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A note on $(m,n)^*$-paranormal operators

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Abstract. In this note, as a further generalization of paranormal operators, we prove some properties of $(m,n)^*$-paranormal operators on Hilbert space. The equality of approximate point spectrum with joint approximate point spectrum and also that of point spectrum with joint point spectrum are proved. We also show that this class possesses SVEP under a given condition.

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1. Introduction

In this note, we present a class of non-normal operators, which is a topic of growing interest. We denote the set of all complex numbers by $\mathbb{C}$, the set of all integers by $\mathbb{Z}$ and the set of all real numbers by $\mathbb{R}$. We assume that $\mathcal{H}$ is an infinite dimensional separable complex Hilbert space with inner product $\langle .,. \rangle$, unless stated otherwise and $B(\mathcal{H})$ is a C^*-algebra of bounded linear operators acting on $\mathcal{H}$. Throughout the manuscript, $\sigma(T)$, $N(T)$ and $R(T)$ denote the spectrum, the null space and the range space of the operator T, respectively. An operator T is said to be hyponormal if $T^*T - TT^* \geq 0$ [8], and paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [6]. For a fixed positive real number m, an operator T is called m^*-paranormal if $\|T^*x\|^2 \leq m\|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [3].

In [5], new classes of operators, namely, (m,n)-paranormal operators and $(m,n)^*$-paranormal operators are introduced. For a positive real number m and a positive integer n, an operator T in $B(\mathcal{H})$ is called (m,n)-paranormal if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, and $(m,n)^*$-paranormal operator if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$.

The paper is organized as follows. In Sect. 2, we prove algebraic properties for the class of $(m,n)^*$-paranormal operators. We prove that the restriction of $(m,n)^*$-paranormal operator to an invariant subspace is again $(m,n)^*$-paranormal. Also, we show that the tensor product of two $(m,n)^*$-paranormal operators need not be $(m,n)^*$-paranormal and the direct sum of countably many $(m,n)^*$-paranormal operators is $(m,n)^*$-paranormal.

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In Sect. 3, we prove spectral properties for the class of $(m,n)^*$-paranormal operators. Under some conditions, we prove the equality of the approximate point spectrum and the joint approximate point spectrum for $(m,n)^*$-paranormal operators. Moreover, we show that the point spectrum coincides with the joint point spectrum for the class of $(m,n)^*$-paranormal operators. We also discuss SVEP for the same class of operators.

2. Algebraic properties of $(m,n)^*$-paranormal operators

First, we give an example of an $(m,n)^*$-paranormal operator.

Example 2.1. Let $\mathcal{H} = l^2(\mathbb{N},\mathbb{C})$ and $T \in B(\mathcal{H})$ be the unilateral shift operator defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$ with the adjoint of T given by $T^*(x_1, x_2, ...) = (x_2, x_3, ...)$. For $m = 5$ and $n = 8$, T is $(5,8)^*$-paranormal.

The following proposition implies that every part of an $(m,n)^*$-paranormal operator is $(m,n)^*$-paranormal.

Proposition 2.2. *Let T be a $(m,n)^*$-paranormal operator and W is an invariant subspace of $\mathcal{H}$ under T. Then the restriction of T on W, $T|_W$ is also $(m,n)^*$-paranormal.*

Proof. This proof is similar to the proof of [9, Proposition 1], so we omit it here. □

The following theorem establishes that the direct sum of countably many $(m,n)^*$-paranormal operators is again $(m,n)^*$-paranormal.

