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General iterative methods for common fixed points of asymptotically nonexpansive mappings

Godwin Chidi Ugwunnadi[1]

Abstract. In this paper, we study an iterative process for approximating a common fixed point of afamily of uniformly asymptotically regular asymptotically nonexpansive mappings with variational inequality problem in uniformly convex Banach space with uniformly Gâteaux differentiable norm. We prove a strong convergence theorem under some suitable conditions. Our result is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ (and consequently in Sobolev spaces). Our results improve and generalize some well-known results in the literature.

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1. Introduction

Let K be a nonempty closed and convex subset of a real Banach space E and E^* the dual space of E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the pairing between the elements of E and those of E^*.

Let $T : E \to E$ be a nonlinear mapping, a point $x \in E$ is called a *fixed point of* T if $Tx = x$. We denote by $F(T)$ the set of all fixed points of T (i.e., $F(T) = \{x \in E : Tx = x\}$). The mapping T is said to be L-*Lipschitz* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in E. \tag{1.2}$$

If in this case, (1.2) is satisfied with $L \in [0, 1)$ (respectively, $L \in [0, 1]$), then the mapping T is called a *contraction* (respectively, *nonexpansive*). The map T is said to be uniformly $L - Lipschitzian$ if there exists $L \geq 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \tag{1.3}$$

[1]Department of Mathematics,University of Eswatini, Private Bay 4, Kwaluseni, Eswatini; Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94, Pretoria 0204, South Africa,
e-mail: gcugwunnadi@uniswa.sz; ugwunnadi4u@yahoo.com

for all $x, y \in E$, $n \in \mathbb{N}$ and T is called *asymptotically nonexpansive* if there exists a sequence $v_n \in [0, \infty)$, $\lim_{n\to\infty} v_n = 0$ such that for all $x, y \in K$

$$||T^n x - T^n y|| \leq (1 + v_n)||x - y|| \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.4}$$

It is clear that every asymptotically nonexpansive mapping is uniformly L- Lipschitzian with $L = \sup_{n\geq 1}\{1 + v_n\}$. It is well known (see for example [10]) that the class of nonexpansive mappings is a proper subclass of the class of asymptotically nonexpansive mappings. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [10] as an important generalization of the class of nonexpansive mappings. Goebel and Kirk [10] proved that if K is a nonempty, bounded, closed and convex subset of a real uniformly convex Banach space and T is a self asymptotically nonexpansive mapping of K, then T has a fixed point. Related problems have been extensively investigated in the literature (see [9, 11, 25, 26, 27, 18, 19]).
Let K be a nonempty closed and convex subset of a real Banach space E. A mapping T is said to be asymptotically regular if

$$\lim_{n\to\infty} ||T^{n+1}x - T^n x|| = 0$$

for all $x \in K$. It is said to be uniformly asymptotically regular if for any bounded subset D of K,

$$\lim_{n\to\infty} \sup_{x\in D} ||T^{n+1}x - T^n x|| = 0.$$

Fixed point theory has played very crucial roles in many different fields of science, which can be witnessed in game theory and optimization problems, approximation problems, differential equations, variational inequalities, complementary problems, equilibrium theory, control theory and economics.

The approximation of fixed points of mappings is exceptionally significant because of its importance in proving the existence of fixed points of mappings. It can be used to prove the solvability of optimization problems, differential equations, variational inequalities, and equilibrium problems. In most cases the basic tool has been the sequence of successive approximations used in the study of fixed point theory. A good deal of work has been associated with the nonexpansive mappings. As the sequence of iterates for a nonexpansive mapping need not always converge therefore several researchers have tried to give techniques for convergence of the sequence of iterates.

Definition 1.1. Let $G : E \to E$ be a nonlinear mapping. Then, a variational inequality problem with respect to K and G is to find $x^* \in K$ such that

$$\langle Gx^*, j(y - x^*)\rangle \geq 0, \text{ for all } y \in K, \quad j(y - x^*) \in J(y - x^*). \tag{1.5}$$

For some positive real numbers δ and λ, the mapping $G : E \to E$ is said to be δ-strongly accretive, if for any $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Gx - Gy, j(x - y)\rangle \geq \delta||x - y||^2 \tag{1.6}$$

and it is called λ-strictly pseudocontractive if

$$\langle Gx - Gy, j(x-y)\rangle \leq ||x-y||^2 - \lambda||(I-G)x-(I-G)y||^2. \tag{1.7}$$

Definition 1.2. Let δ, λ and β be positive real numbers satisfying $\delta + \lambda > 1$ and $\beta \in (0,1)$. Then $G : E \to E$ is δ-strongly accretive and λ-strictly pseudocontractive (see [24]) if, for all $x, y \in E$,

$$||(I-G)x-(I-G)y|| \leq (\sqrt{(1-\delta)/\lambda})||x-y||$$

and

$$||(I-\beta G)x-(I-\beta G)y|| \leq [1-\beta(1-\sqrt{(1-\delta)/\lambda})]||x-y||,$$

that is, $(I-G)$ and $(I-\beta G)$ are contractive mappings.

Variational inequality has developed into a crucial mechanism in studying many problems originating in certain areas of pure and applied sciences. Numerous methods have been developed by many researchers for solving variational inequality problems and related optimization and control problems via approximation of fixed point of mappings, see [6, 12, 14, 36].

Many a problem in pure and applied sciences can be reframed as a problem of finding a fixed point of a nonexpansive mapping. In 1953, Mann [21] introduced an iterative method that converges weakly to a fixed point of a nonexpansive mapping. However, even in a Hilbert space, Mann's iteration may fail to converge strongly. Several attempts have been made to construct an iteration method that guarantees the strong convergence. Halpern [13] proposed the so-called Halpern iteration method which converges strongly to a fixed point of a nonexpansive mapping. Later in 2000, Moudafi [23] introduced the viscosity approximation method to generalize the ideas of Halpern, for nonexpansive mappings in a Hilbert space H, as follows:

Let H be a real Hilbert space and $T : H \to H$ a nonexpansive mapping such that $F(T)$ is nonempty, let f be a contraction on H, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in (0,1). He proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.8) strongly converges to the unique solution x^* in $F(T)$ of the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \text{ for all } x \in F(T).$$

Xu [34] in 2003, proved, under some condition on the real sequence $\{\alpha_n\}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrarily and for any fixed $b \in H$,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.9}$$

converges strongly to $x^* \in F(T)$ which is the unique solution of the minimization problem

$$\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where A is a strongly positive bounded linear operator (i.e. $\exists\, \bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}||x||^2, \quad \forall x \in H$).

Combining the iterative method (1.8) and (1.9), Marino and Xu [22] studied the following general iterative method:

$$x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0. \tag{1.10}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $x^* \in F(T)$ which solves the variational inequality problem

$$\langle (\gamma f - A)x^*, x - x^*\rangle \leq 0 \quad \forall x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada [35] in 2001 introduced the following hybrid iterative method in Hilbert space H:

$$x_{n+1} = Tx_n - \lambda_n \mu G T x_n, \quad n \geq 0, \tag{1.11}$$

where G is a κ-Lipschitzian and η-strongly monotone operator on H with $\kappa > 0, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$ and T is a nonexpansive mapping on H. Under some appropriate conditions, he proved that the sequence $\{x_n\}$ generated by (1.11) converges strongly to the unique solution of the variational inequality problem

$$\langle Gx^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T).$$

Recently, combining (1.10) and (1.11), Tian [31] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T(x_n), \tag{1.12}$$

and proved that the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, x - x^*\rangle \leq 0, \quad \forall x \in F(T).$$

In 2007, Maingé [20] studied the Halpern-type scheme for approximation of a common fixed point of *countable infinite* family of nonexpansive mappings in a real Hilbert space. Define $\mathcal{N}_I := \{i \in \mathbb{N} : T_i \neq I\}$ (I being the identity mapping on H). He proved the following theorems.

