eta(V) + B(\xi)S(V, X) + (n-2)D(V)\eta(X) \\
 & \quad + (n-1)\alpha(X)\eta(V) + (n-2)\gamma(V)\eta(X) + (n-1)\beta(\xi)g(V, X) \\
 & \quad + D(X)\eta(V) + B(X)\eta(V) - B(\xi)g(V, X) \\
 (3.18) \quad & + \gamma(X)\eta(V) + \beta(X)\eta(V) - \beta(\xi)g(V, X).
 \end{aligned}$$

Replacing U by V in (3.7) then (3.18) yields

$$\begin{aligned}
 & [B(\xi) - D(\xi)]S(V, X) \\
 & = [(n-2)\gamma(\xi) - D(\xi) - (n-2)\beta(\xi) + B(\xi)]g(V, X) \\
 (3.19) \quad & + (n-2)[B(V) + \beta(V) - D(V) - \gamma(V)]\eta(X).
 \end{aligned}$$

Now, putting $X = \xi$ in (3.19) we get

$$\begin{aligned}
 & B(V) + \beta(V) - D(V) - \gamma(V) \\
 (3.20) \quad & = [D(\xi) - B(\xi) + \gamma(\xi) - \beta(\xi)]\eta(V).
 \end{aligned}$$

Using (3.20) in (3.19) we get

$$(3.21) \quad S(V, X) = L_1g(V, X) + L_2\eta(V)\eta(X),$$

where $L_1 = \left(\frac{(n-2)\{\gamma(\xi) - \beta(\xi)\} - D(\xi) + B(\xi)}{B(\xi) - D(\xi)} \right)$ and $L_2 = \left(\frac{(n-2)[D(\xi) - B(\xi) + \gamma(\xi) - \beta(\xi)]}{B(\xi) - D(\xi)} \right)$.

This leads to the following:

Theorem 3.4. *Every generalized weakly ϕ -symmetric Lorentzian Para-Sasakian manifold is η -Einstein, provided that $B(\xi) \neq D(\xi)$.*

Theorem 3.5. *Let (M^n, g) be a Lorentzian Para-Sasakian manifold. Then M can not be η -Einstein for each of the curvature restrictions: (i) ϕ -symmetric, (ii) ϕ -recurrent, (iii) generalized ϕ -recurrent, (iv) pseudo ϕ -symmetric, (v) almost pseudo ϕ -symmetric, (vi) generalized pseudo ϕ -symmetric and (vii) generalized almost pseudo ϕ -symmetric.*

Using (3.8) in (3.18) we get

$$\begin{aligned}
 & (n-1)g(\phi X, V) - S(\phi X, V) \\
 &= (n-1)A(X)\eta(V) + B(\xi)S(V, X) + (n-2)D(V)\eta(X) \\
 &\quad + (n-1)\alpha(X)\eta(V) + (n-2)\gamma(V)\eta(X) + (n-1)\beta(\xi)g(V, X) \\
 &\quad + D(X)\eta(V) + B(X)\eta(V) - B(\xi)g(V, X) \\
 (3.22) \quad & + \gamma(X)\eta(V) + \beta(X)\eta(V) - \beta(\xi)g(V, X).
 \end{aligned}$$

Now, setting $X = \xi$ in (3.22) and using (3.12), we get

$$(3.23) \quad [D(\xi) + \gamma(\xi)]\eta(V) = -[D(V) + \gamma(V)].$$

Using (3.21) in (3.14), we obtain

$$(3.24) \quad A(X) + B(X) + D(X) + \alpha(X) + \beta(X) + \gamma(X) = 0$$

Next, in view of $\alpha = \beta = \gamma = 0$, the relation (3.24) yields

$$(3.25) \quad A(X) + B(X) + D(X) = 0.$$

This motivates us to state

Theorem 3.6. *In a weakly ϕ -symmetric Lorentzian Para-Sasakian manifold $(M^n, g)(n > 2)$, the sum of the associated 1-forms is given by (3.25).*

Theorem 3.7. *There does not exist an Lorentzian Para-Sasakian manifold which is*

- (i) ϕ -recurrent,
- (ii) generalized ϕ -recurrent, provided the 1-forms are collinear,
- (iii) pseudo ϕ -symmetric,
- (iv) generalized semi-pseudo ϕ -symmetric, provided the 1-forms are collinear,
- (v) generalized almost pseudo ϕ -symmetric, provided the 1-forms are collinear.