Theorem 2.3. *Let $\mathcal{H}$ be a direct sum of countably indexed family of Hilbert spaces $\{\mathcal{H}_i : i \in I\}$ such that $\mathcal{H}_i \cong \mathcal{H}_j$ for all i, $j \in I$. Suppose for each i, an operator T_i on $\mathcal{H}_i$ is $(m,n)^*$-paranormal. If the operator T is the direct sum of T_i for all $i \in I$, then T is also $(m,n)^*$-paranormal.*

Proof. For each i, T_i is $(m,n)^*$-paranormal, so by [5, Theorem 3.1]

$$m^{\frac{2}{n+1}} T_i^{*n+1} T_i^{n+1} - (n+1)a^n T_i T_i^* + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0,$$

for each $a > 0$, which implies

$$\begin{aligned}
&m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} + m^{\frac{2}{n+1}} n a^{n+1} I \\
&= m^{\frac{2}{n+1}} \left(\oplus_{i\in I} {T_i}^{*n+1}\right) \left(\oplus_{i\in I} {T_i}^{n+1}\right) + m^{\frac{2}{n+1}} n a^{n+1} \left(\oplus_{i\in I} I_i\right) \\
&= m^{\frac{2}{n+1}} \left(\oplus_{i\in I} {T_i}^{*n+1} {T_i}^{n+1}\right) + m^{\frac{2}{n+1}} n a^{n+1} \left(\oplus_{i\in I} I_i\right) \\
&\geq (n+1)a^n \left(\oplus_{i\in I} T_i T_i^*\right) \\
&= (n+1)a^n \left(\oplus_{i\in I} T_i\right) \left(\oplus_{i\in I} {T_i}^*\right) \\
&= (n+1)a^n T T^*.
\end{aligned}$$

Equivalently,

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T T^* + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0,$$

for each $a > 0$. Now, by [5, Theorem 3.1], T is $(m,n)^*$-paranormal. □

Lemma 2.4. *Let T be a shift operator defined by $T(e_k) = w_{k-1}e_{k-1}$ on a Hilbert space $l^2(\mathbb{Z},\mathbb{C})$ with non zero weights (w_k), and orthonormal basis (e_k), where k is any integer. Then T is $(m,n)^*$-paranormal if and only if*

$$|w_k|^n \leq m|w_{k-1}||w_{k-2}|\cdots|w_{k-n}|$$

for all unit vectors and $n \geq 2$.

Proof. The proof is close in spirit to that of [5, Theorem 3.5] and thus is omitted. □

Our next example shows that the inverse of $(m,n)^*$-paranormal operator need not be $(m,n)^*$-paranormal.

Example 2.5. Let T be a weighted shift operator defined by $T(e_k) = w_k e_{k+1}$ on a Hilbert space $l^2(\mathbb{Z},\mathbb{C})$ with non zero weights (w_k), and orthonormal basis (e_k), where

$$w_k = \begin{cases} 5 & \text{if } k \leq 0 \\ 4 & \text{if } k = 1 \\ 3 & \text{if } k \geq 2. \end{cases}$$

Hence,

$$T(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, 3x_{-1}, 3x_0, 4x_1, 5x_2, 5x_3, \ldots).$$

By [5, Theorem 3.5], when $n \geq 2$ and $m \geq 1$, T is $(m,n)^*$-paranormal for all unit vectors. It is easy to see that T is an invertible operator and

$$T^{-1}(\ldots, y_{-1}, y_0, y_1, \ldots) = (\ldots, \frac{y_0}{5}, \frac{y_1}{5}, \frac{y_2}{4}, \frac{y_3}{3}, \frac{y_4}{3}, \ldots),$$

that is,

$$T^{-1}(e_k) = \alpha_{k-1}e_{k-1}.$$

The corresponding weighted sequence (α_k) of T^{-1} is

$$\alpha_k = \begin{cases} \frac{1}{5} & \text{if } k \leq 0 \\ \frac{1}{4} & \text{if } k = 1 \\ \frac{1}{3} & \text{if } k \geq 2. \end{cases}$$

Again by Lemma 2.4, when $n \geq 2$, T^{-1} is $(m,n)^*$-paranormal if and only if

$$|\alpha_k|^n \leq m|\alpha_{k-1}||\alpha_{k-2}|\ldots|\alpha_{k-n}|. \tag{2.1}$$

for all unit vectors. If we choose $k = 2$, $m = 3.2$ and $n = 3$, then (2.1) does not hold. Thus, T^{-1} is not $(3.2,3)^*$-paranormal, whereas T is. Hence, the result holds.

