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On some inequalities for double and path integrals on general domains[1]

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Abstract. In this paper, by the use of the celebrated Green's identity for double and path integrals, we establish some integral inequalities for functions of two variables defined on closed and bounded subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

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double integral, path integral, Green's identity, integral inequalities.

1. Introduction

In paper [1], the authors obtained among others the following results concerning the difference between the double integral on the disk and the values in the center or the path integral on the circle:

Theorem 1. If $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives on $D(C,R)$, the disk centered in the point $C = (a,b)$ with the radius $R > 0$, and

$$\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left|\frac{\partial f(x,y)}{\partial x}\right| < \infty,$$
$$\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty} := \sup_{(x,y)\in D(C,R)} \left|\frac{\partial f(x,y)}{\partial y}\right| < \infty;$$

then

$$\left|f(C) - \frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dx\,dy\right| \leq \frac{4}{3\pi}R\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} + \left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.1}$$

The constant $\frac{4}{3\pi}$ is sharp.

We also have

$$\left|\frac{1}{\pi R^2}\iint_{D(C,R)} f(x,y)\,dx\,dy - \frac{1}{2\pi R}\int_{\sigma(C,R)} f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{3\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} + \left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right], \tag{1.2}$$

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where $\sigma(C,R)$ is the circle centered in $C=(a,b)$ with the radius $R>0$ and

$$\left|f(C)-\frac{1}{2\pi R}\int_{\sigma(C,R)}f(\gamma)\,dl(\gamma)\right| \leq \frac{2R}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty}+\left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right]. \tag{1.3}$$

In the same paper [1] the authors also established the following Ostrowski type inequality:

Theorem 2. If f has bounded partial derivatives on $D(0,1)$, then

$$\begin{aligned}\left|f(u,v)-\frac{1}{\pi}\iint_{D(0,1)}f(x,y)\,dx\,dy\right| &\leq \frac{2}{\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(0,1),\infty}\left(u\arcsin u+\frac{1}{3}\sqrt{1-u^2}\,(2+u^2)\right)\right.\\ &\quad\left.+\left\|\frac{\partial f}{\partial y}\right\|_{D(0,1),\infty}\left(v\arcsin v+\frac{1}{3}\sqrt{1-v^2}\,(2+v^2)\right)\right]\end{aligned} \tag{1.4}$$

for any $(u,v)\in D(0,1)$.

For other integral inequalities for double integrals see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In this paper, by the use of the celebrated Green's identity for double and path integrals, we establish some integral inequalities for functions of two variables defined on closed and bounded subsets of the plane $\mathbb{R}^2$. Some examples for rectangles and disks are also provided.

2. Main Results

Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Let L and M be scalar functions defined at least on an open set containing D. Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem (see for instance https://en.wikipedia.org/wiki/Green%27s_theorem)

$$\int\int_D\left(\frac{\partial M(x,y)}{\partial x}-\frac{\partial L(x,y)}{\partial y}\right)dxdy=\oint_{\partial D}(L(x,y)\,dx+M(x,y)\,dy)\,. \tag{G}$$

Moreover, if the curve ∂D is described by the function $r(t)=(x(t),y(t))$, $t\in[a,b]$, with x, y differentiable on (a,b) then we can calculate the path integral as

$$\oint_{\partial D}(L(x,y)\,dx+M(x,y)\,dy)=\int_a^b[L(x(t),y(t))\,x'(t)+M(x(t),y(t))\,y'(t)]\,dt.$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q.

For a function $f : D \to \mathbb{C}$ having partial derivatives on the domain D we define $\Lambda_{\partial f,D} : D \to \mathbb{C}$ as

$$\Lambda_{\partial f,D}(x,y) := (x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right).$$

We need the following identity, [5]:

Lemma 3. Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Assume that the function $f : D \to \mathbb{C}$ has continuous partial derivatives on the domain D. Then

$$\begin{aligned}(2.1)\qquad &\frac{1}{2}\oint_{\partial D}\left[(x-y)f(x,y)\,dx + (x-y)f(x,y)\,dy\right] - \int\int_D f(x,y)\,dxdy \\ &= \frac{1}{2}\int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy.\end{aligned}$$

Proof. Consider the functions

$$M(x,y) := (x-y)f(x,y) \text{ and } L(x,y) := (x-y)f(x,y)$$

for $(x,y) \in D$.

