Novi Sad J. Math. Vol. 51, No. 2, 2021, 27-37
https://doi.org/10.30755/NSJOM.10382

# Some properties of almost Jordan homomorphisms on Fréchet algebras

**A. Zivari-Kazempour**[1]

**Abstract.** In this paper, we investigate the notion of almost mixed Jordan homomorphisms between Fréchet algebras. We show that if $A$ is a Fréchet algebra and $T : A \longrightarrow \mathbb{C}$ is an almost mixed Jordan homomorphism, then either $T$ is continuous, or it is a 3-homomorphism. Moreover, we prove that every almost Jordan homomorphism from a commutative Fréchet algebra $A$ into $\mathbb{C}$ is almost $n$-multiplicative.

*AMS Mathematics Subject Classification* (2010): 46H40; 47A10

*Key words and phrases:* Almost $n$-multiplicative; almost mixed $n$-Jordan homomorphism; Fréchet algebras; Automatic continuity

## 1. Introduction

An algebra $A$ over the complex field, is called a *Fréchet algebra* if it is a complete metrizable topological linear space, which is also an LMC (locally multiplicatively convex) algebra, i.e., it has a neighbourhood basis of (absolutely) convex sets $V_n$ of zero such that $V_n$ is multiplicative (idempotent) for all natural numbers $n$. The topology of a Fréchet algebra $A$ can be generated by a sequence $(p_i)$ of separating submultiplicative seminorms, i.e.,

$$p_i(xy) \leqslant p_i(x)p_i(y),$$

for all $i \in \mathbb{N}$ and $x, y \in A$, such that $p_i(x) \leqslant p_{i+1}(x)$, whenever $i \in \mathbb{N}$ and $x \in A$. If $A$ is unital then $(p_i)$ can be chosen such that $p_i(1) = 1$ for all $i \in \mathbb{N}$.

A Fréchet algebra $A$ with the above generating sequence of seminorms $(p_i)$ is denoted by $(A, (p_i))$.

Let $(A, (p_i))$ and $(B, (q_j))$ be Fréchet algebras. A linear map $T : A \longrightarrow B$ is called an *almost $n$-multiplicative*, if there exists $\varepsilon > 0$ such that for every $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ with

$$q_j(Ta_1a_2...a_n - Ta_1Ta_2...Ta_n) \leqslant \varepsilon p_i(a_1)p_i(a_2)...p_i(a_n),$$

for all $a_1, ..., a_n \in A$, and $j \in \mathbb{N}$. Also $T$ is called *almost [mixed] $n$-Jordan homomorphism*, if for every $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that

$$q_j(Ta^n - (Ta)^n) \leqslant \varepsilon p_i(a)^n, \quad [q_j(Ta^nb - (Ta)^nTb) \leqslant \varepsilon p_i(a)^n p_i(b)], \quad a, b \in A.$$

[1]Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran,
e-mail: zivari@abru.ac.ir

The *Jacobson radical* of an algebra $A$ is denoted by $radA$, and $A$ is called *semisimple* if $radA = \{0\}$. If $A$ is a commutative Fréchet algebra, then

$$radA = \bigcap \{ker\varphi : \quad \varphi \in \mathfrak{M}(A)\},$$

where $\mathfrak{M}(A)$ is the space of all continuous non-zero multiplicative linear functionals on $A$. See, for example, [6, Proposition 8.1.2].

Let $A$ and $B$ be complex algebras and $T : A \longrightarrow B$ be a linear map. Then, $T$ is called an *n-homomorphism* if for all $a_1, a_2, \cdots, a_n \in A$,

$$T(a_1 a_2 \cdots a_n) = Ta_1 Ta_2 \cdots Ta_n.$$

The concept of an $n$-homomorphism was studied for complex algebras in [7]. Also $T$ is called an *n-Jordan homomorphism* if $Ta^n = (Ta)^n$, for all $a \in A$. This notion was introduced by Herstein in [8]. For the case $n = 2$, these concepts coincide with the classical definitions of homomorphism and Jordan homomorphism, respectively. It is clear that every $n$-homomorphism is an $n$-Jordan homomorphism, but in general, the converse is false.

The following characterization of Jordan homomorphisms on Banach algebras is due to Zelazko.