In [3, p. 127], Arora and Kumar proved that if an operator T is m^*-paranormal, then tensor products $T \otimes I$ and $I \otimes T$ are also m^*-paranormal. It is extended to $(m,n)^*$- paranormal operators as follows:

Theorem 2.6. *Let $T \in B(\mathcal{H})$ be an $(m,n)^*$-paranormal operator. Then $T \otimes I$ and $I \otimes T$ are also $(m,n)^*$-paranormal.*

Proof. By [5, Theorem 3.1], we have

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n TT^* + m^{\frac{2}{n+1}} n a^{n+1} I \geq 0,$$

for each $a > 0$. This implies

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} + m^{\frac{2}{n+1}} n a^{n+1} I \geq (n+1)a^n TT^* \tag{2.2}$$

for each $a > 0$.

Now, by (2.2), consider

$$\begin{aligned} & m^{\frac{2}{n+1}} (T \otimes I)^{*n+1} (T \otimes I)^{n+1} + m^{\frac{2}{n+1}} n a^{n+1} (I \otimes I) \\ &= m^{\frac{2}{n+1}} (T^{*n+1} T^{n+1} \otimes I) + m^{\frac{2}{n+1}} n a^{n+1} (I \otimes I) \\ &= (m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} + m^{\frac{2}{n+1}} n a^{n+1} I) \otimes I \\ &\geq (n+1)a^n TT^* \otimes I \\ &= (n+1)a^n (T \otimes I)(T^* \otimes I). \end{aligned}$$

Therefore, we have

$$\begin{aligned} m^{\frac{2}{n+1}} (T \otimes I)^{*n+1} (T \otimes I)^{n+1} - (n+1)a^n (T \otimes I)(T \otimes I)^* \\ + m^{\frac{2}{n+1}} n a^{n+1} (I \otimes I) \geq 0, \end{aligned}$$

for each $a > 0$. Thus, $T \otimes I$ is $(m,n)^*$-paranormal. In the same way, we can prove that $I \otimes T$ is $(m,n)^*$-paranormal. □

In the following example, we show that the tensor product of two $(m,n)^*$-paranormal operators is not necessarily $(m,n)^*$-paranormal.

Example 2.7. Let $K = \oplus_{k=1}^{\infty} H_k$, where each Hilbert space H_k is isomorphic to $\mathbb{R} \times \mathbb{R}$ for each k. For any positive operators A and B on $\mathbb{R} \times \mathbb{R}$ and for fixed $n \in \mathbb{N}$, define an operator T on K as follows:

$$T(x_1, x_2, ...) = (0, Ax_1, Ax_2, ..., Ax_n, Bx_{n+1}, ...).$$

Thus, the adjoint of T is given by

$$T^*(x_1, x_2, ...) = (Ax_2, Ax_3, ..., Ax_{n+1}, Bx_{n+2}, ...).$$

Assume $x = (\ldots, 0, 0, x_{n+1}, 0, 0, \ldots)$. So, by [5, Theorem 3.1], T is $(2^{\frac{3}{2}}, 2)^*$-paranormal if and only if

$$2B^6 - 3a^2A^2 + 4a^3 I \geq 0,$$

for each $a > 0$. Now, choosing $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$2B^6 - 3a^2A^2 + 4a^3 I = \begin{bmatrix} 4a^3 - 3a^2 + 2 & -3a^2 \\ -3a^2 & 4a^3 - 3a^2 + 2 \end{bmatrix}.$$

The above operator is positive for each $a > 0$. Thus, T is $(2^{\frac{3}{2}}, 2)^*$-paranormal.

