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LP-Sasakian manifolds with generalized symmetric metric connection

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Abstract. The present study initially identifies the generalized symmetric connections of type (α, β), which can be regarded as more generalized forms of quarter and semi-symmetric connections. The quarter and semi-symmetric connections are obtained when $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, respectively. Taking this into account, a new generalized symmetric metric connection is attained on Lorentzian para-Sasakian manifolds. In accordance with this connection, some results are obtained through calculation of tensors belonging to a Lorentzian para-Sasakian manifold involving the curvature tensor, the Ricci tensor and the conformal curvature tensor.

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1. Introduction

A specific metric connection with a torsion different from zero was introduced by Hayden on a Riemannian manifold [11]. The concept of semi-symmetric connection has been introduced by Friedmann and Schouten on a differentiable manifold [8]. Quarter-symmetric connections, being a more generalized form of semi-symmetric connections, were introduced by Golab on a differentiable manifold [9]. These connections have been studied by many authors. For instance we cite ([1], [5]-[6], [10], [16], [25]) and the references therein. Tripathi [27] introduced and studied 17 types of connections which includes the semi-symmetric and quarter-symmetric connections.

On the other hand, Matsumoto [17] introduced Lorentzian para-contact manifolds. Later, many geometers ([2], [7], [12], [18]-[21], [26]) have published different papers in this context. As for the present study, the definition below is presented by taking it a step further and generalizing the quarter symmetric connections as well.

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A linear connection on a (semi-)Riemannian manifold M is a generalized symmetric connection if its torsion tensor T is presented as follows:

$$T(U,V) = \alpha\{u(V)U - u(U)V\} + \beta\{u(V)\varphi U - u(U)\varphi Y\}, \tag{1.1}$$

for all vector fields U and V on M, where α and β are smooth functions on M. φ can be viewed as a tensor of type $(1,1)$ and u is regarded as a 1-form and satisfies $u(X) = g(X,p)$ for a vector field p on M. Furthermore, the connection mentioned is a generalized metric in the case when a Riemannian metric g in M is available as $\overline{\nabla} g = 0$; otherwise, it is non-metric.

In equation (1.1), if $\alpha = 0$, $\beta \neq 0$; $\alpha \neq 0$, $\beta = 0$, then the generalized symmetric connection is called β-quarter-symmetric connection; α-semi-symmetric connection, respectively. Additionally, the generalized symmetric connection reduces to a semi-symmetric, and quarter-symmetric when $(\alpha,\beta) = (1,0)$, and $(\alpha,\beta) = (0,1)$, respectively. Thus, it can be suggested that generalizing semi-symmetric and quarter-symmetric connections paves the way for a generalized symmetric metric connection. These two connections are of great significance both for the study of geometry and applications in physics. Until now many authors have investigated the geometrical and physical aspect of these connections [3], [4], [13], [15], [14],[23], [22], [24].

In the present paper, we define a new connection on a Lorentzian para-Sasakian manifold, which is the generalization of semi-symmetric and quarter-symmetric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection. The preliminaries are presented in Section 2. Section 3 illustrates the generalized symmetric connection on a Lorentzian para-Sasakian manifold. As for Section 4, we calculate curvature tensor and the Ricci tensor of a Lorentzian para-Sasakian manifold with respect to a generalized symmetric metric connection. The last section deals with conformal curvature tensor with respect to a generalized symmetric metric connection.

2. Preliminaries

Let M be a differentiable manifold of dimension n endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and Lorentzian metric g, which satisfies

$$\eta(\xi) = -1, \quad \phi^2(U) = U + \eta(U)\xi, \tag{2.1}$$

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \quad g(U,\xi) = \eta(U), \tag{2.2}$$

$$\nabla_U \xi = \phi U, \quad (\nabla_U \phi)(V) = g(U,V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi \tag{2.3}$$

for all vector fields U, V on M, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Such manifold (M,ϕ,ξ,η,g) is called a Lorentzian para-Sasakian (shortly, LP-Sasakian) manifold [17, 19]. The following are satisfied by an LP-Sasakian manifold:

$$\phi\xi = 0, \; \eta(\phi U) = 0, \; \operatorname{rank}\phi = n-1. \tag{2.4}$$