**Theorem 1.1.** *[19, Theorem 1] Suppose that $A$ is a Banach algebra, which need not be commutative, and $B$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $T : A \longrightarrow B$ is a homomorphism.*

It is well-known that every homomorphism $T : A \longrightarrow B$ between Banach algebras $A$ and $B$ is automatically continuous, where $B$ is commutative and semisimple [3]. For more information about the relationship between $n$-homomorphism, $n$-Jordan homomorphism and their automatic continuity on Banach algebras, we refer the reader to [1, 2, 4, 20, 21], and the references therein.

The notion of *almost multiplicative* functions between normed algebras was introduced by K. Jarosz in [12]. A linear map $T$ between normed algebras $A$ and $B$ is called almost multiplicative if there exists $\varepsilon > 0$ such that for all $a, b \in A$,

$$\|Tab - TaTb\| \leqslant \varepsilon \|a\| \|b\|.$$

Moreover, $T$ is called *almost Jordan* if

$$\|Ta^2 - (Ta)^2\| \leqslant \varepsilon \|a\|^2, \quad a \in A.$$

Some properties of almost multiplicative functionals were investigated in [2], [13] and [14].

For the automatic continuity of homomorphisms between Fréchet and Banach algebras, one may refer to the monographs of Dales [3], M. Fragoulopoulou [5], T. G. Honary [10], T. Husain [11], K. Jarosz [12], and E. A. Michael [15].

In this paper, we investigate almost mixed Jordan homomorphism between Fréchet algebras and we obtain some results on the automatic continuity of such maps.

We show that every almost Jordan homomorphism from commutative Fréchet algebra $A$ into $\mathbb{C}$ is almost $n$-multiplicative.

Throughout this paper, $A$ and $B$ denote Fréchet algebras equipped with the generating sequence of seminorms $(p_i)$ and $(q_j)$, respectively.

## 2. Almost Mixed Jordan homomorphism

We first introduce the concept of an *almost $n$-multiplicative* and *almost [mixed] $n$-Jordan homomorphism* between Fréchet algebras.

**Definition 2.1.** Let $A$ and $B$ be Fréchet algebras. A linear map $T : A \longrightarrow B$ is called *almost $n$-multiplicative* if there exists $\varepsilon > 0$ such that for every $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ with

$$q_j(Ta_1a_2...a_n - Ta_1Ta_2...Ta_n) \leqslant \varepsilon p_i(a_1)p_i(a_2)...p_i(a_n),$$

for every $a_1, ..., a_n \in A$, and $j \in \mathbb{N}$.

**Definition 2.2.** A linear map $T : A \longrightarrow B$ between Fréchet algebras $A$ and $B$ is called an *almost mixed $n$-Jordan homomorphism* if for every $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that

$$q_j(Ta^nb - (Ta)^nTb) \leqslant \varepsilon p_i(a)^n p_i(b),$$

for every $a, b \in A$, and it is called an *almost $n$-Jordan homomorphism* if for every $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that

$$q_j(Ta^n - (Ta)^n) \leqslant \varepsilon p_i(a)^n, \quad a \in A.$$

In the above definitions, if $n = 2$, then we speak about an almost multiplicative and almost mixed Jordan homomorphism, respectively.

The concept of mixed $n$-Jordan homomorphisms was introduced by Neghabi, Bodaghi and Zivari-Kazempour in [16] for Banach algebras. Some significant results concerning almost Jordan homomorphisms on Fréchet algebras were obtained in [17].

Since $(q_j)$ is a separating sequence of seminorms on $B$, hence both definitions turn out to be $n$-multiplicative and [mixed] $n$-Jordan homomorphism, whenever $\varepsilon = 0$, respectively. Moreover, any almost $(n+1)$-multiplicative homomorphism is an almost mixed $n$-Jordan homomorphism, and every almost mixed $n$-Jordan homomorphism is an almost $(n+1)$-Jordan homomorphism for every $\varepsilon \geqslant 0$.

*Remark* 2.3. In the case when $B = \mathbb{C}$, a linear functional $T$ on a Fréchet algebra $A$ is almost $n$-multiplicative, if there exists $m \in \mathbb{N}$ such that

$$|Ta_1a_2...a_n - Ta_1Ta_2...Ta_n| \leqslant \varepsilon p_m(a_1)p_m(a_2)...p_m(a_n),$$

for every $a_1, ..., a_n \in A$. Since the generating sequence $(p_i)$ in the Fréchet algebra $A$ is an increasing sequence, the inequality

$$|Ta_1a_2...a_n - Ta_1Ta_2...Ta_n| \leqslant \varepsilon p_k(a_1)p_k(a_2)...p_k(a_n),$$

holds for all $k \geqslant m$. The same is true for an almost [mixed] $n$-Jordan homomorphism.