Theorem 1.3. (Maingé [20]) *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}$ be countable family of nonexpansive self-mappings of K, $\{t_n\}$ and $\{\sigma_{i,t_n}\}$ be sequences in (0,1) satisfying the following* *conditions: (i)* $\lim t_n = 0$*, (ii)* $\sum_{i\geq 1}\sigma_{i,t_n} = 1-t_n$*, (iii)* $\forall i \in \mathcal{N}_I,\ \lim\limits_{n\to\infty}\frac{t_n}{\sigma_{i,t_n}} = 0$*. Define a fixed point sequence* $\{x_{t_n}\}$ *by*

$$x_{t_n} = t_n C x_{t_n} + \sum_{i\geq 1}\sigma_{i,t_n} T_i x_{t_n}, \quad n \geq 1, \tag{1.13}$$

where $C : K \to K$ *is a strict contraction. Assume* $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$*, then* $\{x_{t_n}\}$ *converges strongly to a unique fixed point of the contraction* $P_F \circ C$*, where* P_F *is a metric projection from* H *onto* F*.*

Theorem 1.4. (Maingé [20]) *Let* K *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\{T_i\}$ *be countable family of nonexpansive self-mappings of* K*,* $\{\alpha_n\}$ *and* $\{\sigma_{i,n}\}$ *be sequences in (0,1) satisfying the following conditions:*

(i) $\sum \alpha_n = \infty$, $\sum_{i\geq 1}\sigma_{i,n} = 1 - \alpha_n$,

(ii)

$$\begin{cases} \frac{1}{\sigma_{i,n}}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \quad or \quad \sum_n \frac{1}{\sigma_{i,n}}|\alpha_{n-1} - \alpha_n| < \infty \\ \frac{1}{\alpha_n}\left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| \to 0, \quad or \quad \sum_n \left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| < \infty \\ \frac{1}{\sigma_{i,n}\alpha_n}\sum_{k\geq 0}|\sigma_{k,n} - \sigma_{k,n-1}| \to 0, \quad or \quad \frac{1}{\sigma_{i,n}}\sum_{k\geq 0}|\sigma_{k,n} - \sigma_{k,n-1}| < \infty. \end{cases}$$

(iii) $\forall i \in \mathcal{N}_I,\ \lim\limits_{n\to\infty}\frac{\alpha_n}{\sigma_{i,n}} = 0$.

Then, the sequence $\{x_n\}$ *defined iteratively by* $x_1 \in K$*,*

$$x_{n+1} = \alpha_n C x_n + \sum_{i\geq 1}\sigma_{i,n} T_i x_n, \quad n \geq 1, \tag{1.14}$$

where $C : K \to K$ *is a strict contraction. Assume* $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to a unique fixed point of the contraction* $P_F \circ C$*, where* P_F *is a metric projection from* H *onto* F*.*

In 2009, Ali [2] studied a modified scheme for approximation of a common fixed point of family of nonexpansive mappings in a real q-uniformly smooth Banach space which is also uniformly convex. He proved the following theorem.

Theorem 1.5. (Ali [2]) *Let* E *be a real* q*-uniformly smooth Banach space which is also uniformly convex. Let* K *be a closed, convex and nonempty subset of* E*. For* $\alpha > 0$*, let* $T_i : K \to K$ $i \in \mathbb{N}$ *and* $A : K \to E$ *be a family of nonexpansive maps and an* α*-inverse strongly accretive map, respectively. Let* P_K *be a nonexpansive projection of* E *onto* K*. For some real numbers* $\delta \in (0,1)$ *and* $\lambda \in (0, (\frac{q\alpha}{d_q})^{\frac{1}{q-1}})$ *define a sequence* $\{x_n\}$ *iteratively by* $x_1, u \in K$*,*

$$x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta\sum_{i\geq 1}\sigma_{in} T_i P_K(x_n - \lambda A x_n), n \geq 1 \tag{1.15}$$

where $\{\alpha_n\}$ *and* $\{\sigma_{in}\}$ *are real sequences in* $(0,1)$ *satisfying the following conditions: (i)* $\lim \alpha_n = 0$, *(ii)* $\sum \alpha_n = \infty$, *(iii)* $\sum_{i\geq 1} \sigma_{in} = 1 - \alpha_n$ *and* $\lim_{n\to\infty} \sum_{i\geq 1} |\sigma_{i,n+1} - \sigma_{in}| = 0$. *Let* $F := [\cap_{i=1}^{\infty} F(T_i)] \cap VI(K,A) \neq \emptyset$. *If either the duality map* j *of* E *admits weak sequential continuity or for at least one* $i \in \mathbb{N}$, $T_i P_K(I - \lambda A)$ *is demicompact, then* $\{x_n\}$ *converges strongly to some element in* F.

Later, Ali et al. [4], extended the result of Tian [31] to q-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. Under some assumption on $\{\alpha_n\}, \gamma, \mu$ and G, they proved that the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, j(x - x^*)\rangle \leq 0, \quad \forall x \in F(T).$$

In 2015, Yolacan [37] studied a hybrid iteration scheme for approximating fixed points of asymptotically nonexpansive mappings and proved convergence theorem for a fixed point of asymptotically nonexpansive mapping in uniformly convex Banach spaces.

In 2017, Jung [15] introduced a modified algorithm of (1.10) for the implicit and the explicit case. Under some control conditions, he established strong convergence of the proposed algorithm to a fixed point of a nonexpansive mapping, which solves certain variational inequality in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm.

Recently, variational inequality and fixed point problem has attracted the attention of many researchers and has been studied mostly in Hilbert spaces (see [6, 12, 14, 36]).

Motivated by the above results, in this paper we study modified iterative scheme for approximating a common fixed point of a family of uniformly asymptotically regular asymptotically nonexpansive mappings with the variational inequality problem in real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Under some mild conditions on the parameters, we prove a strong convergence theorem. Our result is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ (and consequently in Sobolev spaces). Our result extends and improves some recent important results in literature.

2. Preliminaries

Let K be a nonempty, closed, convex and bounded subset of a Banach space E and $S(E) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The space E is said to have *Gâteaux differentiable norm* if for any $x \in S(E)$ the limit

$$\lim_{\lambda\to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \tag{2.1}$$

exists $\forall y \in S(E)$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit (2.1) is attained uniformly for $x \in S(E)$.

Let the diameter of K be defined by $d(K) := \sup\{\|x-y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x-y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [5], see also Lim [16] and the references contained therein) is defined by $N(E) := \inf\{\frac{d(K)}{r(K)}$: K is a closed convex and bounded subset of E with $d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have *uniform normal structure*. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g.,[7, 17]).

The following lemmas are used for our main result.

Lemma 2.1. *[7] Let E be a real normed space and $J_p : E \to E$, $1 < p < \infty$ be the generalized duality map. Then, for any $x, y \in E$, the following inequality holds:*

$$||x+y||^p \leq ||x||^p + p\langle y, j_p(x+y)\rangle,$$

for all $j_p(x+y) \in J_p(x+y)$. In particular, if $p = 2$, then

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 2.2. (Suzuki [28]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||y_n - x_n|| = 0$.*

Lemma 2.3. (See Lemma 16.7 of Chidume [7]) *Let E be a real uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for every $x, y \in B_r(0)$ and $p \in (1, \infty)$, the following inequality holds:*

$$(2.2)\quad 4 \cdot 2^p g(\frac{1}{2}||x+y||) \leq (p \cdot 2^p - 4)||x||^p + p \cdot 2^p \langle y, j_p(x)\rangle + 4||y||^p.$$

Lemma 2.4. (Lim and Xu, [17], Theorem 1) *Suppose E is a Banach space with uniformly normal structure, K is a nonempty bounded subset of E, and $T : K \to K$ is uniformly $k-$Lipschitzian mapping with $k < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset C of K with the following property (P) :*

$$(P)\quad x \in C \textit{ implies } \omega_\omega(x) \subset C,$$

where $\omega_\omega(x)$ is the ω-limit set of T at x, that is, the set

$$\{y \in E : y = weak - \lim_j T^{n_j} x \quad \textit{for some} \quad n_j \to \infty\}.$$

Then, T has a fixed point in C.

Let μ be a linear continuous functional on l^∞ and let $a = (a_1, a_2, \cdots) \in l^\infty$. We will sometimes write $\mu_n(a_n)$ in place of the value $\mu(a)$. A linear continuous functional μ such that $||\mu|| = 1 = \mu(1)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, \cdots) \in l^\infty$ is called a *Banach limit*. It is known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \cdots) \in l^\infty$ (see, for example, [7, 8])

Lemma 2.5. (Xu [33]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \; n \geq 0,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

In the sequel we assume for the sequences $\{\alpha_n\}, \{\sigma_{in}\} \subset (0, 1)$, that $\sum_{i\geq 1} \sigma_{in} := 1 - \alpha_n$ for each $n \in \mathbb{N}$.