If we write

$$\Phi(U,V) = g(\phi U, V) \tag{2.5}$$

for all vector fields U, V on M, then the tensor field Φ is a symmetric $(0,2)$ tensor field [17]. In addition, if η is closed on an LP-Sasakian manifold then we have

$$(\nabla_U \eta)V = \Phi(U,V),\ \ \Phi(U,\xi) = 0, \tag{2.6}$$

for any vector field U and V on M [17, 20]. An LP-Sasakian manifold satisfies the following relations [20, 18]:

$$R(\xi,U)V = g(U,V)\xi - \eta(V)U, \tag{2.7}$$

$$R(U,V)\xi = \eta(V)U - \eta(U)V, \tag{2.8}$$

$$R(\xi,U)\xi = U + \eta(U)\xi, \tag{2.9}$$

$$g(R(U,V)W,\xi) = \eta(R(U,V)W) = g(V,W)\eta(U) - g(U,W)\eta(V), \tag{2.10}$$

$$S(U,\xi) = (n-1)\eta(U), \tag{2.11}$$

$$S(\phi U,\phi V) = S(U,V) + (n-1)\eta(U)\eta(V) \tag{2.12}$$

for all vector fields U, V and W on M, in which R and S can be viewed as the curvature tensor and the Ricci tensor of M, respectively.

An LP-Sasakian manifold M is said to be a generalized η-Einstein if the non-vanishing Ricci tensor S of M satisfies the relation

$$S(U,V) = ag(U,V) + b\eta(U)\eta(V) + cg(\phi U,V) \tag{2.13}$$

for every U, $V \in \Gamma(TM)$, in which a, b and c are viewed as scalar functions on M. If $c = 0$, then M is regarded as an η-Einstein manifold.

On contracting (2.13), we obtain

$$r = na - b. \tag{2.14}$$

In a similar way, setting $U = V = \xi$ in (2.13) and thanks to (2.11), we get

$$-(n-1) = -a + b. \tag{2.15}$$

In view of (2.14) and (2.15), we have

$$a = \frac{r-(n-1)}{n-1},\ \ b = -\frac{n(n-1)-r}{n-1}. \tag{2.16}$$

Hence, the Ricci tensor S of a generalized η-Einstein LP-Sasakian manifold M can be expressed as

$$S(U,V) = \frac{r-(n-1)}{n-1}g(U,V) - \frac{n(n-1)-r}{n-1}\eta(U)\eta(V) + cg(\phi U,V)$$

for every U, $V \in \Gamma(TM)$.

3. Generalized symmetric metric connection in an LP-Sasakian manifold

Assume that $\overline{\nabla}$ is a linear connection and ∇ is a Levi-Civita connection of Lorentzian para-contact metric manifold M such that

$$\overline{\nabla}_U V = \nabla_U V + H(U,V)$$

for all vector fields U and V. The following is obtained so that $\overline{\nabla}$ is a generalized symmetric connection of ∇, in which H is viewed as a tensor of type $(1,2)$;

$$H(U,V) = \frac{1}{2}[T(U,V) + T^{'}(U,V) + T^{'}(V,U)], \tag{3.1}$$

where T is viewed as the torsion tensor of $\overline{\nabla}$ and

$$g(T^{'}(U,V),W) = g(T(W,U),V). \tag{3.2}$$

Thanks to (1.1) and (3.2), we obtain the following;

$$T^{'}(U,V) = \alpha\{\eta(U)V - g(U,V)\xi\} + \beta\{\eta(U)\phi V - g(\phi U,V)\xi\}. \tag{3.3}$$

Using (1.1), (3.1) and (3.3) we obtain

$$H(U,V) = \alpha\{\eta(V)U - g(U,V)\xi\} + \beta\{\eta(V)\phi U - g(\phi U,V)\xi\}.$$

Corollary 3.1. *For an LP-Sasakian manifold, the generalized symmetric metric connection $\overline{\nabla}$ of type (α,β) is given by*

$$\overline{\nabla}_U V = \nabla_U V + \alpha\{\eta(V)U - g(U,V)\xi\} + \beta\{\eta(V)\phi U - g(\phi U,V)\xi\}. \tag{3.4}$$

If $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (0,1)$ are chosen, the generalized symmetric metric connection is diminished to a semi-symmetric metric and a quarter-symmetric metric one, as presented in the following;

$$\overline{\nabla}_U V = \nabla_U V + \eta(V)U - g(U,V)\xi,$$

$$\overline{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U,V)\xi.$$

From (2.3), (2.6) and (3.4) we have the following proposition.