We recall that a Fréchet algebra $A$ is called uniform if $p_i(a^2) = (p_i(a))^2$, for each $i \in \mathbb{N}$ and for all $a \in A$.

The following result has appeared in [6, page 73], without proof. For the proof one may refer to [9, Proposition 2.7].

*Remark* 2.4. Let $A$ and $B$ be Fréchet algebras with generating sequences of seminorms $(p_i)$ and $(q_i)$, respectively. If $\varphi : A \longrightarrow B$ is a linear operator, then $\varphi$ is continuous if and only if for each $j \in \mathbb{N}$ there exist $i \in \mathbb{N}$ and a constant $c_j > 0$ such that

$$q_j(\varphi(a)) \leqslant c_j p_i(a),$$

for every $a \in A$.

In the case that $B$ is a uniform Fréchet algebra and $\varphi : A \longrightarrow B$ is a continuous homomorphism, we may choose $c_j = 1$ for all $j \in \mathbb{N}$.

If $A$ is a Fréchet algebra and $T : A \longrightarrow \mathbb{C}$ is an almost mixed Jordan homomorphism, then there exists smallest $m \in \mathbb{N}$ such that

$$|Ta^2b - (Ta)^2Tb| \leqslant \varepsilon p_m(a)^2 p_m(b), \quad a, b \in A.$$

In the sequel we use this fixed $m$ for every almost mixed Jordan homomorphism.

We commence with the next result, wherein $Z(A) = \{a \in A : \ ax = xa \ \ x \in A\}$ is the center of $A$.

**Theorem 2.5.** *Let $A$ be a Fréchet algebra, and let $f$ be an almost $n$-multiplicative linear functional on $A$. If $f(a) = 1$ for some $a \in Z(A)$, then the linear functional $T : x \longmapsto f(ax)$ is an almost mixed Jordan homomorphism.*

*Proof.* Suppose that $f$ is an almost $n$-multiplicative linear functional. Then there exists $\varepsilon > 0$ such that

$$|fa_1a_2...a_n - fa_1fa_2...fa_n| \leqslant \varepsilon p_m(a_1)p_m(a_2)...p_m(a_n),$$

for every $a_1, ..., a_n \in A$. For each $x, y \in A$, we have

$$\begin{aligned}
&|Tx^2y - (Tx)^2Ty| \\
&\quad = \ |f(ax^2y) - f(ax)f(ax)f(ay)| \\
&\quad = \ |f(ax^2y) \pm f(a^{n-1}x^2ya) - f(ax)f(ax)f(ay)| \\
&\quad \leqslant \ |f(ax^2y) - f(a^{n-1}x^2ya)| + |f(a^{n-1}x^2ya) - f(ax)f(ax)f(ay)| \\
&\quad \leqslant \ |f(a)^{n-2}f(ax^2y)f(a) - f(a^{n-1}x^2ya)| \\
&\qquad\ + |f(a^{n-1}x^2ya) - f(a)^{n-3}f(ax)f(ax)f(ay)| \\
&\quad \leqslant \ \delta p_m(x)^2 p_m(y).
\end{aligned}$$

Thus, $T$ is an almost mixed Jordan homomorphism, for $\delta = 2\varepsilon p_m(a)^n$. $\square$

**Corollary 2.6.** *Let $A$ be Fréchet algebra, and let $f$ be an almost $n$-multiplicative linear functional on $A$. If $f(a) = 1$ for some $a \in Z(A)$, then the linear functional $T(x) := f(ax)$ is an almost 3-Jordan homomorphism.*

The following theorem is similar to Theorem 2.8 of [17].