Theorem 3.1. *Let E be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $G : E \to E$ be an η-strongly accretive and μ-strictly pseudocontractive mapping with $\eta + \mu > 1$ and $f : E \to E$ a contraction with coefficient $\beta \in (0, 1)$. Let $\{T_i\}_{i=1}^\infty$ be a family of uniformly asymptotically regular asymptotically nonexpansive mappings of E into itself with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Assume that $0 < \gamma < \frac{\tau}{2\beta}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\Sigma_{n=0}^\infty \beta_n = \infty$

(C2) $\lim_{n\to\infty} \frac{v_n}{\beta_n} = 0$ *and* $\forall\, i \in \mathcal{N}_I$, $\lim_{n\to\infty} \frac{v_n}{\sigma_{in}} = 0$ *where* $v_n := \sup_{i\geq 1}\{v_{in}\}$

(C3) $\lim_{n\to\infty} \alpha_n = 0$.

For some fixed $\delta \in (0, 1)$, let $\{x_n\}_{n=1}^\infty$ be a sequence defined for $x_0 \in E$ chosen arbitrarily, by

$$\left\{ \begin{array}{l} x_n = [1 - \delta(1 - \alpha_n)]x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n, \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n, \qquad n \geq 0. \end{array} \right. \tag{3.1}$$

Then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, which also solves the following variational inequality:

$$\langle \gamma f(p) - Gp, j(q - p)\rangle \leq 0, \qquad \textit{for all } \; q \in F. \tag{3.2}$$

Proof. First, we show that $\{x_n\}$ defined by (3.1) is well defined. For all $n \in \mathbb{N}$, let

$$T_n^f x := [1-\delta(1-\alpha_n)]x + \delta \sum_{i\geq 1} \sigma_{in} T_i^n[\beta_n \gamma f(x) + (I-\beta_n G)x].$$

Then for all $x, y \in E$, we get

$$\begin{aligned}
&||T_n^f x - T_n^f y|| \\
&\leq [1-\delta(1-\alpha_n)]||x-y|| + \delta \sum_{i\geq 1} \sigma_{in} ||T_i^n[\beta_n\gamma f(x) + (I-\beta_n G)x] \\
&\quad -T_i^n[\beta_n \gamma f(y) + (I-\beta_n)Gy]|| \\
&\leq [1-\delta(1-\alpha_n)]||x-y|| + \delta(1-\alpha_n)(1+v_n) \\
&\quad \times[\beta_n\gamma||f(x)-f(y)|| + ||(I-\beta_n G)x - (I-\beta_n G)y||] \\
&\leq [1-\delta(1-\alpha_n)]||x-y|| + \delta(1-\alpha_n)(1+v_n) \\
&\quad \times[\beta_n\gamma\beta||x-y|| + (1-\tau\beta_n)||x-y||] \\
&= \Big(1-\delta(1-\alpha_n) + \delta(1-\alpha_n)(1+v_n)[1-\beta_n(\tau-\gamma\beta)]\Big)||x-y|| \\
&= \Big(1-\delta(1-\alpha_n)[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n]\Big)||x-y||.
\end{aligned}$$

Since $\lim_{n\to\infty}[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] = 0$, then there exists $n_0 \in \mathbb{N}$ such that $[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] < \epsilon \in (0,1)$ for all $n \geq n_0$. Thus, for $n \geq n_0$, we have

$$\delta(1-\alpha_n)[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] < \delta(1-\alpha_n)\epsilon < 1,$$

therefore, for $n \geq n_0$, we obtain

$$1-\delta(1-\alpha_n)[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] < 1.$$

Hence,

$$||T_n^f x - T_n^f y|| < ||x-y||.$$

That is, $\{x_n\}$ defined by (3.1) is well defined.
Therefore, by the contraction mapping principle, there exists a unique fixed point $x_n \in E$ of T_n^f which satisfies (3.1).

Furthermore, using the same method that was used in Bashir [3], from the choice of the parameter γ, it is easy to see that the mapping $(G-\gamma f): E \to E$ is strongly accretive and so the variational inequality (3.2) has a unique solution in F.
Next, let $p \in F$, since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, then $\frac{v_n}{(1+v_n)\beta_n} \to 0$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ such that $\frac{v_n}{(1+v_n)\beta_n} < \frac{\tau-\gamma\beta}{2}$, for all $n \geq n_0$. Thus

$$\begin{aligned}
||y_n - p|| &= ||\beta_n(\gamma f(x_n) - Gp) + (I-\beta_n G)(x_n-p)|| \\
&\leq \beta_n||\gamma f(x_n) - Gp|| + (1-\beta_n\tau)||x_n-p|| \\
&\leq \Big(1-\beta_n(\tau-\gamma\beta)\Big)||x_n-p|| + \beta_n||\gamma f(p)-Gp||, \qquad (3.3)
\end{aligned}$$

and by (3.1), we obtain

$$\begin{aligned}
||x_n - p|| &= ||[1-\delta(1-\alpha_n)](x_n - p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)|| \\
&\leq [1-\delta(1-\alpha_n)]||x_n - p|| + \delta(1-\alpha_n)(1+v_n)||y_n - p|| \\
&\leq \Big[1-\delta(1-\alpha_n) + \delta(1-\alpha_n)(1+v_n)[1-\beta_n(\tau-\gamma\beta)]\Big]||x_n - p|| \\
&\quad +\delta(1-\alpha_n)(1+v_n)\beta_n||\gamma f(p) - Gp|| \\
&= \Big[1+\delta(1-\alpha_n)v_n - \beta_n\delta(1-\alpha_n)(1+v_n)(\tau-\gamma\beta)\Big]||x_n - p|| \\
&\quad +\delta(1-\alpha_n)(1+v_n)\beta_n||\gamma f(p) - Gp|| \\
&\leq \Big[1-\beta_n\delta(1-\alpha_n)(1+v_n)\Big((\tau-\gamma\beta) - \frac{v_n}{(1+v_n)\beta_n}\Big)\Big]||x_n - p|| \\
&\quad +\beta_n\delta(1-\alpha_n)(1+v_n)\Big((\tau-\gamma\beta) - \frac{v_n}{(1+v_n)\beta_n}\Big)\frac{2||\gamma f(p) - Gp||}{\tau-\gamma\beta}.
\end{aligned}$$

Therefore

$$||x_n - p|| \leq \frac{2||\gamma f(p) - Gp||}{\tau-\gamma\beta}, \quad \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded. Also $\{f(x_n)\}, \{G(x_n)\}, \{y_n\}, \{T_i^n x_n\}$ and $\{T_i^n y_n\}$ are all bounded for each $i \geq 1$.

Also by (3.1),

$$||y_n - x_n|| = \beta_n||\gamma f(x_n) - G(x_n)|| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Using Lemma 2.3, by letting $x^* \in F$ and $1 + v_n\theta_n := (1+v_n)^p$, where θ_n is some quantity in terms of v_n and p, we obtain the following estimate

$$\begin{aligned}
4\cdot 2^p g(\frac{1}{2}||T_i^n y_n - y_n||) &= 4\cdot 2^p g(\frac{1}{2}||T_i^n y_n - x^* + x^* - y_n||) \\
&\leq (p\cdot 2^p - 4)||x^* - y_n||^p + p\cdot 2^p\langle T_i^n y_n - x^* j_p(x^* - y_n)\rangle + 4||T_i^n y_n - x^*||^p \\
&\leq (p\cdot 2^p - 4)||x^* - y_n||^p + p\cdot 2^p\langle T_i^n y_n - y_n + y_n - x^*, j_p(x^* - y_n)\rangle \\
&\quad +4(1+v_n)^p||y_n - x^*||^p \\
&\leq (p\cdot 2^p - 4)||x^* - y_n||^p + p\cdot 2^p\langle T_i^n y_n - y_n, j_p(x^* - y_n)\rangle \\
&\quad -p\cdot 2^p\langle y_n - x^*, j_p(y_n - x^*)\rangle + 4(1+v_n\theta_n)||y_n - x^*||^p \\
&\leq p\cdot 2^p\langle y_n - T_i^n y_n, j_p(y_n - x^*)\rangle + 4v_n\theta_n||y_n - x^*||^p.
\end{aligned} \tag{3.5}$$

Therefore, by (3.1), we obtain

$$\begin{aligned}
\frac{4}{p}\delta\sum_{i\geq 1}\sigma_{in} g(\frac{1}{2}||T_i^n y_n - y_n||) &\leq \delta\sum_{i\geq 1}\sigma_{in}\langle y_n - T_i^n y_n, j_p(y_n - x^*)\rangle \\
&\quad +4v_n\theta_n||y_n - x^*||^p \\
&= \delta(1-\alpha_n)\langle y_n - x_n, j_p(y_n - x^*)\rangle \\
&\quad +4v_n\theta_n||y_n - x^*||^p \\
&\leq \delta(1-\alpha_n)||y_n - x_n||||j_p(y_n - x^*)|| \\
&\quad +4v_n\theta_n||y_n - x^*||^p.
\end{aligned}$$