4. Generalized weakly Ricci ϕ -symmetric Lorentzian Para-Sasakian manifolds

In this section we consider a generalized weakly Ricci ϕ -symmetric Lorentzian Para-Sasakian manifold. Then by the virtue of (2.1), (1.7) yields

$$\begin{aligned}
 & (\nabla_X Q)(U) + \eta((\nabla_X Q)(U))\xi \\
 &= A^*(X)QU + B^*(U)QX + S(U, X)\varrho^* \\
 (4.1) \quad & + \alpha^*(X)U + \beta^*(U)X + g(U, X)\sigma^*.
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 & g(\nabla_X Q(U), V) - S(\nabla_X U, V) + \eta((\nabla_X Q)(U))\eta(V) \\
 &= A^*(X)S(U, V) + B^*(U)S(V, X) + D^*(V)S(U, X) \\
 (4.2) \quad & + \alpha^*(X)g(U, V) + \beta^*(U)g(V, X) + \gamma^*(V)g(U, X).
 \end{aligned}$$

Putting $U = \xi$ in (4.2) and using (2.9),(2.11) we get

$$(4.3) \quad \begin{aligned} & (n-1)g(\phi X, V) - S(\phi X, V) \\ &= A^*(X)(n-1)\eta(V) + B^*(\xi)S(V, X) + D^*(V)(n-1)\eta(X) \\ & \quad + \alpha^*(X)\eta(V) + \beta^*(\xi)g(V, X) + \gamma^*(V)\eta(X). \end{aligned}$$

Setting $X = V = \xi$, $X = \xi$ and $V = \xi$ successively in (4.3), we get

$$(4.4) \quad (n-1)\{A^*(\xi) + B^*(\xi) + D^*(\xi)\} + \alpha^*(\xi) + \beta^*(\xi) + \gamma^*(\xi) = 0,$$

$$(4.5) \quad (n-1)D^*(V) + \gamma^*(V) = (n-1)\{A^*(\xi) + B^*(\xi)\} + \alpha^*(\xi) + \beta^*(\xi)$$

and

$$(4.6) \quad (n-1)A^*(X) + \alpha^*(X) = (n-1)\{B^*(\xi) + D^*(\xi)\} + \beta^*(\xi) + \gamma^*(\xi),$$

respectively.

Next using (4.4), (4.5) and (4.6) in (4.3), we get

$$(4.7) \quad \begin{aligned} & (n-1)g(\phi X, V) - S(\phi X, V) \\ &= B^*(\xi)S(V, X) + \beta^*(\xi)g(V, X) + [(n-1)B^*(\xi) + \beta^*(\xi)]\eta(X)\eta(V). \end{aligned}$$

Replacing X by ϕX in (4.7) and then using (4.7) we obtain

$$(4.8) \quad \begin{aligned} & \{B^*(\xi)B^*(\xi) - 1\}S(X, V) \\ &= \{(n-1)B^*(\xi) + \beta^*(\xi)\}g(\phi X, V) - \{B^*(\xi)\beta^*(\xi) + 1\}g(X, V) \\ & \quad - B^*(\xi)[(n-1)B^*(\xi) + \beta^*(\xi)]\eta(X)\eta(V). \end{aligned}$$

Thus we infer

Theorem 4.1. *In a generalized weakly Ricci ϕ -symmetric Lorentzian Para-Sasakian manifold, the Ricci tensor S takes the form (4.8).*

If we put $(n-1)B^*(\xi) = -\beta^*(\xi)$ in (4.8), then we get

$$S(X, V) = \left(\frac{B^*(\xi)\beta^*(\xi) + 1}{1 - B^*(\xi)B^*(\xi)} \right) g(X, V),$$

which implies that the manifold under consideration is Einstein.

Corollary 4.2. *Every generalized weakly Ricci ϕ -symmetric Lorentzian Para-Sasakian manifold is an Einstein manifold, provided $(n-1)B^*(\xi) = -\beta^*(\xi)$.*

5. Generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime

We now assume that the matter distribution of a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime is a perfect fluid. Then the Einstein's field equation without cosmological constant is given by

$$S(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y)$$

for all vector fields X, Y , where S is the Ricci tensor of type $(0, 2)$, r is the scalar curvature, k is the gravitational constant and T is the energy-momentum tensor of type $(0, 2)$. In a perfect fluid spacetime, the energy-momentum tensor admits the following

$$T(X, Y) = pg(x, y) + (\sigma + p)\eta(X)\eta(Y);$$

where σ and ρ are, respectively, the energy density and the isotropic pressure, and ξ denotes the flow vector field of the fluid.