Proposition 3.2. *The following relations are obtained when M is an LP-Sasakian manifold with generalized metric connection:*

$$\begin{aligned}
(\overline{\nabla}_U \phi)V &= [(1-\beta)g(U,V) + (2-2\beta)\eta(U)\eta(V) - \alpha\Phi(U,V)]\xi \\
&\quad + (1-\beta)\eta(V)U - \alpha\eta(V)\phi U, \\
\overline{\nabla}_U \xi &= (1-\beta)\phi U - \alpha U - \alpha\eta(U)\xi, \\
(\overline{\nabla}_U \eta)V &= (1-\beta)\Phi(U,V) - \alpha g(\phi U,\phi V)
\end{aligned}$$

for every U, $V \in \Gamma(TM)$.

Example 3.3. A 3-dimensional manifold $M = \{(u, v, w) \in R^3\}$ is considered, in which (u, v, w) are regarded as the standard coordinates in R^3. Suppose that ν_1, ν_2, ν_3 are linearly independent global frame on M as presented below

$$\nu_1 = e^w \frac{\partial}{\partial v}, \quad \nu_2 = e^w (\frac{\partial}{\partial u} + \frac{\partial}{\partial v}), \quad \nu_3 = \frac{\partial}{\partial w}.$$

Assume that g is a Lorentzian metric defined as

$$g(\nu_1, \nu_2) = g(\nu_1, \nu_3) = g(\nu_2, \nu_3) = 0, \ g(\nu_1, \nu_1) = g(\nu_2, \nu_2) = -g(\nu_3, \nu_3) = 1.$$

When we consider that η is a 1-form represented as $\eta(U) = g(U, \nu_3)$ for every $U \in TM$ and ϕ is the $(1, 1)$ tensor field presented as $\phi\nu_1 = -\nu_1, \phi\nu_2 = -\nu_2$ and $\phi\nu_3 = 0$, we thereby get $\eta(\nu_3) = -1$, $\phi^2 U = U + \eta(U)\nu_3$ and $g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V)$ for all $U, V \in TM$ through use of linearity of ϕ and g. Therefore for $\nu_3 = \xi$, (ϕ, ξ, η, g) describes a Lorentzian para-contact structure on M. Therefore, ∇ is the Levi-Civita connection concerning the Riemannian metric g. The following are obtained:

$$[\nu_1, \nu_2] = 0, \qquad [\nu_1, \nu_3] = -\nu_1, \qquad [\nu_2, \nu_3] = -\nu_2.$$

By means of using Koszul's formula, the following can be calculated in an easy way

$$\begin{gathered}\nabla_{\nu_1}\nu_1 = -\nu_3, \qquad \nabla_{\nu_1}\nu_2 = 0. \qquad \nabla_{\nu_1}\nu_3 = -\nu_1, \\ \nabla_{\nu_2}\nu_1 = 0, \qquad \nabla_{\nu_2}\nu_2 = -\nu_3, \qquad \nabla_{\nu_2}\nu_3 = -\nu_2, \\ \nabla_{\nu_3}\nu_1 = 0, \qquad \nabla_{\nu_3}\nu_2 = 0, \qquad \nabla_{\nu_3}\nu_3 = 0.\end{gathered}$$

The relations presented above remark that (ϕ, ξ, η, g) is an LP-Sasakian structure on M [26].

Now, we can make similar calculations for generalized symmetric metric connection. Using (3.4) in the above equations, we get

$$\begin{aligned}&\overline{\nabla}_{\nu_1}\nu_1 = (-1 - \alpha + \beta)\nu_3, \quad \overline{\nabla}_{\nu_1}\nu_2 = 0, \quad \overline{\nabla}_{\nu_1}\nu_3 = (-1 - \alpha + \beta)\nu_1, \\ &\overline{\nabla}_{\nu_2}\nu_1 = 0, \quad \overline{\nabla}_{\nu_2}\nu_2 = (-1 - \alpha + \beta)\nu_3, \quad \overline{\nabla}_{\nu_2}\nu_3 = (-1 - \alpha + \beta)\nu_2, \\ &\overline{\nabla}_{\nu_3}\nu_1 = 0, \quad \overline{\nabla}_{\nu_3}\nu_2 = 0, \quad \overline{\nabla}_{\nu_3}\nu_3 = 0.\end{aligned} \tag{3.5}$$

We can easily see that (3.5) holds the relation (1.1). Also, we obtain $\overline{\nabla} g = 0$. Thus, $\overline{\nabla}$ is a generalized symmetric metric connection on M.