**Theorem 2.7.** *Let $T : A \longrightarrow B$ be an almost mixed Jordan homomorphism between two commutative Fréchet algebras. Then for all $a, b, c \in A$,*

$$q_j(Tabc - TaTbTc) \leqslant \varepsilon \ p_i(c)(p_i(a)^2 + p_i(b)^2).$$

*Proof.* Suppose that $A$ and $B$ are commutative. Then for all $a, b, c \in A$,

$$T(a+b)^2c-(T(a+b))^2Tc+(Ta)^2Tc-Ta^2c+(Tb)^2Tc-Tb^2c = 2\big(Tabc-TaTbTc\big).$$

Since $2p_i(a)p_i(b) \leqslant p_i(a)^2 + p_i(b)^2$, hence by the assumption for each $j$ there exists $i$ such that

$$\begin{aligned}
2\big(q_j(Tabc - TaTbTc)\big) &= q_j\big((T(a+b))^2c - T(a+b)^2Tc\big) \\
&+ q_j((Ta)^2Tc - Ta^2c) + q_j((Tb)^2Tc - Tb^2c) \\
&\leqslant \varepsilon \ p_i(c) \ \big(p_i(a+b)^2 + p_i(a)^2 + p_i(b)^2\big) \\
&\leqslant \varepsilon \ p_i(c) \ \big([p_i(a) + p_i(b)]^2 + p_i(a)^2 + p_i(b)^2\big) \\
&\leqslant \varepsilon \ p_i(c) \ \big(2p_i(a)p_i(b) + 2p_i(a)^2 + 2p_i(b)^2\big) \\
&\leqslant 3\varepsilon \ p_i(c) \ (p_i(a)^2 + p_i(b)^2).
\end{aligned}$$

Therefore,

$$q_j(Tabc - TaTbTc) \leqslant \varepsilon_1 \ p_i(c)(p_i(a)^2 + p_i(b)^2),$$

for $\varepsilon_1 = \frac{3}{2}\varepsilon$ and every $a, b, c \in A$. This completes the proof. □

For the proof of the following result we adopt the same method as in [9, Lemma 2.8].

**Lemma 2.8.** *Let $A$ be a commutative Fréchet algebra and $T : A \longrightarrow \mathbb{C}$ be an almost mixed Jordan homomorphism. Then, for every $a, b, c, x \in A$ we have*

$$|Tx|^2 \cdot |Tabc - TaTbTc| \leqslant \varepsilon p_m(x)^2 p_m(c)[(p_m(a) + p_m(b))^2 + 2|TaTb|]. \tag{2.1}$$

*Proof.* Using Theorem 2.7, for every $a, b, c, x \in A$ we have

$$\begin{aligned}
|Tx|^2|Tabc - TaTbTc| &= |Tabc(Tx)^2 - TaTbTc(Tx)^2| \\
&\leqslant |Tabc(Tx)^2 - Tabcx^2| + |Tabcx^2 - TaTbTcx^2| \\
&\quad + |TaTbTcx^2 - TaTbTc(Tx)^2| \\
&\leqslant \varepsilon p_m(abc)\big(p_m(x)^2 + p_m(x)^2\big) \\
&\quad + \varepsilon p_m(cx^2)\big(p_m(a)^2 + p_m(b)^2\big) \\
&\quad + \varepsilon |TaTb| \ p_m(c)\big(p_m(x)^2 + p_m(x)^2\big) \\
&\leqslant \varepsilon p_m(x)^2 p_m(c)[2p_m(a)p_m(b) \\
&\quad + p_m(a)^2 + p_m(b)^2 + 2|TaTb|] \\
&\leqslant \varepsilon p_m(x)^2 p_m(c)[(p_m(a) + p_m(b))^2 + 2|TaTb|],
\end{aligned}$$

as required. □

**Corollary 2.9.** *Let $A$ be a commutative Fréchet algebra and $T : A \longrightarrow \mathbb{C}$ be an almost mixed Jordan homomorphism. If there exists $x \in A$ such that $p_m(x) = 0$ but $Tx \neq 0$, then $T$ is a $(2n+1)$-homomorphism for all $n \in \mathbb{N}$.*

*Proof.* It follows from Lemma 2.8, that $Tabc = TaTbTc$, for all $a, b, c \in A$. Thus, $T$ is a 3-homomorphism and so it is $(2n+1)$-homomorphism for all $n \in \mathbb{N}$. $\square$

The next result, which is the main result of this paper, characterizes almost mixed Jordan homomorphisms. However, its proof is similar to [9, Theorem 3.3].