Definition 5.1. Ricci tensor S of a Lorentzian Para-Sasakian manifold (M^n, g) is named cyclic parallel if

$$(\nabla_X S)(U, V) + (\nabla_U S)(V, X) + (\nabla_V S)(X, U) = 0$$

Analogously with the work of [26] and the equation (3.21) we infer

Theorem 5.2. ([26], Theorem 2.1. Page-310) *In a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime with cyclic parallel Ricci tensor, the associated scalars L_1 and L_2 are constants.*

Theorem 5.3. ([26], Theorem 2.2. Page-311) *In a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime with cyclic parallel Ricci tensor, the energy-momentum tensor is cyclic parallel.*

Theorem 5.4. ([26], Theorem 2.3. Page-311) *In a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime with constant associated scalars L_1 and L_2 , if the energy-momentum tensor is cyclic parallel, then the Ricci tensor is cyclic parallel.*

Theorem 5.5. ([26], Theorem 3.1. Page-314) *If a perfect fluid generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime obeys Einstein equation without cosmological constant and the square of the length of the Ricci operator is $\frac{1}{3}(\frac{1}{r^2})$, then the spacetime can not contain pure matter. Also in such a spacetime without pure matter the pressure of the fluid is positive or negative according as $L_1 < \frac{L_2}{4}$ or $L_1 > \frac{L_2}{4}$.*

Definition 5.6. The Ricci tensor of a Lorentzian Para-Sasakian spacetime is said to be of Codazzi type if

$$(\nabla_X S)(U, V) = (\nabla_U S)(X, V).$$

Theorem 5.7. ([26], Theorem 4.1., Page-315) *In a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime, the energy-momentum tensor is of Codazzi type if and only if its Ricci tensor is of Codazzi type.*

Theorem 5.8. ([26], Theorem 4.2., Page-315) *If the energy-momentum tensor of a perfect fluid in a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then the integral curves of the flow vector field are geodesics.*

Theorem 5.9. ([26], Theorem 4.3., Page-318) *If the energy-momentum tensor of a perfect fluid generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then both the energy density and isotropic pressure of the fluid are constants over a hypersurface orthogonal to ξ .*

Theorem 5.10. ([26], Theorem 4.4., Page-318) *In a perfect fluid generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime with Codazzi type energy-momentum tensor, the fluid has vanishing vorticity and vanishing shear.*

Theorem 5.11. ([26], Theorem 4.5., Page-319) *If the energy-momentum tensor of a perfect fluid generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.*

Theorem 5.12. ([26], Theorem 4.6., Page-319) *If a perfect fluid generalized weakly ϕ -symmetric Lorentzian Para-Sasakian spacetime with Codazzi type energy-momentum tensor admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N.*

6. Example of a generalized weakly ϕ -symmetric Lorentzian Para-Sasakian manifold

(see [4], p-286-287) Let $M^3(\phi, \xi, \eta, g)$ be a Lorentzian Para-Sasakian manifold with a ϕ -basis

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = \phi e_1 = e^{z-\alpha x} \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z},$$

where α is non-zero constant. Then from Koszul's formula for Lorentzian metric g , we can obtain the Levi-Civita connection as follows

$$\begin{aligned} \nabla_{e_1} e_3 &= e_2, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_1 &= -e_3, \\ \nabla_{e_2} e_3 &= e_1, & \nabla_{e_2} e_2 &= \alpha e^z e_3, & \nabla_{e_2} e_1 &= \alpha e^z e_2, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_1 &= 0. \end{aligned}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor R (up to symmetry and skew-symmetry)

$$\begin{aligned} R(e_1, e_2)e_1 &= -(1 - \alpha^2 e^{2z})e_2, \\ R(e_1, e_2)e_2 &= (1 - \alpha^2 e^{2z})e_1, \\ R(e_1, e_3)e_1 &= -e_3, R(e_1, e_3)e_3 = e_1, \\ R(e_2, e_3)e_2 &= -e_3, R(e_2, e_3)e_3 = e_2. \end{aligned}$$

Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $X, Y, U, V \in \chi(M)$ can be written as

$$X = \sum_1^3 a_i e_i, \quad Y = \sum_1^3 b_i e_i, \quad Z = \sum_1^3 c_i e_i.$$

This implies that

$$R(X, Y)Z = le_1 + me_2 + ne_3,$$

where

$$\begin{aligned} l &= (a_1 b_3 - a_3 b_1) c_3 + c_2 (a_1 b_2 - a_2 b_1) (1 - \alpha^2 e^{2z}), \\ m &= (a_2 b_3 - a_3 b_2) c_3 - c_1 (a_1 b_2 - a_2 b_1) (1 - \alpha^2 e^{2z}), \\ n &= -(a_1 b_3 - a_3 b_1) c_1 - c_2 (a_2 b_3 - a_3 b_2). \end{aligned}$$

Also,

$$\begin{aligned} R(e_1, Y)Z &= \{(1 - \alpha^2 e^{2z}) b_2 c_2 + b_3 c_3\} e_1 - b_2 c_1 (1 - \alpha^2 e^{2z}) e_2 - b_3 c_1 e_3, \\ R(e_2, Y)Z &= -b_1 c_2 (1 - \alpha^2 e^{2z}) e_1 + \{(1 - \alpha^2 e^{2z}) b_1 c_1 - b_3 c_3\} e_2 - b_3 c_2 e_3, \\ R(e_3, Y)Z &= -b_1 c_3 e_1 - b_2 c_3 e_2 + (b_1 c_1 + b_2 c_2) e_3, \\ R(X, e_1)Z &= -\{a_3 c_3 + (1 - \alpha^2 e^{2z}) a_2 c_2\} e_1 + a_2 c_1 (1 - \alpha^2 e^{2z}) e_2 + a_3 c_1 e_3, \\ R(X, e_2)Z &= a_1 c_2 (1 - \alpha^2 e^{2z}) e_1 + \{a_3 c_3 - (1 - \alpha^2 e^{2z}) a_1 c_1\} e_2 + a_3 c_2 e_3, \\ R(X, e_3)Z &= a_1 c_3 e_1 + a_2 c_3 e_2 - (a_1 c_1 + a_2 c_2) e_3, \\ R(X, Y) e_1 &= -(a_1 b_2 - a_2 b_1) (1 - \alpha^2 e^{2z}) e_2 - (a_1 b_3 - a_3 b_1) e_3, \\ R(X, Y) e_2 &= (a_1 b_2 - a_2 b_1) (1 - \alpha^2 e^{2z}) e_1 - (a_2 b_3 - a_3 b_2) e_3, \\ R(X, Y) e_3 &= (a_3 b_1 - a_1 b_3) e_1 + (a_3 b_2 - a_2 b_3) e_2, \end{aligned}$$

Again,

$$G(X, Y)Z = pe_1 + qe_2 + re_3,$$

where

$$\begin{aligned} p &= (a_1 b_2 - a_2 b_1) c_2 + (a_3 b_1 - a_1 b_3) c_3, \\ q &= (a_2 b_1 - a_1 b_2) c_1 + (a_3 b_2 - a_2 b_3) c_3, \\ r &= (a_3 b_1 - a_1 b_3) c_1 + (a_3 b_2 - a_2 b_3) c_2. \end{aligned}$$

Also, we have

$$\begin{aligned} G(e_1, Y)Z &= (b_2 c_2 - b_3 c_3) e_1 - b_2 c_1 e_2 - b_3 c_1 e_3, \\ G(e_2, Y)Z &= -b_1 c_2 e_1 + (b_1 c_1 - b_3 c_3) e_2 - b_3 c_2 e_3, \\ G(e_3, Y)Z &= b_1 c_3 e_1 + b_2 c_3 e_2 - (b_1 c_1 + b_2 c_2) e_3, \\ G(X, e_1)Z &= -(a_2 c_2 - a_3 c_3) e_1 + a_2 c_1 e_2 + a_3 c_1 e_3, \\ G(X, e_2)Z &= a_1 c_2 e_1 - (a_1 c_1 - a_3 c_3) e_2 + a_3 c_2 e_3, \\ G(X, e_3)Z &= a_1 c_3 e_1 + a_2 c_3 e_2 - (a_1 c_1 + a_2 c_2) e_3, \\ G(X, Y) e_1 &= -(a_1 b_2 - a_2 b_1) e_2 + (a_3 b_1 - a_1 b_3) e_3, \\ G(X, Y) e_2 &= (a_1 b_2 - a_2 b_1) e_1 + (a_3 b_2 - a_2 b_3) e_3, \\ G(X, Y) e_3 &= (a_3 b_1 - a_1 b_3) e_1 + (a_3 b_2 - a_2 b_3) e_2 \end{aligned}$$