4. Curvature Tensor

Consider that M is an n-dimensional LP-Sasakian manifold, then the following can define the curvature tensor $\overline{R}$ of the generalized metric connection $\overline{\nabla}$ on M.

$$\overline{R}(U, V)W = \overline{\nabla}_U \overline{\nabla}_V W - \overline{\nabla}_V \overline{\nabla}_U W - \overline{\nabla}_{[U,V]} W. \tag{4.1}$$

When Proposition 3.2 is used, through (3.4) and (4.1), we obtain

$$\begin{aligned}\overline{R}(U,V)W &= R(U,V)W + K_1(V,W)U - K_1(U,W)V + K_2(V,W)\phi U \\ &\quad -K_2(U,W)\phi V + \{K_3(U,V)W - K_3(V,U)W\}\xi,\end{aligned} \tag{4.2}$$

where

$$K_1(V,W) = (\alpha\beta - \alpha)\Phi(V,W) + \alpha^2 g(V,W) + (\alpha^2 + \beta - \beta^2)\eta(V)\eta(W), \tag{4.3}$$

$$K_2(V,W) = (\beta^2 - 2\beta)\Phi(V,W) - \alpha(1-\beta)g(V,W), \tag{4.4}$$

$$K_3(U,V)W = \{(\alpha^2 + \beta)g(V,W) + \alpha\beta\Phi(V,W)\}\eta(U). \tag{4.5}$$

From (2.1)-(2.3), (2.7), (2.8) and (4.2)-(4.5), we have the following lemma

Proposition 4.1. *When M is an n-dimensional LP-Sasakian manifold with generalized symmetric metric connection, we have the following equations:*

$$\begin{aligned}&\overline{R}(U,V)\xi \\ &\quad = (1-\beta+\beta^2)(\eta(V)U - \eta(U)V) + \alpha(1-\beta)(\eta(U)\phi V - \eta(V)\phi U), \\ &\overline{R}(\xi,V)W \\ &\quad = \{-a\Phi(V,W) + (1-\beta)g(V,W) - \beta^2\eta(V)\eta(W)\}\xi \\ &\qquad -(1-\beta+\beta^2)\eta(W)V + \alpha(1-\beta)\eta(W)\phi V, \\ &\overline{R}(\xi,V)\xi \\ &\quad = (1-\beta+\beta^2)(\eta(V)\xi + V) + \alpha(\beta-1)\phi V\end{aligned}$$

for every U, V, $W \in \Gamma(TM)$.

In the following, the Ricci tensor $\overline{S}$ and the scalar curvature $\overline{r}$ of an *LP*-Sasakian manifold is presented with generalized symmetric metric connection $\overline{\nabla}$

$$\overline{S}(U,V) = \sum_{i=1}^{n} \epsilon_i g(\overline{R}(\nu_i,U)V,\nu_i),$$

and

$$\overline{r} = \sum_{i=1}^{n} \epsilon_i \overline{S}(\nu_i,\nu_i),$$

respectively, where $\epsilon_i = g(\nu_i,\nu_i)$, in which $U, V \in \Gamma(TM)$, $\{\nu_1, \nu_2, ..., \nu_n\}$ is viewed as orthonormal frame. Then by using (2.3) and (4.2) we obtain

$$\begin{aligned}\overline{S}(V,W) = \sum_{i=1}^{n} \varepsilon_i \{ & g(R(\nu_i,V)W,\nu_i) - K_1(\nu_i,W)g(V,e_i) \\ & +K_1(V,W)\varepsilon_i\} + K_2(V,W)g(\phi\nu_i,\nu_i) - K_2(\nu_i,W)g(\phi V,\nu_i) \\ & +\{K_3(\nu_i,V)W - K_3(V,\nu_i)W\}\eta(\nu_i\}.\end{aligned} \tag{4.6}$$