**Theorem 2.10.** *Let $A$ be a commutative Fréchet algebra and $T : A \longrightarrow \mathbb{C}$ be an almost mixed Jordan homomorphism. Then, at least one of the following holds:*

*(i) $T$ is a 3-homomorphism,*

*(ii) $T$ is continuous.*

*Proof.* Suppose that $T$ is an almost mixed Jordan homomorphism for some $\varepsilon > 0$. Then, by Theorem 2.7, we have

$$|Tabc - TaTbTc| \leqslant \varepsilon\, p_m(c)(p_m(a)^2 + p_m(b)^2), \tag{2.2}$$

for all $a, b, c \in A$. Set $\xi = 1 + \frac{1+\sqrt{1+4\varepsilon}}{2}$. If for all $a \in A$,

$$|Ta| \leqslant \xi p_m(a), \tag{2.3}$$

then $T$ is continuous. If (2.3) does not hold for some $u \in A$, then we have

$$|Tu| > \xi p_m(u), \tag{2.4}$$

and hence $T$ is not continuous by Remark 2.4. Thus, $Tu \neq 0$. If $p_m(u) = 0$, then $T$ is a 3-homomorphism by Corollary 2.9. Now assume that $p_m(u) \neq 0$ in (2.4). We may assume without loss of generality that $p_m(u) = 1$ and $|Tu| > \xi$. Therefore, we can write $|Tu| = \xi + r$, for some $r > 0$. Since $p_m(u) = 1$ by (2.2) we have

$$|Tu^3 - (Tu)^3| \leqslant 2\varepsilon. \tag{2.5}$$

On the other hand,

$$|Tu^3| = |(Tu)^3 - \big((Tu)^3 - Tu^3\big)| \geqslant |(Tu)^3| - |Tu^3 - (Tu)^3|. \tag{2.6}$$

Since $\xi > 2$ and $\xi^3 - \xi \geqslant 2\varepsilon$, so by (2.5) and (2.6) we get

$$|Tu^3| \geqslant |Tu|^3 - 2\varepsilon = (\xi + r)^3 - 2\varepsilon > \xi^3 + 3\xi^2 r + \xi - \xi^3 > \xi + r. \tag{2.7}$$

We prove by induction that

$$|T(u^{3^n})| > \xi + nr. \tag{2.8}$$

To see this

$$\begin{aligned}|Tu^{3^{n+1}}| &= |Tu^{3^n}Tu^{3^n}Tu^{3^n} - \big(Tu^{3^n}Tu^{3^n}Tu^{3^n} - T(u^{3^n}u^{3^n}u^{3^n})\big)| \\ &\geqslant |(Tu^{3^n})^3| - 2\varepsilon \geqslant |Tu^{3^n}|^3 + \xi - \xi^3,\end{aligned}$$

Hence,

$$\begin{aligned}|Tu^{3^{n+1}}| &\geqslant \big(\xi + nr\big)^3 + \xi - \xi^3 \\ &> \xi + 3\xi^2 nr \\ &> \xi + 3nr \\ &\geqslant \xi + (n+1)r.\end{aligned}$$

Thus, (2.8) holds for all $\mathbb{N}$. To prove that $T$ is a 3-homomorphism, let $a, b, c \in A$ and note that $Tu^{3^n} \neq 0$ for each $n \in \mathbb{N}$, by (2.8). Let $\alpha = [(p_m(a) + p_m(b))^2 + 2|TaTb|]$. By taking $x = u^{3^n}$ in (2.1), it follows from (2.8) that

$$\begin{aligned}|Tabc - TaTbTc| &\leqslant \frac{\varepsilon p_m(u^{3^n})^2 p_m(c)\alpha}{|Tu^{3^n}|^2} \\ &\leqslant \frac{\varepsilon [p_m(u)^{3^n}]^2 p_m(c)\alpha}{|Tu^{3^n}|^2} \\ &\leqslant \frac{\varepsilon p_m(c)\alpha}{(\xi + nr)^2}. \qquad (2.9)\end{aligned}$$

Letting $n \to \infty$ in (2.9), we obtain that $Tabc = TaTbTc$. Therefore, $T$ is a 3-homomorphism. □

**Corollary 2.11.** *Let $T : A \longrightarrow B$ be an almost mixed Jordan homomorphism between commutative Fréchet algebras. If $B$ is semisimple, then $T$ is either an 3-homomorphism or it is continuous.*

*Proof.* Suppose that $T$ is not a 3-homomorphism. Let $\varphi \in \mathfrak{M}(B)$, then it is routine to check that $\varphi \circ T : A \longrightarrow \mathbb{C}$ is an almost mixed Jordan homomorphism, and hence it is continuous by Theorem 2.10.