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

$$\begin{aligned}
(\nabla_{e_1} R)(X, Y)Z &= -le_3 + ne_2 + a_1 R(e_3, Y)Z - a_3 R(e_2, Y)Z \\
&+ b_1 R(X, e_3)Z - b_3 R(X, e_2)Z + c_1 R(X, Y)e_3 - c_3 R(X, Y)e_2 \\
&= -le_3 + ne_2 + a_1 \{b_1 c_3 e_1 + b_2 c_3 e_2 - (b_1 c_1 + b_2 c_2) e_3\} \\
&- a_3 \{-b_1 c_2 (1 - \alpha^2 e^{2z}) e_1 + \{(1 - \alpha^2 e^{2z}) b_1 c_1 - b_3 c_3\} e_2 - b_3 c_2 e_3\} \\
&+ b_1 \{-a_1 c_3 e_1 - a_2 c_3 e_2 + (a_1 c_1 + a_2 c_2) e_3\} - b_3 \{a_1 c_2 (1 - \alpha^2 e^{2z}) e_1 \\
&+ \{a_3 c_3 - (1 - \alpha^2 e^{2z}) a_1 c_1\} e_2 + a_3 c_2 e_3\} + c_1 \{(a_3 b_1 - a_1 b_3) e_1 \\
&+ (a_3 b_2 - a_2 b_3) e_2\} - c_3 \{(a_1 b_2 - a_2 b_1) (1 - \alpha^2 e^{2z}) e_1 - (a_2 b_3 - a_3 b_2) e_3\},
\end{aligned}$$

$$\begin{aligned}
(\nabla_{e_2} \bar{R})(X, Y)Z &= \alpha e^z (le_2 + me_3) + ne_1 - \alpha e^z \{a_1 R(e_2, Y)Z + a_2 R(e_3, Y)Z\} \\
&- a_3 R(e_1, Y)Z - \alpha e^z \{b_1 R(e_2, Y)Z + b_2 R(e_3, Y)Z\} - b_3 R(e_1, Y)Z \\
&- \alpha e^z \{c_1 R(e_2, Y)Z + c_2 R(e_3, Y)Z\} - c_3 R(e_1, Y)Z, \\
&= \alpha e^z (le_2 + me_3) + ne_1 - \alpha e^z (a_1 + b_1 + c_1) \{-b_1 c_2 (1 - \alpha^2 e^{2z}) e_1 \\
&+ \{(1 - \alpha^2 e^{2z}) b_1 c_1 - b_3 c_3\} e_2 - b_3 c_2 e_3\} \\
&- \alpha e^z (a_2 + b_2 + c_2) \{b_1 c_3 e_1 + b_2 c_3 e_2 - (b_1 c_1 + b_2 c_2) e_3\} \\
&+ \alpha e^z (a_3 + b_3 + c_3) \{ \{(1 - \alpha^2 e^{2z}) b_2 c_2 - b_3 c_3\} e_1 - b_2 c_1 (1 - \alpha^2 e^{2z}) e_2 - b_3 c_1 e_3 \} \\
&= [n + \alpha e^z b_1 c_2 (a_1 + b_1 + c_1) (1 - \alpha^2 e^{2z}) - \alpha e^z b_1 c_3 (a_2 + b_2 + c_2) \\
&+ \alpha e^z (a_3 + b_3 + c_3) \{(1 - \alpha^2 e^{2z}) b_2 c_2 - b_3 c_3\}] e_1 \\
&+ [\alpha e^z l - \alpha e^z (a_1 + b_1 + c_1) \{(1 - \alpha^2 e^{2z}) b_1 c_1 - b_3 c_3\} \\
&- \alpha e^z b_2 c_3 (a_2 + b_2 + c_2) - b_2 c_1 \alpha e^z (a_3 + b_3 + c_3) (1 - \alpha^2 e^{2z})] e_2 \\
&+ [\alpha e^z m + \alpha e^z b_3 c_2 (a_1 + b_1 + c_1) + \alpha e^z (a_2 + b_2 + c_2) (b_1 c_1 + b_2 c_2) \\
&- \alpha e^z b_3 c_1 (a_3 + b_3 + c_3)] e_3,
\end{aligned}$$