Then by using (4.3), (4.4), (4.5) and (4.6) we obtain

$$\begin{aligned}\overline{S}(V,W) &= S(V,W) + \{-\alpha\beta + (n-2)(\alpha\beta - \alpha)\\ &\quad +(\beta^2 - 2\beta)trace\Phi\}\Phi(V,W) + \{-2\alpha^2 + \beta - \beta^2 + n\alpha^2\\ &\quad +(\alpha\beta - \alpha)trace\Phi\}g(V,W) + \{-2\alpha^2 + n(\alpha^2 + \beta - \beta^2)\}\eta(V)\eta(W).\end{aligned} \tag{4.7}$$

Due to the fact that the Ricci tensor S of the Levi-connection is symmetric, (4.7) provides us with the following:

Corollary 4.2. *Consider that M is an n-dimensional LP-Sasakian manifold equipped with a generalized symmetric metric connection $\overline{\nabla}$. The Ricci tensor $\overline{S}$ with respect to the generalized symmetric metric connection $\overline{\nabla}$ is symmetric.*

Proposition 4.3. *Let M be an n-dimensional LP-Sasakian manifold admitting a generalized symmetric metric connection $\overline{\nabla}$. Then we have*

$$\overline{S}(V,\xi) = \{(n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\}\eta(V), \tag{4.8}$$

$$\overline{S}(\phi V,\phi W) = \overline{S}(V,W) + \{(n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\}\eta(V)\eta(W) \tag{4.9}$$

for any V, $W \in \Gamma(TM)$.

Proof. Using (2.1), (2.4) and (2.11) in the equation (4.7), we get (4.8). By using (2.3), (2.4) and (2.12) in the equation (4.7), we have (4.9). □

Now, we will make similar computations for $3-$dimensional case. In a $3-$dimensional the curvature tensor $\widetilde{R}$ is defined as

$$\begin{aligned}\overline{R}(U,V)W &= \overline{S}(V,W)U - g(U,W)\overline{Q}V + g(V,W)\overline{Q}U - \overline{S}(U,W)V\\ &\quad -\frac{\overline{r}}{2}(g(V,W)U - g(U,W)V),\end{aligned} \tag{4.10}$$

where $\overline{R}$, $\overline{Q}$, $\overline{S}$ and $\overline{r}$ are the curvature tensor, the Ricci operator, the Ricci tensor and the scalar curvature with respect to generalized symmetric metric connection, respectively.

Putting $W=\xi$ in the equation (4.10) and using proposition 4.1, we obtain

$$\begin{aligned}&(1-\beta+\beta^2)(\eta(V)U - \eta(U)V) + \alpha(1-\beta)(\eta(U)\phi V - \eta(V)\phi U)\\ &= \overline{S}(V,\xi)U - \eta(U)\overline{Q}V + \eta(V)\overline{Q}U - \overline{S}(U,\xi)V - \frac{\overline{r}}{2}(\eta(V)U - \eta(U)V).\end{aligned}$$

In this equation, if we write $V=\xi$ and using (4.8), we have

$$\begin{aligned}&(1-\beta+\beta^2)(-U - \eta(U)\xi) + \alpha(1-\beta)\phi U\\ &\quad = -\overline{Q}U - \eta(U)\overline{Q}\xi + \left\{2(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi - \frac{\overline{r}}{2}\right\}(-U - \eta(U)\xi).\end{aligned} \tag{4.11}$$

Moreover, from (4.8) we obtain $\overline{Q}\xi = \{(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi\}\xi$. If we use this equation in (4.11), we get

$$\begin{aligned}\overline{Q}U &= -\{3(1-\beta+\beta^2)+2\alpha(\beta-1)trace\Phi-\frac{\overline{r}}{2}\}\eta(U)\xi \\ &\quad -\{1-\beta+\beta^2+\alpha(\beta-1)trace\Phi-\frac{\overline{r}}{2}\}U-\alpha(1-\beta)\phi U. \qquad (4.12)\end{aligned}$$