Hence $\forall\ i \in \mathcal{N}_I$, we obtain

$$\begin{aligned} g(\frac{1}{2}||T_i^n y_n - y_n||) &\leq \frac{p(1-\alpha_n)}{4\sigma_{in}}||y_n - x_n||||j_p(y_n - x^*)|| \\ &+ \frac{p}{\delta}\frac{v_n}{\sigma_{in}}\theta_n||y_n - x^*||^p. \end{aligned}$$

It follows by $\forall\ i \in \mathcal{N}_I$, $\lim_{n\to\infty}\frac{v_n}{\sigma_{in}} = 0$ and (3.4) that $\lim_{n\to\infty} g(\frac{1}{2}||T_i^n y_n - y_n||) = 0$ for each $i \geq 1$. By the property of g and for each $i \geq 1$, we have

$$\lim_{n\to\infty}||T_i^n y_n - y_n|| = 0. \tag{3.6}$$

Since

$$\begin{aligned} ||T_i y_n - y_n|| &\leq ||T_i y_n - T_i(T_i^n y_n)|| + ||T_i(T_i^n y_n) - T_i^n y_n|| + ||T_i^n y_n - y_n|| \\ &\leq (L_i + 1)||y_n - T_i^n y_n|| + ||T_i^{n+1} y_n - T_i^n y_n||. \end{aligned} \tag{3.7}$$

From (3.6) and asymptotical regularity of T_i, for each $i \geq 1$ we obtain

$$\lim_{n\to\infty}||T_i y_n - y_n|| = 0. \tag{3.8}$$

Also

$$\begin{aligned} ||T_i^n x_n - x_n|| &\leq ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n|| \\ &\leq (v_n + 1)||y_n - x_n|| + ||T_i^n y_n - y_n||, \end{aligned}$$

from (3.4) and (3.6), for each $i \geq 1$, we obtain

$$\lim_{n\to\infty}||T_i^n x_n - x_n|| = 0. \tag{3.9}$$

Following the same argument of (3.7) - (3.8) and using (3.9), we obtain

$$\lim_{n\to\infty}||T_i x_n - x_n|| = 0, \quad \text{for each} \quad i \in \mathcal{N}_I. \tag{3.10}$$

Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n||y_n - y||^2, \quad \text{for all } y \in E,$$

where μ_n denote a Banach limit. Then by coercivity of the functional ϕ, $\phi(y) \to \infty$ as $||y|| \to \infty$, ϕ is continuous and convex (see Takahashi [29] and Thesis by Abdulrashid [1] for more detail). Thus, since E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u\in E}\phi(u)$ (see [1, 29] for more detail). Define the set

$$K^* := \{y \in E : \phi(y) = \min_{u\in E}\phi(u)\}.$$

Because E is a reflexive Banach space, K^* is a nonempty bounded closed and convex subset of E (see [1, 29] for more detail). Next, we show that $\forall i \in N_I$, T_i has a fixed point in K^*. By (3.8), $\lim_{n\to\infty}||T_i y_n - y_n|| = 0$, $\forall i \in \mathcal{N}_I$.

We show that $\lim_{n\to\infty}||T_i^m y_n - y_n|| = 0, \quad \forall i \in \mathcal{N}_I$ and $m \geq 1$. We prove by induction. For $m = 1$, the result follows by (3.8). Assume that for $m = k$, $\lim_{n\to\infty}||T_i^k y_n - y_n|| = 0, \quad \forall i \in \mathcal{N}_I$. Then

$$\begin{aligned}
\lim_{n\to\infty}||T_i^{k+1}y_n - y_n|| &= \lim_{n\to\infty}||T_i(T_i^k)y_n - y_n|| \\
&\leq \lim_{n\to\infty}[||T_i(T_i^k)y_n - T_i y_n|| + ||T_i y_n - y_n||] \\
&\leq \lim_{n\to\infty}[L_i||T_i^k y_n - y_n|| + ||T_i y_n - y_n||] = 0.
\end{aligned}$$

Now, for $y \in K^*$ and $v := \omega - \lim_j T_i^{m_j} y$, using the weak lower semicontinuity of ϕ and the properties of the Banach limit (see [7, 29, 30] for more detail), we have

$$\begin{aligned}
\phi(v) &\leq \liminf_{j\to\infty}\phi\Big(T_i^{m_j} y\Big) \leq \limsup_{m\to\infty}\phi\Big(T_i^m x\Big) \\
&= \limsup_{m\to\infty}\Big(\mu_n||y_n - T_i^m y||^2\Big) \\
&= \limsup_{m\to\infty}\Big(\mu_n||y_n - T_i^m y_n + T_i^m y_n - T_i^m y||^2\Big) \\
&\leq \limsup_{m\to\infty}\Big(\mu_n||T_i^m y_n - T_i^m y||^2\Big) \\
&\leq \limsup_{m\to\infty}\Big(\mu_n[(1+v_m)||y_n - y||]^2\Big) = \phi(y) \\
&\leq \inf_{u\in E}\phi(u).
\end{aligned}$$

So, $v \in K^*$ by Lemma 2.4, T_i has a fixed point in K^* for all $i \in \mathcal{N}_I$ and so $K^* \cap F \neq \emptyset$.

Let $p \in K^* \cap F$ and let $t \in (0,1)$. Then, it follows that $\phi(p) \leq \phi(p - t(G - \gamma f)p)$ and using Lemma 2.1, we obtain that

$$||y_n - p + t(G - \gamma f)p||^2 \leq ||y_n - p||^2 + 2t\langle (G - \gamma f)p, j(y_n - p + t(G - \gamma f)p)\rangle,$$

which implies that

$$\mu_n\langle(\gamma f - G)p, j(y_n - p + t(G - \gamma f)p)\rangle \leq 0.$$

Moreover,

$$\begin{aligned}
&\mu_n\langle(\gamma f - G)p, j(y_n - p)\rangle \\
&\quad = \mu_n\langle(\gamma f - G)p, j(y_n - p) - j(y_n - p + t(G - \gamma f)p)\rangle \\
&\qquad + \mu_n\langle(\gamma f - G)p, j(y_n - p + t(G - \gamma f)p)\rangle \\
&\quad \leq \mu_n\langle(\gamma f - G)p, j(y_n - p) - j(y_n - p + t(G - \gamma f)p)\rangle.
\end{aligned}$$

Since j is norm-to-weak* uniformly continuous on bounded subsets of E and monotone, we have that

$$\mu_n\langle(\gamma f - G)p, j(y_n - p)\rangle \leq 0. \tag{3.11}$$

Now, from (3.1), we obtain

$$\begin{aligned}
||y_n - p||^2 &= ||\beta_n \gamma f(x_n) + (I - \beta_n G)x_n - p||^2 \\
&= ||(I - \beta_n G)x_n - (I - \beta_n G)p + \beta_n(\gamma f(x_n) - Gp)||^2 \\
&\leq ||(I - \beta_n G)x_n - (I - \beta_n G)p||^2 + 2\beta_n \langle \gamma f(x_n) - Gp, j(y_n - p)\rangle \\
&\leq (1 - \tau\beta_n)^2 ||x_n - p||^2 + 2\beta_n \langle \gamma f(x_n) - \gamma f(p), j(y_n - p)\rangle \\
&\quad + 2\beta_n \langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
&\leq (1 - \tau\beta_n)||x_n - p||^2 + 2\beta_n \gamma\beta ||x_n - p|| ||y_n - p|| \\
&\quad + 2\beta_n \langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
&\leq [1 - \beta_n(\tau - \gamma\beta)]||x_n - p||^2 + \beta_n\gamma\beta||y_n - p||^2 \\
&\quad + 2\beta_n \langle \gamma f(p) - Gp, j(y_n - p)\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
(3.12)\quad & ||y_n - p||^2 \\
&\leq \frac{1 - \beta_n(\tau - \gamma\beta)}{1 - \beta_n\gamma\beta}||x_n - p||^2 + \frac{2\beta_n}{1 - \beta_n\gamma\beta}\langle \gamma f(p) - G(p), j(y_n - p)\rangle \\
&= \Big(1 - \frac{\beta_n[\tau - 2\gamma\beta]}{1 - \beta_n\gamma\beta}\Big)||x_n - p||^2 \\
(3.13)\quad &\quad + \frac{2\beta_n}{1 - \beta_n\gamma\beta}\langle \gamma f(p) - G(p), j(y_n - p)\rangle.
\end{aligned}$$