Equivalently, $T \otimes T$ is $(2^{\frac{3}{2}}, 2)^*$-paranormal if and only if

$$2(B^6 \otimes B^6) - 3a^2(A^2 \otimes A^2) + 4a^3(I \otimes I) \geq 0,$$

for each $a > 0$. Choosing $a = 1$, we have

$$2(B^6 \otimes B^6) - 3a^2(A^2 \otimes A^2) + 4a^3(I \otimes I) = \begin{bmatrix} 3 & -3 & -3 & -3 \\ -3 & 3 & -3 & -3 \\ -3 & -3 & 3 & -3 \\ -3 & -3 & -3 & 3 \end{bmatrix}.$$

The above operator is not positive for $x = (1,1,1,1)$. So, $T \otimes T$ is not $(2^{\frac{3}{2}}, 2)^*$-paranormal. This proves our claim.

Example 2.8. In this example, we show that the sum of two $(m,n)^*$-paranormal operators need not be $(m,n)^*$-paranormal.

Choose $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. By [5, Theorem 3.1], A is $(25,3)^*$-paranormal if and only if

$$5{A^*}^4 A^4 - 4a^3 AA^* + 15a^4 I \geq 0,$$

for each $a > 0$. So,

$$5{A^*}^4 A^4 - 4a^3 AA^* + 15a^4 I = \begin{bmatrix} 5 - 4a^3 + 15a^4 & 0 \\ 0 & 5 - 4a^3 + 15a^4 \end{bmatrix}$$

is positive for each $a > 0$. Similarly, by [5, Theorem 3.1], B is $(25,3)^*$-paranormal if and only if $5{B^*}^4 B^4 - 4a^3 BB^* + 15a^4 I \geq 0$, for each $a > 0$. Thus,

$$5{B^*}^4 B^4 - 4a^3 BB^* + 15a^4 I = \begin{bmatrix} 5 - 8a^3 + 15a^4 & 20 - 4a^3 \\ 20 - 4a^3 & 85 - 4a^3 + 15a^4 \end{bmatrix}$$

is positive for each $a > 0$.

Now, let $T = A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Again, T is $(25,3)^*$-paranormal if and only if $5{T^*}^4 T^4 - 4a^3 TT^* + 15a^4 I \geq 0$ for each $a > 0$. Now, for $a = \frac{1}{5}$, $5{T^*}^4 T^4 - 4a^3 TT^* + 15a^4 I = \begin{bmatrix} -4a^3 + 15a^4 & 0 \\ 0 & 15a^4 \end{bmatrix} < 0$. Thus, the sum of two $(m,n)^*$-paranormal operator is not $(m,n)^*$-paranormal.

The next example shows that the class of $(m,n)^*$-paranormal operators is independent from the class of (m,n)-paranormal operators, and vice versa.

Example 2.9. Here we show that an operator T is $(m,n)^*$-paranormal but not (m,n)-paranormal.

Suppose that the operator T is a weighted shift on the Hilbert space $l^2(\mathbb{Z},\mathbb{C})$ with non zero weights (w_k), and orthonormal basis (e_k), defined as: $T(e_k) = w_k e_{k+1}$ for all k. Choose

$$w_k = \begin{cases} 1 & \text{if } k \leq 1 \\ 3 & \text{if } k = 2 \\ 2 & \text{if } k = 3 \\ 3 & \text{if } k = 4 \\ 5 & \text{if } k \geq 5 \end{cases}$$

By [5, Theorem 3.5], T is $\left(\frac{7}{5},3\right)^*$-paranormal if and only if

$$|w_{k-1}|^3 \leq \frac{7}{5}|w_k||w_{k+1}||w_{k+2}| \tag{2.3}$$

for all unit vectors and for all k. Then T satisfies (2.3) for all k. Thus, T is $\left(\frac{7}{5},3\right)^*$-paranormal.

Similarly, by [5, Theorem 2.9], T is $\left(\frac{7}{5},3\right)$-paranormal if and only if

$$|w_k|^2 \leq \frac{7}{5}|w_{k+1}||w_{k+2}| \tag{2.4}$$

for all unit vectors and for all k. If we put $k = 2$ in (2.4), then T does not satisfy the inequality (2.4). This proves our assertion.