$$\text{and } (\nabla_{e_3} \bar{R})(X, Y)Z = 0.$$

With the help of the above relations one can easily bring out the following

$$\begin{aligned}
&\phi^2 ((\nabla_{e_1} R)(X, Y)Z) \\
&= [(1 - \alpha^2 e^{2z}) \{a_3 b_1 - a_1 b_3\} c_2 + c_1 (a_3 b_1 - a_1 b_3) \\
&- c_3 (a_1 b_2 - a_2 b_1) (1 - \alpha^2 e^{2z})] e_1 \\
&+ [n + a_1 b_2 c_3 - (1 - \alpha^2 e^{2z}) a_3 b_1 c_1 - a_2 b_1 c_3 \\
&+ (1 - \alpha^2 e^{2z}) a_1 b_3 c_1 + c_1 (a_3 b_2 - a_2 b_3)] e_2,
\end{aligned}$$

$$\begin{aligned} &\phi^2((\nabla_{e_2}\bar{R})(X, Y)Z) \\ &= [n + \alpha e^z b_1 c_2 (a_1 + b_1 + c_1)(1 - \alpha^2 e^{2z}) - \alpha e^z b_1 c_3 (a_2 \\ &\quad + b_2 + c_2) + \alpha e^z (a_3 + b_3 + c_3)\{(1 - \alpha^2 e^{2z})b_2 c_2 \\ &\quad - b_3 c_3\}]e_1 + [\alpha e^z l - \alpha e^z (a_1 + b_1 + c_1)\{(1 - \alpha^2 e^{2z})b_1 c_1 \\ &\quad - b_3 c_3\} - \alpha e^z b_2 c_3 (a_2 + b_2 + c_2) - b_2 c_1 \alpha e^z (a_3 \\ &\quad + b_3 + c_3)(1 - \alpha^2 e^{2z})]e_2, \end{aligned}$$

$$\phi^2((\nabla_{e_3}\bar{R})(X, Y)Z) = 0.$$

For the following choice of the 1-forms

$$\begin{aligned} A(e_1) &= \frac{1}{b_1 + c_1}, A(e_2) = \frac{1}{b_2 + c_2}, A(e_3) = \frac{1}{b_3 + c_3}, \\ \alpha(e_1) &= -\frac{1}{b_1 + c_1}, \alpha(e_2) = -\frac{1}{b_2 + c_2}, \alpha(e_3) = \frac{1}{b_3 + c_3}, \\ B(e_1) &= \frac{1}{a_1 + b_1}, B(e_2) = \frac{1}{a_2 + b_2}, B(e_3) = \frac{1}{a_3 + b_3}, \\ \beta(e_1) &= -\frac{1}{a_1 + b_1}, \beta(e_2) = -\frac{1}{a_2 + b_2}, \beta(e_3) = -\frac{1}{a_3 + b_3}, \\ D(e_1) &= \gamma(e_1) = 0 = D(e_2) = \gamma(e_2) = D(e_3) = \gamma(e_3), \end{aligned}$$

one can easily verify the relations

$$\begin{aligned} \phi^2((\nabla_{e_i}R)(X, Y)Z) &= A(e_i)R(X, Y)Z + B(X)R(e_i, Y)Z \\ &\quad + B(Y)R(X, e_i)Z + D(Z)R(X, Y)e_i \\ &\quad + g(R(X, Y)Z, e_i)\rho + \alpha(e_i)G(X, Y)Z \\ &\quad + \beta(X)G(e_i, Y)Z + \beta(Y)G(X, e_i)Z \\ &\quad + \gamma(U)G(X, Y, e_i, V) + g(G(X, Y)Z, e_i)\sigma, \end{aligned}$$

provided $a_i = kb_i = -2kc_i$ for $i = 1, 2, 3$. From the above, we can state that

Theorem 6.1. *There exist a Lorentzian Para-Sasakian manifold (M^3, g) which is generalized weakly ϕ -symmetric.*

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