From (3.1) and (3.12) by denoting w_n for $2v_n + v_n^2$, we obtain

$$\begin{aligned}
||x_n - p||^2 &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta \sum_{i\geq 1} \sigma_{in}(T_i^n y_n - p)||^2 \\
&\leq [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta \sum_{i\geq 1} \sigma_{in}||T_i^n y_n - p||^2 \\
&\leq [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta(1 - \alpha_n)(1 + v_n)^2||y_n - p||^2 \\
&= [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta(1 - \alpha_n)(1 + w_n)||y_n - p||^2 \\
&\leq [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta(1 - \alpha_n)(1 + w_n)\Big(1 - \frac{\beta_n[\tau - 2\gamma\beta]}{1 - \beta_n\gamma\beta}\Big)||x_n - p||^2 \\
&\quad + \delta(1 - \alpha_n)(1 + w_n)\frac{2\beta_n}{1 - \beta_n\gamma\beta}\langle \gamma f(p) - G(p), j(y_n - p)\rangle \\
&= \Big[1 - \delta(1 - \alpha_n)(1 + w_n) \times \frac{\beta_n}{1 - \beta_n\gamma\beta}\Big([\tau - 2\gamma\beta] - \frac{w_n(1 - \beta_n\gamma\beta)}{(1 + w_n)\beta_n}\Big)\Big]||x_n - p||^2 \\
&\quad + \delta(1 - \alpha_n)(1 + w_n)\frac{2\beta_n}{1 - \beta_n\gamma\beta}\langle \gamma f(p) - G(p), j(y_n - p)\rangle.
\end{aligned}$$

Since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, it implies that $\frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n} \leq \frac{\tau - 2\gamma\beta}{2}$, for all $n \geq n_0$. This implies

$$||x_n - p||^2 \leq \frac{2\langle \gamma f(p) - G(p), j(y_n - p)\rangle}{\Big([\tau - 2\gamma\beta] - \frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n}\Big)},$$

then by (3.11), we obtain

$$\mu_n||x_n - p||^2 \le 0.$$

Thus there exists a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = p$, from (3.4) we obtain $\lim_{k\to\infty} y_{n_k} = p$. Next, we show that p solves the variational inequality (3.2). Indeed, from the relation (3.1)

$$y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n$$

we get

$$(G(y_n) - \gamma f(x_n)) = -\frac{1}{\beta_n}(I - \beta_n G)(y_n - x_n).$$

So, for any $z \in F$ we obtain

$$\begin{aligned}
\langle (G(y_n) - \gamma f(x_n)), j(y_n - z)\rangle &= -\frac{1}{\beta_n}\langle (I - \beta_n G)(y_n - x_n), j(y_n - z)\rangle \\
= & -\frac{1}{\beta_n}\langle y_n - x_n, j(y_n - z)\rangle + \langle G(y_n - x_n), j(y_n - z)\rangle \\
= & -\frac{1}{\beta_n\delta(1-\alpha_n)}\langle \delta\sum_{i\ge 1}\sigma_{in}(y_n - T_i^n y_n), j(y_n - z)\rangle \\
& +\langle G(y_n - x_n), j(y_n - z)\rangle \\
= & -\frac{1}{\beta_n\delta(1-\alpha_n)}\langle \delta\sum_{i\ge 1}\sigma_{in}(y_n - T_i^n y_n) - \delta\sum_{i\ge 1}\sigma_{in}(I - T_i^n)z, j(y_n - z)\rangle \\
& +\langle G(y_n - x_n), j(y_n - z)\rangle \\
= & -\frac{1}{\beta_n\delta(1-\alpha_n)}\langle \delta(1-\alpha_n)(y_n - z) - \delta\sum_{i\ge 1}\sigma_{in}(T_i^n y_n - T_i^n z), j(y_n - z)\rangle \\
& +\langle G(y_n - x_n), j(y_n - z)\rangle \\
= & -\frac{1}{\beta_n\delta(1-\alpha_n)}\Big(\delta(1-\alpha_n)\langle y_n - z, j(y_n - z)\rangle \\
& -\delta\sum_{i\ge 1}\sigma_{in}\langle T_i^n y_n - T_i^n z, j(y_n - z)\rangle\Big) \\
& +\langle G(y_n - x_n), j(y_n - z)\rangle \\
\le & -\frac{1}{\beta_n}||y_n - z||^2 + \frac{1}{\beta_n}(1 + v_n)||y_n - z||^2 \\
& +||G||||y_n - x_n||||y_n - z||.
\end{aligned}$$

Therefore

$$\langle (G(y_n) - \gamma f(x_n)), j(y_n - z)\rangle \le \frac{v_n}{\beta_n}||y_n - z||^2 + ||G||||y_n - x_n||||y_n - z||,$$

and so

$$\langle (G(y_{n_k}) - \gamma f(x_{n_k})), j(y_{n_k} - z)\rangle \le \frac{v_{n_k}}{\beta_{n_k}}||y_{n_k} - z||^2 + ||G||||y_{n_k} - x_{n_k}||||y_{n_k} - z||.$$

Taking the limit as $k \to \infty$ on both sides of the above inequality, and using the fact that $x_{n_k} \to p$ and $y_{n_k} \to p$ as $k \to \infty$ we obtain $\langle (G(p) - \gamma f(p)), j(p - z)\rangle \leq 0 \quad \forall z \in F$. This implies that $p \in F$ is a solution of the variational inequality (3.2). Now assume there exists another subsequence of $\{x_n\}$ and $\{y_n\}$ say $\{x_{n_j}\}$ and $\{y_{n_j}\}$ respectively such that $\lim_{j\to\infty} x_{n_j} = p^* = \lim_{j\to\infty} y_{n_j}$. Then, using (3.6) and (3.10) we have $p^* \in F$. Repeating the above argument with p replaced by p^* we can easily obtain that p^* is also a solution of the variational inequality (3.2). By the uniqueness of the solution of the variational inequality, we obtain that $p = p^*$. This completes the proof. □

Now, we prove a strong convergence of explicit scheme to a common fixed point of family of asymptotically nonexpansive maps which is also a unique solution of some variational inequality problem in uniformly convex Banach space.

Theorem 3.2. *Let E be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $G : E \to E$ be an η-strongly accretive and μ-strictly pseudocontractive mapping with $\eta+\mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\beta \in (0, 1)$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of E with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\gamma \in \Big(0, \min\{\frac{\tau}{2\beta}, \eta\}\Big)$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \beta_n = 0$, $\Sigma_{n=0}^{\infty}\beta_n = \infty$ *and* $\forall\ i \in N_I$, $\lim_{n\to\infty} \frac{\beta_n}{\sigma_{in}} = 0$

(C2) $\lim_{n\to\infty} \frac{v_n}{\beta_n} = 0$ *and* $\forall\ i \in \mathcal{N}_I$, $\lim_{n\to\infty} \frac{v_n}{\sigma_{in}} = 0$ *where* $v_n := \sup_{i\geq 1}\{v_{in}\}$

(C3) $\lim_{n\to\infty} \alpha_n = 0$.

For some fixed $\delta \in (0, 1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in E$ chosen arbitrarily,

$$(3.14) \qquad \begin{cases} x_{n+1} = [1 - \delta(1 - \alpha_n)]x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n, \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n, \qquad n \geq 0. \end{cases}$$

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, where p is the unique solution of the variational inequality problem

$$(3.15) \qquad \langle \gamma f(p) - Gp, j(q - p)\rangle \leq 0, \quad \textit{for all } q \in F.$$

Proof. By the choice of γ, $(G-\gamma f)$ is strongly accretive, thus the variational inequality (1.5) has a unique solution in F. Next, we show that $\{x_n\}$ is bounded. Let $p \in F$, since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, then $\frac{v_n}{(1+v_n)\beta_n} \to 0$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ such that $\frac{v_n}{(1+v_n)\beta_n} < \frac{\tau-\gamma\beta}{2}$, for all $n \geq n_0$.

$$\begin{aligned} ||y_n - p|| &= ||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)|| \\ &\leq \beta_n||\gamma f(x_n) - Gp|| + (1 - \beta_n \tau)||x_n - p|| \\ &\leq \Big(1 - \beta_n(\tau - \gamma\beta)\Big)||x_n - p|| + \beta_n||\gamma f(p) - Gp||. \end{aligned} \tag{3.16}$$