Then have the following theorem

Theorem 4.4. *In a 3-dimensional LP-Sasakian manifold with generalized symmetric metric connection, the Ricci tensor $\overline{S}$ and the curvature tensor $\overline{R}$ satisfy the following:*

$$\overline{S}(U,V) = k_1\eta(U)\eta(V)+k_2 g(U,V)-\alpha(1-\beta)g(\phi U,V) \qquad (4.13)$$

$$\begin{aligned}&\overline{R}(U,V)W \\ &= (2k_2-\frac{\overline{r}}{2})\{g(V,W)U-g(U,W)V\} \\ &+ k_1\{g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi+\eta(V)\eta(W)U-\eta(U)\eta(W)V\} \qquad (4.14)\\ &+ \alpha(1-\beta)\{g(U,W)\phi V-g(V,W)\phi U+g(\phi U,W)V-g(\phi V,W)U\},\end{aligned}$$

for any $U,V,W\in\Gamma(TM)$, where $k_1 = -\{3(1-\beta+\beta^2)+2\alpha(\beta-1)trace\Phi-\frac{\overline{r}}{2}\}$ and $k_2 = -\{1-\beta+\beta^2+\alpha(\beta-1)trace\Phi-\frac{\overline{r}}{2}\}$.

Proof. From (4.12), we have (4.13). If we use (4.12) and (4.13) in the equation (4.11), we can easily obtain the equation (4.14). □

Taking account of (4.14), we write

$$\begin{aligned}&\overline{R}(U,V,W,Z) \qquad (4.15)\\ &= (2k_2-\frac{\overline{r}}{2})\{g(V,W)g(U,Z)-g(U,W)g(V,Z)\} \\ &\quad +k_1\{g(V,W)\eta(U)g(\xi,Z)-g(U,W)\eta(V)g(\xi,Z) \\ &\quad +\eta(V)\eta(W)g(U,Z)-\eta(U)\eta(W)g(V,Z)\} \\ &\quad +\alpha(1-\beta)\{g(U,W)g(\phi V,Z)-g(V,W)g(\phi U,Z) \\ &\quad +g(\phi U,W)g(V,Z)-g(\phi V,W)g(U,Z)\},\end{aligned}$$

for any $U,V,W,Z\in\Gamma(TM)$. Interchanging U and V in (4.15) yields

$$\begin{aligned}&\overline{R}(V,U,W,Z) \\ &= (2k_2-\frac{\overline{r}}{2})\{g(U,W)g(V,Z)-g(V,W)g(U,Z)\} \\ &\quad +k_1\{g(U,W)\eta(V)g(\xi,Z)-g(V,W)\eta(U)g(\xi,Z) \qquad (4.16)\\ &\quad +\eta(U)\eta(W)g(V,Z)-\eta(V)\eta(W)g(U,Z)\} \\ &\quad +\alpha(1-\beta)\{g(V,W)g(\phi U,Z)-g(U,W)g(\phi V,Z) \\ &\quad +g(\phi V,W)g(U,Z)-g(\phi U,W)g(V,Z)\}, \quad \forall U,V,W,Z\in\Gamma(TM).\end{aligned}$$

From (4.15) and (4.16), we get

$$\overline{R}(U,V,W,Z)+\overline{R}(V,U,W,Z)=0, \quad \forall\, U,V,W,Z\in\Gamma(TM). \tag{4.17}$$

Again interchanging W and Z in (4.15) and adding the newly obtained equation to (4.15) produces

$$\begin{aligned}(4.18)\qquad & \overline{R}(U,V,W,Z)+\overline{R}(U,V,Z,W)\\ &= k_1\{g(V,W)\eta(V)\eta(Z)-\eta(U)\eta(Z)g(V,W)\}.\end{aligned}$$

Next, let us interchange pair slots in (4.15) to get

$$\begin{aligned}(4.19)\qquad & \overline{R}(W,Z,U,V)\\ &= (2k_2-\frac{\overline{r}}{2})\{g(Z,U)g(W,V)-g(W,U)g(Z,V)\}\\ &\quad +k_1\{g(Z,U)\eta(W)g(\xi,V)-g(W,U)\eta(Z)g(\xi,V)\\ &\quad +\eta(Z)\eta(U)g(W,V)-\eta(W)\eta(U)g(Z,V)\}\\ &\quad +\alpha(1-\beta)\{g(W,U)g(\phi Z,V)-g(Z,U)g(\phi W,V)\\ &\quad +g(\phi W,U)g(Z,V)-g(\phi Z,U)g(W,V)\},\\ &\qquad \forall\, U,V,W,Z\in\Gamma(TM).\end{aligned}$$

Using (4.15) and (4.19), we obtain

$$\overline{R}(U,V,W,Z)-\overline{R}(W,Z,U,V)=0, \tag{4.20}$$

and

$$\begin{aligned}(4.21)\qquad & \overline{R}(U,V,W,Z)+\overline{R}(W,Z,U,V)\\ &= 2(2k_2-\frac{\widetilde{r}}{2})\{g(V,W)g(U,Z)-g(U,W)g(V,Z)\}\\ &\quad +2k_1\{g(Z,U)\eta(V)\eta(W)-g(W,U)\eta(Z)\eta(V)\\ &\quad +\eta(Z)\eta(U)g(W,V)-\eta(W)\eta(U)g(Z,V)\},\end{aligned}$$

for all $U,V,W,Z\in\Gamma(TM)$.