Now, suppose that $a_n \longrightarrow 0$ in $A$ and $Ta_n \longrightarrow b$ in $B$. Then, by the continuity of $\varphi \circ T$, we have $(\varphi \circ T)(a_n) \longrightarrow 0$. On the other hand, it follows from the continuity of $\varphi : B \longrightarrow \mathbb{C}$ that

$$(\varphi \circ T)(a_n) = \varphi(Ta_n) \longrightarrow \varphi(b).$$

Consequently, $\varphi(b) = 0$ and since $\varphi \in \mathfrak{M}(B)$ was arbitrary, we get $b = 0$. Therefore, $T$ is continuous by the Closed Graph Theorem. □

The next result follows from Theorem 2.5 and Theorem 2.10.

**Corollary 2.12.** *Suppose that $A$ is a commutative Fréchet algebra, and $f$ is an almost $n$-multiplicative linear functional. If $f(a) = 1$ for some $a \in A$, then the linear functional $T : x \longmapsto f(ax)$ is either a 3-homomorphism or it is continuous.*

**Example 2.13.** *Let $K_i = [-i, i]$ for $i \in \mathbb{N}$, and consider the Fréchet algebra $A = C(\mathbb{R})$, the algebra of continuous complex-valued functions on $\mathbb{R}$ with the compact open topology, equipped with the seminorms*

$$p_i(f) = sup\{|f(x)| : \quad x \in K_i\}.$$

*Let $\xi = \frac{1+\sqrt{1+4\delta}}{2}$, where $\delta$ is positive. For a fixed $a \in \mathbb{R}$, we define $T : A \longrightarrow \mathbb{C}$ by $Tf = \xi f(a)$. Then there exists $m \in \mathbb{N}$ such that $a \in K_m$. Since $\xi^2 = \xi + \delta$, hence for all $f, g \in A$,*

$$\begin{aligned}
|Tf^2g - (Tf)^2Tg| &= \big|\xi(f^2g)a - (\xi fa)^2(\xi ga)\big| \\
&= \big|\xi(fa)^2(ga) - (\xi+\delta)\xi(fa)^2ga\big| \\
&= \big| - \delta\xi(fa)^2(ga) - \delta(fa)^2(ga)\big| \\
&\leqslant (\xi+1)\delta p_m(f)^2 p_m(g).
\end{aligned}$$

*Therefore, $T$ is an almost mixed Jordan homomorphism for $\varepsilon = (\xi+1)\delta$. Since $T$ is not a 3-homomorphism, hence, by Theorem 2.10, $T$ is continuous.*

**Theorem 2.14.** *Let $A$ be a commutative Fréchet algebra, and $T : A \longrightarrow \mathbb{C}$ be an almost Jordan homomorphism. Then $T$ is almost $n$-multiplicative, for all $n \geqslant 2$.*

*Proof.* Let $T$ be an almost Jordan homomorphism. Then by Theorem 2.11 of [17], either $T$ is a homomorphism or it is continuous. If $T$ is a homomorphism, then it is $n$-multiplicative. In particular, $T$ is almost $n$-multiplicative for every $\varepsilon \geqslant 0$. Now we assume that $T$ is continuous. By Remark 2.4, for all $a \in A$,

$$|Ta| \leqslant p_m(a). \tag{2.10}$$

It follows from Theorem 2.8 of [17], that $T$ is almost multiplicative. Hence there exists $\varepsilon_1 > 0$ such that

$$|Tab - TaTb| \leqslant \varepsilon_1 p_m(a)p_m(a), \quad a, b \in A. \tag{2.11}$$

By (2.10) and (2.11), for all $a, b, c \in A$, we have

$$\begin{aligned}
|Tabc - TaTbTc| \leqslant& |Tabc - TabTc| + |TabTc - TaTbTc| \\
\leqslant& \varepsilon_1 p_m(ab)p_m(c) + |Tab - TaTb||Tc| \\
\leqslant& \varepsilon_1 p_m(a)p_m(b)p_m(c) + \varepsilon_1 p_m(a)p_m(b)|Tc| \\
\leqslant& \varepsilon' p_m(a)p_m(b)p_m(c).
\end{aligned}$$