Using (3.16), we obtain

$$\begin{aligned} ||x_{n+1} - p|| &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta \sum_{i\geq 1} \sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p|| + \delta(1 - \alpha_n)(1 + v_n)||y_n - p|| \\ &\leq \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + v_n)[1 - \beta_n(\tau - \gamma\beta)]\Big]||x_n - p|| \\ &\quad +\delta(1 - \alpha_n)(1 + v_n)\beta_n||\gamma f(p) - Gp|| \\ &= \Big[1 + \delta(1 - \alpha_n)v_n - \beta_n\delta(1 - \alpha_n)(1 + v_n)(\tau - \gamma\beta)\Big]||x_n - p|| \\ &\quad +\delta(1 - \alpha_n)(1 + v_n)\beta_n||\gamma f(p) - Gp|| \\ &\leq \Big[1 - \beta_n\delta(1 - \alpha_n)(1 + v_n)\Big((\tau - \gamma\beta) - \frac{v_n}{(1 + v_n)\beta_n}\Big)\Big]||x_n - p|| \\ &\quad +\beta_n\delta(1 - \alpha_n)(1 + v_n)\Big((\tau - \gamma\beta) - \frac{v_n}{(1 + v_n)\beta_n}\Big)\frac{2||\gamma f(p) - Gp||}{\tau - \gamma\beta} \\ &\leq \max\Big\{||x_n - p||, \frac{2||\gamma f(p) - Gp||}{\tau - \gamma\beta}\Big\}. \end{aligned}$$

By induction, we obtain

$$||x_n - p|| \leq \max\Big\{||x_{n_0} - p||, \frac{2||\gamma f(p) - Gp||}{\tau - \gamma\beta}\Big\} \quad \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded. Also $\{f(x_n)\}, \{G(x_n)\}, \{y_n\}, \{T_i^n x_n\}$ and $\{T_i^n y_n\}$ are all bounded.

Define two sequences by $\gamma_n := (1 - \delta)\alpha_n + \delta$ and $z_n := \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n}$. From the recursion formula (3.14), observe that

$$z_n = \frac{\delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n + \alpha_n x_n}{\gamma_n}$$

which implies

$$\begin{aligned} z_{n+1} - z_n &= \frac{\delta \sum_{i\geq 1} \sigma_{in+1} T_i^{n+1} y_{n+1} + \alpha_{n+1} x_{n+1}}{\gamma_{n+1}} \\ &\quad - \frac{\delta \sum_{i\geq 1} \sigma_{in} T_i^{n} y_{n} + \alpha_{n} x_{n}}{\gamma_{n}} \\ &= \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} \Big(T_i^{n+1} y_{n+1} - T_i^{n+1} y_n \Big) \\ &\quad + \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} \Big(T_i^{n+1} y_{n} - T_i^{n} y_n \Big) \\ &\quad + \Big(\frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} - \frac{\delta \sum_{i\geq 1} \sigma_{in}}{\gamma_{n}} \Big) T_i^n y_n \\ &\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}} x_{n+1} - \frac{\alpha_n}{\gamma_n} x_n, \end{aligned}$$

therefore

$$\begin{aligned} ||z_{n+1} - z_n|| &\leq \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} \Big\| T_i^{n+1} y_{n+1} - T_i^{n+1} y_n \Big\| \\ &\quad + \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} \Big\| T_i^{n+1} y_{n} - T_i^{n} y_n \Big\| \\ &\quad + \Big| \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} - \frac{\delta \sum_{i\geq 1} \sigma_{in}}{\gamma_{n}} \Big| ||T_i^n y_n|| \\ &\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}} ||x_{n+1}|| + \frac{\alpha_n}{\gamma_n} ||x_n|| \\ &\leq \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}} (1 + v_{n+1}) \|y_{n+1} - y_n\| \\ &\quad + \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} \Big\| T_i^{n+1} y_{n} - T_i^{n} y_n \Big\| \\ &\quad + \Big| \frac{\delta \sum_{i\geq 1} \sigma_{in+1}}{\gamma_{n+1}} - \frac{\delta \sum_{i\geq 1} \sigma_{in}}{\gamma_{n}} \Big| ||T_i^n y_n|| \\ &\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}} ||x_{n+1}|| + \frac{\alpha_n}{\gamma_n} ||x_n||. \end{aligned} \tag{3.17}$$

But

$$\begin{aligned} y_{n+1} - y_n &= \beta_{n+1} \gamma \Big(f(x_{n+1}) - f(x_n) \Big) + \Big(\beta_{n+1} - \beta_n \Big) \gamma f(x_n) \\ &\quad + \Big((I - \beta_{n+1} G) x_{n+1} - (I - \beta_{n+1} G) x_n \Big) \\ &\quad + \Big((I - \beta_{n+1} G) x_n - (I - \beta_n G) x_n \Big), \end{aligned}$$

so that

$$\begin{aligned} ||y_{n+1}-y_n|| &\le \beta_{n+1}\gamma\beta||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||\gamma f(x_n)|| \\ &\quad +(1-\beta_{n+1}\tau)||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||G(x_n)|| \\ &= [1-\beta_{n+1}(\tau-\gamma\beta)]||x_{n+1}-x_n|| \\ &\quad +|\beta_{n+1}-\beta_n|\big[||\gamma f(x_n)||+||G(x_n)||\big]. \end{aligned} \tag{3.18}$$

Using (3.18) in (3.17), we obtain that

$$\begin{aligned} &||z_{n+1}-z_n||-||x_{n+1}-x_n|| \\ &\le \Big(\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})[1-\beta_{n+1}(\tau-\gamma\beta)]-1\Big)||x_{n+1}-x_n|| \\ &\quad +\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})|\beta_{n+1}-\beta_n|\big[||\gamma f(x_n)||+||G(x_n)||\big] \\ &\quad +\frac{\delta\sum_{i\ge1}\sigma_{in+1}\Big\|T_i^{n+1}y_n-T_i^n y_n\Big\|}{\gamma_{n+1}} \\ &\quad +\Big|\frac{\delta\sum_{i\ge1}\sigma_{in+1}||T_i^n y_n||}{\gamma_{n+1}}-\frac{\delta\sum_{i\ge1}\sigma_{in}||T_i^n y_n||}{\gamma_n}\Big| \\ &\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n|| \\ &\le \Big(\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})[1-\beta_{n+1}(\tau-\gamma\beta)]-1\Big)||x_{n+1}-x_n|| \\ &\quad +\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})|\beta_{n+1}-\beta_n|\big[||\gamma f(x_n)||+||G(x_n)||\big] \\ &\quad +\frac{\delta\sum_{i\ge1}\sigma_{in+1}\Big\|T_i^{n+1}y_n-T_i^n y_n\Big\|}{\gamma_{n+1}} \\ &\quad +\Big|\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}-\frac{\delta(1-\alpha_n)}{\gamma_n}\Big|M^* \\ &\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n||. \end{aligned}$$

for some $M^*>0$ and this implies

$$\limsup_{n\to\infty}(||z_{n+1}-z_n||-||x_{n+1}-x_n||)\le 0,$$

and from Lemma 2.2, we have

$$\lim_{n\to\infty}||z_n-x_n||=0.$$

Hence

$$||x_{n+1}-x_n||=(1-\gamma_n)||z_n-x_n||\to 0 \text{ as } n\to\infty. \tag{3.19}$$

Next, we show that $\lim_{n\to 0}||T_i^n y_n - x_n|| = 0$, for any $x^* \in F$. Using the same argument as in (3.5), from (3.14) and Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{4}{p}\delta\sum_{i\geq 1}\sigma_{in}g(\frac{1}{2}||T_i^n y_n - x_n||)\\
\leq\ & 4(1+v_n\theta_n)||y_n - x^*||^p + \delta\sum_{i\geq 1}\sigma_{in}\langle x_n - T_i^n y_n, j_p(x_n - x^*)\rangle\\
&-4||x^* - x_n||^p\\
\leq\ & 4(1+v_n\theta_n)[||x_n - x^*||^p + \beta_n p\langle(\gamma f - G)x_n, j_p(x_n - p)\rangle]\\
&-4||x^* - x_n||^p + \delta\sum_{i\geq 1}\sigma_{in}\langle x_n - T_i^n y_n, j_p(x_n - x^*)\rangle\\
=\ & \delta(1-\alpha_n)\langle x_{n+1} - x_n, j_p(x_n - x^*)\rangle + 4v_n\theta_n||x_n - x^*||^p\\
&+4\beta_n p(1+v_n\theta_n)\langle(\gamma f - G)x_n, j_p(x_n - p)\rangle\\
\leq\ & \delta(1-\alpha_n)||x_{n+1} - x_n||||j_p(x_n - x^*)|| + 4v_n\theta_n||x_n - x^*||^p\\
&+4\beta_n p(1+v_n\theta_n)\langle(\gamma f - G)x_n, j_p(x_n - p)\rangle.
\end{aligned}$$