Example 2.10. We choose the operator T as defined in Example 2.7. Choose $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^2 = C = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. By [5, Theorem 2.1], an easy calculation provides us that T is $(3^{\frac{3}{2}},2)$-paranormal if and only if $3B^6 - 3a^2B^2 + 6a^3I \geq 0$ for each $a > 0$. Now,

$$3B^6 - 3a^2B^2 + 6a^3I = \begin{bmatrix} 507 - 15a^2 + 6a^3 & 210 - 6a^2 \\ 210 - 6a^2 & 87 - 3a^2 + 6a^3 \end{bmatrix}$$

is positive for each $a > 0$. Thus, T is $(3^{\frac{3}{2}},2)$-paranormal.

Similarly, by [5, Theorem 3.1], it is clear that T is $(3^{\frac{3}{2}},2)^*$-paranormal if and only if $3B^6 - 3a^2A^2 + 6a^3I \geq 0$ for each $a > 0$. Now,

$$3B^6 - 3a^2A^2 + 6a^3I = \begin{bmatrix} 507 - 12a^2 + 6a^3 & 210 \\ 210 & 87 + 6a^3 \end{bmatrix}$$

is not positive at $a = \frac{1}{5}$. Consequently, T is not $(3^{\frac{3}{2}},2)^*$-paranormal.

Example 2.11. We choose the operator T defined as in Example 2.9 along with non zero weights $w_k = \frac{1}{3^{|k|}}$ for all integers k. Now, by [5, Thorem 3.5], T is $(m,3)^*$-paranormal if and only if

$$|w_{k-1}|^3 \leq m|w_k||w_{k+1}||w_{k+2}| \tag{2.5}$$

for all unit vectors and for all k. Then (2.5) holds for $m \geq 729$. By [5, Theorem 2.9], T is $(m,3)$-paranormal if and only if

$$|w_k|^2 \leq m|w_{k+1}||w_{k+2}| \tag{2.6}$$

for all unit vectors and for all k. Therefore, (2.6) holds for $m \geq 27$.

Now, if we choose m such that $27 \leq m < 729$, then T is $(m,3)$-paranormal but not $(m,3)^*$-paranormal. If we choose $m < 27$, then T is neither $(m,3)$-paranormal nor $(m,3)^*$-paranormal. For $m \geq 729$, T is both $(m,3)$-paranormal and $(m,3)^*$-paranormal.

An operator T is said to be $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$. The class of $(m,n)^*$-paranormal operators is a generalization of the class of paranormal and $*$-paranormal operators. Note that for $m = 1$ and $n = 1$, the class of $(m,n)^*$-paranormal operators coincides with the class of $*$-paranormal operators. But there is no inclusion relation between $(m,n)^*$-paranormal operators and $*$-paranormal operators. Now, we give an example which is $(m,n)^*$-paranormal but not $*$-paranormal.

Example 2.12. Let T be an operator defined on a Hilbert space $\mathcal{H} = \mathbb{R} \otimes \mathbb{R}$. We choose $T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. It is known [2] that T is $*$-paranormal if and only if ${T^*}^2T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$ for each positive number λ. Consider the operator

$$ {T^*}^2T^2 - 2\lambda TT^* + \lambda^2 I = \begin{bmatrix} 1 - 20\lambda + \lambda^2 & 6 - 6\lambda \\ 6 - 6\lambda & 37 - 2\lambda + \lambda^2 \end{bmatrix} $$

is not positive for $\lambda = 1$, hence T is not $*$-paranormal.