Thus, $T$ is almost 3-multiplicative for $\varepsilon' = 2\varepsilon_1$. Now let $T$ be almost $n$-multiplicative for some fixed $n \in \mathbb{N}$. Then there exists $\varepsilon_2 > 0$ such that

$$|Ta_1a_2...a_n - Ta_1Ta_2...Ta_n| \leqslant \varepsilon_2 p_m(a_1)p_m(a_2)...p_m(a_n), \tag{2.12}$$

for all $a_1, a_2, ..., a_n \in A$. Hence by (2.10), (2.11) and (2.12), we get

$$\begin{aligned}|Ta_1a_2...a_{n+1} - Ta_1Ta_2...Ta_{n+1}| \leqslant &|Ta_1a_2...a_{n+1} - Ta_1a_2Ta_3...Ta_nTa_{n+1}| \\ &+|Ta_1a_2Ta_3...Ta_nTa_{n+1} - Ta_1Ta_2...Ta_{n+1}| \\ \leqslant &\varepsilon_2 p_m(a_1a_2)p_m(a_3)...p_m(a_{n+1}) \\ &+|Ta_1a_2 - Ta_1Ta_2|\big(|Ta_3|...|Ta_{n+1}|\big) \\ \leqslant &\varepsilon_2 p_m(a_1)p_m(a_2)...p_m(a_{n+1}) \\ &+\varepsilon_1 p_m(a_1)p_m(a_2)\big(p_m(a_3)...p_m(a_{n+1})\big) \\ \leqslant &\varepsilon p_m(a_1)p_m(a_2)...p_m(a_{n+1}).\end{aligned}$$

Consequently, $T$ is almost $(n+1)$-multiplicative for $\varepsilon = \varepsilon_1 + \varepsilon_2$. This finishes the proof. □

It is known that every Jordan homomorphism is an $n$-Jordan homomorphism [18]. The next result generalizes this property for almost Jordan homomorphisms.

**Corollary 2.15.** *Let $A$ be a commutative Fréchet algebra, and $T : A \longrightarrow \mathbb{C}$ be an almost Jordan homomorphism. Then $T$ is an almost $n$-Jordan homomorphism for all $n \geqslant 2$.*

**Proposition 2.16.** *Let $T : A \longrightarrow B$ be a linear map between Fréchet algebras such that*

$$q_j(Ta^nb - (Ta)^nTb) \leqslant \varepsilon(p_i(a) + p_i(b)), \tag{2.13}$$

*for all $a, b \in A$ and for some $\varepsilon > 0$. Then, $T$ is an $(n+1)$-Jordan homomorphism.*

*Proof.* Assume that the inequality (2.13) holds for all $a, b \in A$. Replacing $b$ by $a$ in (2.13), we find

$$q_j(Ta^{n+1} - (Ta)^{n+1}) \leqslant 2\varepsilon p_i(a), \tag{2.14}$$

for all $a \in A$. Setting $a = 2^m x$, we obtain

$$q_j(Tx^{n+1} - (Tx)^{n+1}) \leqslant \frac{\varepsilon 2^{m+1}}{2^{m(n+1)}} p_i(x), \tag{2.15}$$

for all $x \in A$. Letting $m \to \infty$, we obtain $Tx^{n+1} = (Tx)^{n+1}$ and so $T$ is an $(n+1)$-Jordan homomorphism. □

*Remark* 2.17. Let $A$ and $B$ be Fréchet algebras and $T : A \longrightarrow B$ be a linear map such that

$$q_j(Ta^nb - (Ta)^nTb) \leqslant \varepsilon p_i(a)^{np} p_i(b), \tag{2.16}$$

for some $\varepsilon > 0$ and $a, b \in A$. If $p < 1$, then $T$ is an $(n+1)$-Jordan homomorphism. Indeed, replacing $a$ by $2^m x$ in (2.16), we get

$$q_j(Tx^nb - (Tx)^nTb) \leqslant \varepsilon 2^{nm(p-1)} p_i(x)^{np} p_i(b).$$

Since $(q_j)$ is a separating sequence of seminorms on $B$, by letting $m \to \infty$, for all $x, b \in A$, we get $T(x^n b) = (Tx)^n Tb$. Consequently, $T$ is mixed $n$-Jordan, and so it is an $(n+1)$-Jordan homomorphism.