Hence $\forall\ i \in \mathcal{N}_I$, we obtain

$$\begin{aligned}
g(\frac{1}{2}||T_i^n y_n - x_n||) \leq\ & \frac{p(1-\alpha_n)}{4\sigma_{in}}||x_{n+1} - x_n||||j_p(x_n - x^*)||\\
&+\frac{p}{\delta}\frac{v_n}{\sigma_{in}}\theta_n||x_n - x^*||^p\\
&+\frac{p^2}{\delta}\frac{\beta_n}{\sigma_{in}}(1+v_n\theta_n)\langle(\gamma f - G)x_n, j_p(x_n - x^*)\rangle.
\end{aligned}$$

It follows from the fact that $\lim_{n\to\infty}\frac{v_n}{\sigma_{in}} = 0,\ \forall\ i \in \mathcal{N}_i$, $\lim_{n\to\infty}\frac{\beta_n}{\sigma_{in}} = 0,\ \forall i \in \mathcal{N}_I$ and (3.19) that

$$g(\frac{1}{2}||T_i^n y_n - x_n||) \to 0 \quad \text{as} \quad n \to \infty,$$

for all $i \in \mathcal{N}_I$. Then by the property of g, it implies that for each $i \in \mathcal{N}_I$,

$$\lim_{n\to\infty}||T_i^n y_n - x_n|| = 0. \tag{3.20}$$

Also, from the recursion formula (3.14), we obtain

$$||y_n - x_n|| = \beta_n||\gamma f(x_n) - G(x_n)|| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.21}$$

which implies that for each $i \geq 1$,

$$||T_i^n y_n - y_n|| \leq ||T_i^n y_n - x_n|| + ||x_n - y_n|| \to 0 \ \text{ as } \ n \to \infty. \tag{3.22}$$

Furthermore

$$\begin{aligned}
||T_i^n x_n - x_n|| \leq\ & ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - x_n||\\
\leq\ & (1+v_n)||x_n - y_n|| + ||T_i^n y_n - x_n||.
\end{aligned}$$

It follows from (3.19) and (3.21), for each $i \geq 1$, we obtain

$$\lim_{n\to\infty} ||T_i^n x_n - x_n|| = 0 \tag{3.23}$$

Therefore

$$\begin{aligned} ||T_i x_n - x_n|| &\leq ||T_i x_n - T_i^{n+1} x_n|| + ||T_i^{n+1} x_n - T_i^{n+1} x_{n+1}|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| \\ &\leq L_i ||x_n - T_i^n x_n|| + (2 + v_{n+1}) ||x_{n+1} - x_n|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}||, \end{aligned}$$

for each $i \geq 1$, also by using (3.20) and (3.23), we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0 \quad \text{for each} \quad i \geq 1. \tag{3.24}$$

we also have

$$\begin{aligned} ||T_i y_n - y_n|| &\leq ||T_i y_n - T_i x_n|| + ||T_i x_n - x_n|| + ||x_n - y_n|| \\ &\leq (1 + L_i) ||y_n - x_n|| + ||T_i x_n - x_n||. \end{aligned}$$

This implies that,

$$\lim_{n\to\infty} ||T_i y_n - y_n|| = 0 \quad \text{for each} \quad i \geq 1. \tag{3.25}$$

Next we show that

$$\limsup_{n\to\infty} \langle (\gamma f - G)p, j(y_n - p) \rangle \leq 0.$$

For each $m \geq 0$, let $z_m \in E$ be the unique fixed point of the contraction mapping, T_n^f. Then $z_m = [1 - \delta(1 - \alpha_m)] z_m + \delta \sum_{i\geq 1} \sigma_{im} T_i^m y_m$, where $y_m =$

$\beta_m \gamma f(z_m) + (I - \beta_m G) z_m$ (see Theorem 3.1) and that $p = \lim_{m\to\infty} z_m$, so that

$$
\begin{aligned}
||z_m - y_n||^2 &\leq [1-\delta(1-\alpha_m)]||z_m - y_n||^2 + \delta \sum_{i\geq 1} \sigma_{im} ||T_i^m y_m - y_n||^2 \\
&\leq [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} \Big[||T_i^m y_m - T_i^m y_n|| + ||T_i^m y_n - y_n|| \Big]^2 \\
&\leq [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} \Big[||T_i^m y_m - T_i^m y_n||^2 + 2||T_i^m y_m - T_i^m y_n|| ||T_i^m y_n - y_n|| \\
&\quad + ||T_i^m y_n - y_n||^2 \Big]^2 \\
&\leq [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} \Big[(1+w_m)||y_m - y_n||^2 + 2(1+v_m)||y_m - y_n|| ||T_i^m y_n - y_n|| \\
&\quad + ||T_i^m y_n - y_n||^2 \Big]^2 \\
&= [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad + \delta(1-\alpha_m)(1+w_m)||y_m - y_n||^2 \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} [(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||] ||T_i^m y_n - y_n|| \\
&= [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad + \delta(1-\alpha_m)(1+w_m)||\beta_m(\gamma f(z_m) - G(z_m)) + z_m - y_n||^2 \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} [(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||] ||T_i^m y_n - y_n||
\end{aligned}
$$

$$
\begin{aligned}
&\leq [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad + \delta(1-\alpha_m)(1+w_m) \Big[||z_m - y_n||^2 \\
&\quad + 2\beta_m \langle \gamma f(z_m) - G(z_m), j(y_m - y_n) \rangle \Big] \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} [(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||] ||T_i^m y_n - y_n|| \\
&= ||z_m - y_n||^2 + \delta(1-\alpha_m) w_m ||z_m - y_n||^2 \\
&\quad + 2\beta_m \delta(1-\alpha_m)(1+w_m) \langle \gamma f(z_m) - G(z_m), j(y_m - y_n) \rangle \\
&\quad + \delta \sum_{i\geq 1} \sigma_{im} [(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||] ||T_i^m y_n - y_n||.
\end{aligned}
$$

Therefore

$$\langle \gamma f(z_m) - G(z_m), j(y_n - y_m)\rangle \leq \frac{w_m}{2\beta_m(1+w_m)}||z_m - y_n||^2 + \frac{\delta\sum_{i\geq 1}\sigma_{im}[(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||]||T_i^m y_n - y_n||}{2\beta_m(1+w_m)}.$$

Now, taking limit superior as $n \to \infty$ firstly, and then as $m \to \infty$, we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \leq 0 \tag{3.26}$$

Moreover, we note that

$$\begin{aligned}
\langle \gamma f(p) - Gp, j(y_n - p)\rangle &= \langle \gamma f(p) - Gp, j(y_n - p)\rangle - \langle \gamma f(p) - Gp, j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(p) - Gp, j(y_n - z_m)\rangle - \langle \gamma f(p) - Gz_m, j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(p) - Gz_m, j(y_n - z_m)\rangle - \langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \\
&= \langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle \\
&\quad + \langle Gz_m - Gp, j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(z_m) - \gamma f(p), j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle. \qquad (3.27)
\end{aligned}$$

Taking limit superior as $n \to \infty$ in (3.27), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle \\
&\quad + ||Gz_m - Gp||\limsup_{n\to\infty}||y_n - z_m|| \\
&\quad + ||\gamma f(z_m) - \gamma f(p)||\limsup_{n\to\infty}||y_n - z_m|| \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle \\
&\quad + \Big((1 + \frac{1}{\mu}) + \beta\gamma\Big)\|z_m - p\|\limsup_{n\to\infty}||y_n - z_m|| \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle.
\end{aligned}$$

By Theorem 3.1, $z_m \to p \in F$ as $m \to \infty$.
Since j is $norm - to - weak^*$ uniformly continuous on bounded subset of E, with uniformly Gâteaux differentiable norm, so we conclude that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle = 0,$$

therefore, using (3.26) we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(y_n - p)\rangle &\leq \limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(y_n - p)\rangle \\
&\leq 0.
\end{aligned}$$