Thanks to (4.17), (4.18), (4.20) and (4.21), we have the following result.

Theorem 4.5. *In a 3-dimensional LP-Sasakian manifold with generalized symmetric metric connection, the following hold:*

(i) $\overline{R}(U,V,W,Z)+\overline{R}(V,U,W,Z)=0$,

(ii) $\overline{R}(U,V,W,Z)+\overline{R}(U,V,Z,W)=k_1\{g(V,W)\eta(V)\eta(Z)-\eta(U)\eta(Z)g(V,W)\}$,

(iii) $\overline{R}(U,V,W,Z)-\overline{R}(W,Z,U,V)=0$,

(iv) $\overline{R}(U,V,W,Z)+\overline{R}(W,Z,U,V)=2(2k_2-\frac{\overline{r}}{2})\{g(V,W)g(U,Z)$ $-g(U,W)g(V,Z)\}+2k_1\{g(Z,U)\eta(V)\eta(W)-g(W,U)\eta(Z)\eta(V)$ $+\eta(Z)\eta(U)g(W,V)-\eta(W)\eta(U)g(Z,V)\}$,

for any $U,V,W,Z\in\Gamma(TM)$, *where* $k_1=-\{3(1-\beta+\beta^2)+2\alpha(\beta-1)trace\Phi-\frac{\overline{r}}{2}\}$ *and* $k_2=-\{1-\beta+\beta^2+\alpha(\beta-1)trace\Phi-\frac{\overline{r}}{2}\}$.

5. Conformal Curvature Tensor

Let M be an n-dimensional LP-Sasakian manifold admitting a generalized symmetric metric connection $\overline{\nabla}$ and $\overline{Q}$ denotes the Ricci operator with respect to the generalized symmetric metric connection satisfying the relation $g(\overline{Q}U, V) = \widetilde{S}(U, V)$. Then the conformal curvature tensor $\overline{C}$ of M is defined by

$$\begin{aligned} &\overline{C}(U, V)W \\ &= \overline{R}(U, V)W \\ &\quad -\frac{1}{n-2}\{\overline{S}(V, W)U - \overline{S}(U, W)V + g(V, W)QU - g(U, W)QV\} \\ &\quad +\frac{\overline{r}}{(n-1)(n-2)}\{g(V, W)U - g(U, W)V\}, \end{aligned} \tag{5.1}$$

where $\overline{r}$ is used to represent the scalar curvature with respect to the generalized symmetric metric connection.

Definition 5.1. An n-dimensional LP-Sasakian manifold M admitting a generalized symmetric metric connection is said to be conformally flat if conformal curvature tensor $\overline{C} = 0$.

We prove the following result.

Theorem 5.2. *A conformally flat LP-Sasakian manifold of dimension n equipped with a generalized symmetric metric connection is a generalized η-Einstein manifold of quasi constant curvature.*

Proof. Let us suppose that M be conformally flat n-dimensional LP-Sasakian manifold equipped with a generalized symmetric metric connection. Then, in view of (5.1), we have

$$\begin{aligned} &\overline{R}(U, V)W \\ &= \frac{1}{n-2}\{\overline{S}(V, W)U - \overline{S}(U, W)V + g(V, W)QU - g(U, W)QV\} \\ &\quad +\frac{\overline{r}}{(n-1)(n-2)}\{g(V, W)U - g(U, W)V\}. \end{aligned} \tag{5.2}$$

Thanks to (2.2) and (4.8), we obtain

$$\begin{aligned} &g(\overline{R}(U, V)W, \xi) \\ &= \frac{1}{n-2}[\overline{S}(V, W)\eta(U) - \overline{S}(U, W)\eta(V) \\ &\quad +g(V, W)\{(n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\}\eta(U) \\ &\quad -g(U, W)\{(n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\}\eta(V)] \\ &\quad +\frac{\overline{r}}{(n-1)(n-2)}\{g(V, W)\eta(U) - g(U, W)\eta(V)\}. \end{aligned} \tag{5.3}$$