Now, we show that T is $(m,n)^*$-paranormal. By [5, Theorem 3.1], T is $(100^{\frac{3}{2}}, 2)^*$-paranormal if and only if $100{T^*}^3T^3 - 3a^2TT^* + 200a^3 I \geq 0$, for each $a > 0$. Consider the operator equation

$$100{T^*}^3T^3 - 3a^2TT^* + 200a^3 I = \begin{bmatrix} 100 - 30a^2 + 200a^3 & 900 - 9a^2 \\ 900 - 9a^2 & 8200 - 3a^2 + 200a^3 \end{bmatrix}.$$

It is clear that the above operator is positive for each $a > 0$. Therefore, T is $(100^{\frac{3}{2}}, 2)^*$-paranormal.

3. Spectral properties of $(m,n)^*$-paranormal operators

In this section, we write $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_a(T)$ and $\sigma_{ja}(T)$ of the operator T as the point spectrum, the joint point spectrum, the approximate point spectrum and the joint approximate point spectrum, respectively (see [12]). Some authors showed that the point spectrum and the joint point spectrum are the same for some classes of non normal operators. Similarly, the approximate point spectrum and the joint approximate point spectrum are also the same for certain classes of operators [1, 4, 11–13]. We extend these results for the class of $(m,n)^*$-paranormal operators.

Theorem 3.1. *If $T \in B(\mathcal{H})$ is a $(m,n)^*$-paranormal operator, then $\sigma_a(T) = \sigma_{ja}(T)$ for all unit vectors.*

Proof. It is easy to see that $\sigma_{ja}(T) \subseteq \sigma_a(T)$.

To prove the reverse inequality, for unit vectors (x_n), it is sufficient to prove that the norm $\|(T-\lambda I)^* x_n\| \to 0$ as $n \to \infty$. Since T is $(m,n)^*$-paranormal, we have

$$\|T^* x\|^{n+1} \leq m\|T^{n+1}x\|,$$

that is,

$$\|T^* x\|^2 \leq m^{\frac{2}{n+1}} \|T^{n+1}x\|^{\frac{2}{n+1}} \tag{3.1}$$

By assumption, we have
$(T-\lambda)x_n \to 0$, it follows that $(T^{n+1} - \lambda^{n+1})x_n \to 0$ for all positive integers n. Since $|\|T^{n+1}x_n\| - |\lambda^{n+1}|| \leq \|(T^{n+1}-\lambda^{n+1})x_n\|$, implies $\|T^{n+1}x_n\| \to |\lambda|^{n+1}$

By using (3.1), we get

$$\begin{aligned}\|(T-\lambda I)^* x_n\|^2 &= \|T^* x_n\|^2 - \langle T^* x_n, \bar{\lambda} x_n\rangle - \langle \bar{\lambda} x_n, T^* x_n\rangle + |\lambda|^2 \\ &\leq m^{\frac{2}{n+1}} \|T^{n+1}x_n\|^{\frac{2}{n+1}} - |\lambda|^2 \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

Thus, $\|(T-\lambda I)^* x_n\| = 0$ for the sequence (x_n) of unit vectors. Thus, $\sigma_a(T) = \sigma_{ja}(T)$. □

As a consequence, we have the following:

Corollary 3.2. *Let $m \leq 1$ be such that T is an $(m,n)^*$-paranormal operator for unit vectors. If λ and μ are two distinct eigenvalues of T corresponding to eigenvectors x and y, then $\langle x, y\rangle = 0$.*

Corollary 3.3. *For a fixed $m \leq 1$, if an operator T is an $(m,n)^*$-paranormal for all unit vectors such that λ and μ are two distinct points in $\sigma_a(T)$ and $\sigma_{ja}(T)$, respectively, corresponding to sequences (x_n) and (y_n), then $\langle x_n, y_n\rangle =$* 0.

Theorem 3.4. *For $m \leq 1$, if T is $(m,n)^*$-paranormal operator for all unit vectors, then $\sigma_p(T) = \sigma_{jp}(T)$.*

Definition 3.5. An operator T is said to have single valued extension property (abbreviated as SVEP) at $\gamma_0 \in \mathbb{C}$, if for every open neighborhood G of γ_0, the only analytic function $f : G \to \mathcal{H}$ which satisfies the equation $(T - \gamma I)f(\gamma) = 0$ for all $\gamma \in G$ is the function $f = 0$.