We have

$$\frac{\partial}{\partial x}\left[(x-y)f(x,y)\right] = f(x,y) + (x-y)\frac{\partial f(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\left[(y-x)f(x,y)\right] = f(x,y) + (y-x)\frac{\partial f(x,y)}{\partial y}$$

for $(x,y) \in D$.

If we add these two equalities, then we get

$$(2.2)\qquad \frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y} = 2f(x,y) + \Lambda_{\partial f,D}(x,y)$$

for $(x,y) \in D$.

If we integrate this equality on D, then we obtain

$$\begin{aligned}(2.3)\qquad &\int\int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right)dxdy \\ &= 2\int\int_D f(x,y)\,dxdy + \int\int_D \Lambda_{\partial f,D}(x,y)\,dxdy.\end{aligned}$$

From Green's identity we also have

$$\int\int_D \left(\frac{\partial M(x,y)}{\partial x} - \frac{\partial L(x,y)}{\partial y}\right) dxdy \tag{2.4}$$
$$= \oint_{\partial D} (L(x,y)\,dx + M(x,y)\,dy)$$
$$= \oint_{\partial D} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy].$$

By employing (2.3) and (2.4) we deduce the desired equality (2.1). □

Corollary 4. With the assumptions of Lemma 3 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then

$$\frac{1}{2}\int_a^b (x(t) - y(t)) f(x(t), y(t)) (x'(t) + y'(t))\,dt - \int\int_D f(x,y)\,dxdy \tag{2.5}$$
$$= \frac{1}{2}\int\int_D \Lambda_{\partial f, D}(x,y)\,dxdy.$$

We consider the following Lebesgue norms for a measurable function $g : D \to \mathbb{C}$

$$\|g\|_{D,p} := \left(\int\int_D |g(x,y)|^p\,dxdy\right)^{1/p} < \infty \text{ for } p \geq 1$$

and

$$\|g\|_{D,\infty} := \sup_{(x,y)\in D} |g(x,y)| < \infty \text{ for } p = \infty.$$

We have the following result:

Theorem 5. Let ∂D be a simple, closed counterclockwise curve in the xy-plane, bounding a region D. Assume that the function $f : D \to \mathbb{C}$ has continuous partial derivatives on the domain D. Then

$$\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\oint_{\partial D} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy]\right| \tag{2.6}$$
$$\leq \frac{1}{2}\int\int_D |x-y| \left|\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right| dxdy$$
$$\leq \frac{1}{2}\begin{cases} \left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{D,\infty} \int\int_D |x-y|\,dxdy; \\ \\ \left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{D,p} \left(\int\int_D |x-y|^q\,dxdy\right)^{1/q} \\ \text{where } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{D,1} \sup_{(x,y)\in D} |x-y|. \end{cases}$$

Proof. From the identity (2.1) we have

$$\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\oint_{\partial D}\left[(x-y)f(x,y)\,dx + (x-y)f(x,y)\,dy\right]\right| \le \frac{1}{2}\int\int_D \left|(x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right)\right| dxdy. \tag{2.7}$$

Using Hölder's integral inequality we have

$$\int\int_D \left|(x-y)\left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right)\right| dxdy$$
$$\le \begin{cases} \sup_{(x,y)\in D}\left|\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right| \int\int_D |x-y|\,dxdy; \\ \left(\int\int_D \left|\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right|^p dxdy\right)^{1/p} \left(\int\int_D |x-y|^q\,dxdy\right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \sup_{(x,y)\in D}|x-y| \int\int_D \left|\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right| dxdy \end{cases}$$

$$= \begin{cases} \left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{D,\infty} \int\int_D |x-y|\,dxdy; \\ \left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{D,p} \left(\int\int_D |x-y|^q\,dxdy\right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \left\|\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\|_{D,1} \sup_{(x,y)\in D}|x-y| \end{cases}$$

and by (2.7) we get the desired result (2.6). □

Corollary 6. With the assumptions of Theorem 5 and if there exists a constant $L>0$ such that

$$\left|\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right| \le L|x-y| \text{ for } (x,y)\in D, \tag{2.8}$$

then we have

$$\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\oint_{\partial D}\left[(x-y)f(x,y)\,dx + (x-y)f(x,y)\,dy\right]\right| \le \frac{1}{2}L\int\int_D (x-y)^2\,dxdy. \tag{2.9}$$