Replacing U by ξ and taking into account (2.1),(2.2) and Proposition 4, we get

$$\begin{aligned}\overline{S}(V,W) &= \frac{1}{(n-1)^2}[-(n-1)K_3+\overline{r}]g(V,W)\\ &\quad+\frac{1}{(n-1)^2}[-2(n-1)K_3+\overline{r}]\eta(W)\eta(V),\end{aligned}\tag{5.4}$$

where $K_3=\{(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi\}$.

Let us set $a=\frac{1}{(n-1)^2}[-(n-1)K_3+\overline{r}]$ and $b=\frac{1}{(n-1)^2}[-2(n-1)K_3+\overline{r}]$, then equation (5.4) takes the following form

$$\overline{S}(V,W) = ag(V,W)+b\eta(V)\eta(W)+cg(\phi V,W).\tag{5.5}$$

Finally, in the light of (5.2) and (5.4), we obtain

$$\begin{aligned}&g(\overline{R}(U,V)W,Z)\\ &= \frac{2}{(n-1)^2(n-2)}K_4[g(V,W)g(U,Z)-g(U,W)g(V,Z)]\\ &\quad+\frac{1}{(n-1)^2(n-2)}K_5[\eta(W)\eta(V)g(U,Z)-\eta(U)\eta(W)g(V,Z)\\ &\quad+\eta(U)\eta(Z)g(V,W)-\eta(V)\eta(Z)g(U,W)],\end{aligned}\tag{5.6}$$

where $K_4=-(n-1)^2(1-\beta+\beta^2)-\alpha(n-1)(\beta-1)trace\Phi+(2-n)\widetilde{r}$ and $K_5=-2(n-1)^2(1-\beta+\beta^2)-2\alpha(n-1)(\beta-1)trace\Phi+\widetilde{r}$.

Thus in the light of (5.4) and (5.6) our result follows. □

We note that an n-dimensional *LP*-Sasakian manifold M admitting a generalized symmetric metric connection is said to be quasi conformally flat if

$$g(\overline{C}(U,V)W,Z)=0,\forall\, U,V,W,Z\in\Gamma(TM).\tag{5.7}$$

We have the following theorem.

Theorem 5.3. *A quasi conformally flat LP-Sasakian manifold M of dimension n equipped with a generalized symmetric metric connection is a generalized η-Einstein manifold.*

Proof. Let us suppose that M be quasi conformally flat n-dimensional *LP*-Sasakian manifold equipped with a generalized symmetric metric connection. Then, in view of (5.1), we have

$$\begin{aligned}&g(\overline{R}(U,V)W,\phi Z)-\frac{1}{n-2}\{\overline{S}(V,W)g(U,\phi Z)-\overline{S}(U,W)g(V,\phi Z)\\ &\qquad+g(V,W)g(QU,\phi Z)-g(U,W)g(QV,\phi Z)\}\\ &\quad+\frac{\overline{r}}{(n-1)(n-2)}\{g(V,W)g(U,\phi Z)-g(U,W)g(V,\phi Z)\}=0,\end{aligned}\tag{5.8}$$

for all $U, V, W, Z \in \Gamma(TM)$. Setting $V = W = \xi$ and taking account of (2.1),(2.2),(2.4) and (4.8), we obtain

$$\overline{S}(U,\phi Z) = \frac{-1}{(n-1)}[(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi + \frac{\overline{r}}{(n-1)}]g(U,\phi Z), \quad \forall\, U,V,W,Z \in \Gamma(TM). \tag{5.9}$$

Putting $Z = \phi Z$ and applying of (2.1),(2.2) and (4.9), we arrive at

$$\overline{S}(U,Z) = \frac{1}{(n-1)}[-K_7 + \frac{\overline{r}}{(n+1)}]g(U,Z) + \frac{1}{(n-1)}[-nK_7 + \frac{\overline{r}}{(n+1)}]\eta(U)\eta(Z), \quad \forall\, U,V,W,Z \in \Gamma(TM), \tag{5.10}$$

where $K_7 = (n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi$. □

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