An operator T has SVEP if T has SVEP at every $\gamma \in \mathbb{C}$.

The following theorem can be proved in similar way as in [10, Theorem 8].

Theorem 3.6. *Let $T \in B(\mathcal{H})$ be a $(m,n)^*$-paranormal operator, where $m \leq 1$. Then T has SVEP.*

Using the same method as that in [7, Lemma 2.5], we prove the next result.

Proposition 3.7. *Let $T \in B(\mathcal{H})$ be an $(m, n)^*$-paranormal and hyponormal operator. Then $N(T - \lambda I) \subseteq N(T^* - \bar{\lambda} I)$ for all unit vectors, for all $\lambda \in \mathbb{C}$ and $m \leq 1$.*

Proof. First, choose $\lambda \in \mathbb{C}$ and $x \in N(T - \lambda I)$. Since T is $(m, n)^*$-paranormal, we have

$$\|T^* x\|^{n+1} \leq m\|T^{n+1}x\| = m\|\lambda^{n+1}x\| = m|\lambda^{n+1}|,$$

that is,

$$\|T^* x\| \leq |\lambda|.$$

In the sequel, consider

$$\begin{aligned} \|(T^* - \bar{\lambda} I)x\|^2 &= \|T^* x\|^2 - \lambda\langle x, Tx\rangle - \bar{\lambda}\langle Tx, x\rangle + |\lambda|^2 \\ &\leq \|Tx\|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 \\ &= 0. \end{aligned}$$

Thus, $\|(T^* - \bar{\lambda} I)x\| = 0$. Hence, $x \in N(T^* - \bar{\lambda} I)$. This completes our claim. □

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References

[1] Aluthge, A., and Wang, D. The joint approximate point spectrum of an operator. *Hokkaido Math. J. 31*, 1 (2002), 187–197.

[2] Arora, S. C., and Thukral, J. K. On a class of operators. *Glas. Mat. Ser. III 21(41)*, 2 (1986), 381–386.

[3] Arora, S. C., and Thukral, J. K. M^*-paranormal operators. *Glas. Mat. Ser. III 22(42)*, 1 (1987), 123–129.

[4] Chō, M., and Yamazaki, T. An operator transform from class A to the class of hyponormal operators and its application. *Integral Equations Operator Theory 53*, 4 (2005), 497–508.

[5] Dharmarha, P., and Ram, S. (m, n)-paranormal operators and $(m, n)^*$-paranormal operators. *Commun. Korean Math. Soc. 35*, 1 (2020), 151–159.

[6] Furuta, T. On the class of paranormal operators. *Proc. Japan Acad. 43* (1967), 594–598.

[7] Han, Y. M., and Kim, A.-H. A note on $*$-paranormal operators. *Integral Equations Operator Theory 49*, 4 (2004), 435–444.

[8] Istrăţescu, V. On some hyponormal operators. *Pacific J. Math. 22* (1967), 413–417.

[9] Kubrusly, C. S., and Duggal, B. P. A note on k-paranormal operators. *Oper. Matrices 4*, 2 (2010), 213–223.

[10] Tanahashi, K., and Uchiyama, A. A note on $*$-paranormal operators and related classes of operators. *Bull. Korean Math. Soc. 51*, 2 (2014), 357–371.

[11] Uchiyama, A., Tanahashi, K., and Lee, J. I. Spectrum of class $A(s,t)$ operators. *Acta Sci. Math. (Szeged) 70*, 1-2 (2004), 279–287.

[12] Yang, C., and Yuan, J. Spectrum of class $wF(p,r,q)$ operators for $p+r \leq 1$ and $q \geq 1$. *Acta Sci. Math. (Szeged) 71*, 3-4 (2005), 767–779.

[13] Zuo, F., and Shen, J. A note on $*$-n-paranormal operators. *Adv. Math. (China) 42*, 2 (2013), 153–158.

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