We now conclude by showing that $x_n \to p$ as $n \to \infty$. Since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, if we denote by w_n the value $2v_n + v_n^2$, it implies that $\frac{w_n}{(1+w_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{w_n}{(1+w_n)\beta_n} \le \frac{\tau - 2\gamma\beta}{2}$, for all $n \ge n_0$. By recursion formula (3.14), we obtain

$$
\begin{aligned}
||x_{n+1} - p||^2 &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta \sum_{i \ge 1} \sigma_{in}(T_i^n y_n - p)||^2 \\
\le\ & [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta \sum_{i \ge 1} \sigma_{in} ||T_i^n y_n - p||^2 \\
\le\ & [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta(1 - \alpha_n)(1 + v_n)^2 ||y_n - p||^2 \\
\le\ & [1 - \delta(1 - \alpha_n)]||x_n - p||^2 \\
& + \delta(1 - \alpha_n)(1 + w_n)\Big[||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)||^2\Big] \\
\le\ & [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta(1 - \alpha_n)(1 + w_n)\Big[(1 - \beta_n \tau)||x_n - p||^2 \\
& + 2\beta_n \langle \gamma f(x_n) - Gp, j(y_n - p)\rangle\Big] \\
\le\ & \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + w_n)(1 - \beta_n \tau)\Big]||x_n - p||^2 \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\gamma\beta ||x_n - p|| ||y_n - p|| \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
\le\ & \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + w_n)(1 - \beta_n \tau)\Big]||x_n - p||^2 \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\gamma\beta ||x_n - p||^2 \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\gamma\beta ||x_n - p|| ||y_n - x_n|| \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
\le\ & \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + w_n)(1 - \beta_n[\tau - 2\gamma\beta])\Big]||x_n - p||^2 \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\gamma\beta ||x_n - p|| ||y_n - x_n|| \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
=\ & \Big[1 - \beta_n \delta(1 - \alpha_n)(1 + w_n)\Big((\tau - 2\gamma\beta) - \frac{w_n}{(1 + w_n)\beta_n}\Big)\Big]||x_n - p||^2 \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
& + 2\beta_n \delta(1 - \alpha_n)(1 + w_n)\gamma\beta ||x_n - p|| ||y_n - x_n|| \\
=\ & \Big[1 - \beta_n \delta(1 - \alpha_n)(1 + w_n)\Big((\tau - 2\gamma\beta) - \frac{w_n}{(1 + w_n)\beta_n}\Big)\Big]||x_n - p||^2 \\
& + \beta_n \delta(1 - \alpha_n)(1 + w_n)\Big((\tau - 2\gamma\beta) - \frac{w_n}{(1 + w_n)\beta_n}\Big) \times \\
& \frac{2[\langle \gamma f(p) - Gp, j(y_n - p)\rangle + \gamma\alpha ||x_n - p|| ||y_n - x_n||]}{\Big((\tau - 2\gamma\beta) - (w_n/(1 + w_n)\beta_n)\Big)}.
\end{aligned}
$$

Observe that $\sum \beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\beta)-\frac{w_n}{(1+w_n)\beta_n}\Big)=\infty$ and

$$\limsup\Big(\frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\beta||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\beta)-(w_n/(1+w_n)\beta_n)\Big)}\Big)\leq 0$$

Applying Lemma 2.5, we obtain $||x_n-p||\to 0$ as $n\to\infty$. This completes the proof.

□

Corollary 3.3. *Let E be a real uniformly convex Banach space whose duality mapping J is weakly sequentially continuous. Let $G: E\to E$, $f: E\to E$, $\{T_i\}_{i=1}^{\infty}$ F, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Theorem 3.2, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p\in F$, which is also the unique solution of the variational inequality*

$$\langle\gamma f(p)-Gp, j(q-p)\rangle\leq 0,\ \textit{for all } q\in F.$$

Corollary 3.4. *Let H be a real Hilbert space, $\{z_t\}_{t\in(0,1)}$, be as in Theorem 3.1. Then $\{z_t\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{\infty}$ say p, which is a unique solution of the variational inequality*

$$\langle(G-\gamma f)p, q-p\rangle\geq 0,\quad \textit{for all } q\in F.$$

Corollary 3.5. *Let H be a real Hilbert space and let C a nonempty closed convex subset of H. Let $G: H\to H$, $f: H\to H$, $\{T_i\}_{i=1}^{\infty}$ F, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Theorem 3.2. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p\in F$, which is also the unique solution of the variational inequality*

$$\langle\gamma f(p)-Gp, q-p\rangle\leq 0,\ \textit{for all } q\in F.$$

Here we present detailed example of an asymptotically nonexpansive self mapping which is also uniformly asymptotically regular by Goebel and Kirk [10], (see also [32]).

Example 3.6. Let B denote the unit ball in the Hilbert space ℓ^2. Then

$$B:=\{x=(x_1,x_2,\cdots)\in\ell^2: |x_1|\leq 1/2 \text{ and } |x_i|\leq 1 \text{ for } i=2,3,\cdots\},$$

with norm $||x||=\max\limits_{1\leq i\leq n}\{|x_i|,\ \text{ for } n=1,2,3,\cdots\}$. Define a mapping $T: B\to B$ by

$$Tx=(0,2x_1,a_2x_2,a_3x_3,\cdots)$$

where $\{a_i\}$ is a sequence of numbers such that $0<a_i<1$ and $\prod_{i=2}^{\infty}=\frac{1}{2}$. Then T is Lipschitzian and $||Tx-Ty||\leq 2||x-y||$, for all $x,y\in B$. Moreover, if $u=(1/2,0,0,\cdots)$, $v=(0,0,0,\cdots)\in B$. Then, we get

$$||Tu-Tv||=||(0,1,0,0,\cdots)||=1>1/2=||u-v||.$$

Hence, T is not nonexpansive. Now, let $x = (x_1, x_2, x_3, \cdots)$ and $y = (y_1, y_2, y_3, \cdots)$ be in B. Then

$$T^m(x) = \Big(0, 0, \cdots, 0, 2\prod_{i=2}^{m} a_i x_1, \prod_{i=2}^{m+1} a_i x_2, \cdots, \prod_{i=k}^{m+k-1} a_i x_k, \cdots\Big).$$

Thus, for $m \geq n$, we have $||T^m x - T^m y|| = 0$. And for any $k > 0$ and $n > k$, if $n > m$, then $m = n - k$. With this we obtain

$$\begin{aligned} & ||T^m x - T^m y|| \\ \leq\ & \max\Big\{2\prod_{i=2}^{m} a_i|x_1 - y_1|, \prod_{i=2}^{m+1} a_i|x_2 - y_2|, \cdots, \prod_{i=k}^{m+k-1} a_i|x_k - y_k|\Big\} \\ \leq\ & \max\Big\{2\prod_{i=2}^{m} a_i, \prod_{i=2}^{m+1} a_i, \cdots, \prod_{i=k}^{m+k-1} a_i\Big\}||x - y|| \\ \leq\ & 2\prod_{i=2}^{m} a_i||x - y|| = k_m||x - y||, \end{aligned}$$

where $k_m = 2\prod_{i=2}^{m} a_i$, $k_m \to 1$ as $m \to \infty$. Hence T is asymptotically nonexpansive mapping. Also, it is easy to see that T is uniformly asymptotic regular on B.

Remark 3.7. A prototype of the sequence $\{a_i\}$ in Example 3.6 such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} = \frac{1}{2}$ is given by $a_i := \left(\frac{1}{2}\right)^{\frac{1}{i^2}}$. That is

$$\prod_{i=2}^{\infty} a_i = \prod_{i=2}^{\infty} \left(\frac{1}{2}\right)^{\frac{1}{i^2}} = \left(\frac{1}{2}\right)^{\sum_{i=2}^{\infty}\frac{1}{i^2}}.$$

But

$$\begin{aligned} \sum_{i=2}^{\infty}\frac{1}{i^2} &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ &= \Big(\frac{1}{2^2} + \frac{1}{3^2}\Big) + \Big(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\Big) \\ &\quad + \Big(\frac{1}{8^2} + \cdots + \frac{1}{15^2}\Big) + \Big(\frac{1}{16^2} + \cdots + \frac{1}{31^2}\Big) + \cdots \\ &\leq \Big(\frac{1}{2^2} + \frac{1}{2^2}\Big) + \Big(\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}\Big) \\ &\quad + \Big(\frac{1}{8^2} + \frac{1}{8^2} + \cdots + \frac{1}{8^2}\Big) + \Big(\frac{1}{16^2} + \frac{1}{16^2} + \cdots + \frac{1}{16^2}\Big) + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \\ &= \frac{1/2}{1 - 1/2} = 1. \end{aligned}$$

Therefore

$$\prod_{i=2}^{\infty} a_i = \frac{1}{2}.$$

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