We recall that this situation fails for $p = 1$. For example, let $A = C(\mathbb{R})$, and let $T : A \longrightarrow \mathbb{C}$ defined by $Tf = \xi f(a)$ as in Example 2.13. Then $T$ satisfies (2.16) with $n = 2$, $p = 1$ and $\varepsilon = (\xi + 1)\delta$. However, $T$ is not an 3-Jordan homomorphism.

## Acknowledgement

The author would like to thank the referees for providing valuable comments to improve the content of this paper.

This research was partially supported by the Ayatollah Borujerdi University with Grant No. 15664-171288.

## References

[1] An, G. Characterizations of $n$-Jordan homomorphisms. *Linear Multilinear Algebra 66*, 4 (2018), 671–680.

[2] Ansari-Piri, E., Shayanpour, H., and Heidarpour, Z. Approximately $n$-multiplicative and approximately additive functions in normed algebras. *Bull. Math. Anal. Appl. 7*, 1 (2015), 12–19.

[3] Dales, H. G. *Banach algebras and automatic continuity*, vol. 24 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 2000. Oxford Science Publications.

[4] Eshaghi Gordji, M., Jabbari, A., and Karapinar, E. Automatic continuity of surjective $n$-homomorphisms on Banach algebras. *Bull. Iranian Math. Soc. 41*, 5 (2015), 1207–1211.

[5] Fragoulopoulou, M. *Topological algebras with involution*, vol. 200 of *North-Holland Mathematics Studies.* Elsevier Science B.V., Amsterdam, 2005.

[6] Goldmann, H. *Uniform Fréchet algebras*, vol. 162 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1990.

[7] Hejazian, S., Mirzavaziri, M., and Moslehian, M. S. $n$-homomorphisms. *Bull. Iranian Math. Soc. 31*, 1 (2005), 13–23, 88.

[8] Herstein, I. N. Jordan homomorphisms. *Trans. Amer. Math. Soc. 81* (1956), 331–341.

[9] Honary, T. G. Omidi, M., and Sanatpour, A. H. Almost $n$-multiplicative maps between Fréchet algebras. *Int. J. Nonlinear Anal. Appl. 8*, 1 (2017), 187–195.

[10] Honary, T. G. Automatic continuity of homomorphisms between Banach algebras and Fréchet algebras. *Bull. Iranian Math. Soc. 32*, 2 (2006), 1–11, 77.

[11] Husain, T. *Multiplicative functionals on topological algebras*, vol. 85 of *Research Notes in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1983.

[12] Jarosz, K. *Perturbations of Banach algebras*, vol. 1120 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1985.

[13] Johnson, B. E. Approximately multiplicative functionals. *J. London Math. Soc. (2) 34*, 3 (1986), 489–510.

[14] Johnson, B. E. Approximately multiplicative maps between Banach algebras. *J. London Math. Soc. (2) 37*, 2 (1988), 294–316.

[15] Michael, E. A. Locally multiplicatively-convex topological algebras. *Mem. Amer. Math. Soc. 11* (1952), 79.

[16] Neghabi, M., Bodaghi, A., and Zivari-Kazempour, A. Characterization of mixed $n$-Jordan and pseudo $n$-Jordan homomorphisms. *Filomat 34*, 6 (2020), 1989–2002.

[17] Omidi, M. R., Oulfatian, B., and Zivari-Kazempour, A. Almost Jordan homomorphisms on Fréchet algebras. *Quaest. Math. 43*, 10 (2020), 1467–1476.

[18] Palmer, T. W. *Banach algebras and the general theory of $^*$-algebras. Vol. I*, vol. 49 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1994. Algebras and Banach algebras.

[19] Żelazko, W. A characterization of multiplicative linear functionals in complex Banach algebras. *Studia Math. 30* (1968), 83–85.

[20] Zivari-Kazempour, A. A characterization of 3-Jordan homomorphism on Banach algebras. *Bull. Aust. Math. Soc. 93*, 2 (2016), 301–306.

[21] Zivari-Kazempour, A. Automatic continuity of $n$-Jordan homomorphisms on Banach algebras. *Commun. Korean Math. Soc. 33*, 1 (2018), 165–170.

*Received by the editors December 13, 2019*
*First published online November 1, 2020*