Theorem 7. With the assumptions of Theorem 5 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a,b]$, with x, y differentiable on (a,b), then

$$\left| \int\int_D f(x,y)\,dxdy - \frac{1}{2}\int\int_D (x-y)\left(\frac{\partial f(x,y)}{\partial y} - \frac{\partial f(x,y)}{\partial x}\right)dxdy \right| \le \frac{1}{2}\int_a^b |f(x(t),y(t))|\,|x(t)-y(t)|\,|x'(t)+y'(t)|\,dt =: M(f,\partial D). \tag{2.10}$$

We also have the bounds

$$M(f,\partial D) \le \frac{1}{2}\begin{cases} \sup_{t\in[a,b]} |f(x(t),y(t))| \int_a^b |x(t)-y(t)|\,|x'(t)+y'(t)|\,dt; \\ \left(\int_a^b |f(x(t),y(t))|^p\,dt\right)^{1/p} \left(\int_a^b |x(t)-y(t)|^q\,|x'(t)+y'(t)|^q\,dt\right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \int_a^b |f(x(t),y(t))|\,dt \sup_{t\in[a,b]} \left[|x(t)-y(t)|\,|x'(t)+y'(t)|\right] \end{cases} \tag{2.11}$$

and the bound

$$M(f,\partial D) \le \frac{\sqrt{2}}{2}\int_{\partial D} |f(x,y)|\,|x-y|\,d\ell =: N(f,\partial D), \tag{2.12}$$

where the integral is taken as an arc-lenght integral, namely

$$\int_{\partial D} |f(x,y)|\,|x-y|\,d\ell = \int_a^b |f(x(t),y(t))|\,|x(t)-y(t)|\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt.$$

Moreover, we have

$$N(f,\partial D) \le \frac{\sqrt{2}}{2}\begin{cases} \|f\|_{\partial D,\infty} \int_{\partial D} |x-y|\,d\ell; \\ \|f\|_{\partial D,p} \left(\int_{\partial D} |x-y|^q\,d\ell\right)^{1/q} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \|f\|_{\partial D,1} \sup_{(x,y)\in\partial D} |x-y|, \end{cases} \tag{2.13}$$

where the norms $\|\cdot\|_{\partial D,p}$ are defined by

$$\|f\|_{\partial D,p} := \left(\int_{\partial D} |f(x,y)|^p\,d\ell\right)^{1/p},\ p \ge 1$$

and

$$\|f\|_{\partial D,\infty} \sup_{(x,y)\in\partial D} |f(x,y)|.$$

Proof. From the identity (2.1) we have

$$\begin{aligned}
&\left|\int\int_D f(x,y)\,dxdy - \frac{1}{2}\int\int_D (x-y)\left(\frac{\partial f(x,y)}{\partial y} - \frac{\partial f(x,y)}{\partial x}\right)dxdy\right| \\
&= \frac{1}{2}\left|\oint_{\partial D}\left[(x-y)\,f(x,y)\,dx + (x-y)\,f(x,y)\,dy\right]\right| \\
&\le \frac{1}{2}\int_a^b |f(x(t),y(t))|\,|x(t)-y(t)|\,|x'(t)+y'(t)|\,dt = M(f,\partial D).
\end{aligned}$$

By Hölder's integral inequality we have

$$\begin{aligned}
&\int_a^b |f(x(t),y(t))|\,|x(t)-y(t)|\,|x'(t)+y'(t)|\,dt \\
&\le \begin{cases}
\sup_{t\in[a,b]}|f(x(t),y(t))|\int_a^b |x(t)-y(t)|\,|x'(t)+y'(t)|\,dt; \\[2ex]
\left(\int_a^b |f(x(t),y(t))|^p\,dt\right)^{1/p}\left(\int_a^b |x(t)-y(t)|^q\,|x'(t)+y'(t)|^q\,dt\right)^{1/q} \\
\text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\int_a^b |f(x(t),y(t))|\,dt\,\sup_{t\in[a,b]}\left[|x(t)-y(t)|\,|x'(t)+y'(t)|\right],
\end{cases}
\end{aligned}$$

which proves the inequality (2.11).

By the elementary inequality

$$|z+w| \le \sqrt{2}\sqrt{z^2+w^2} \text{ for } z,\ w\in\mathbb{R}$$

we have

$$|x'(t)+y'(t)| \le \sqrt{2}\sqrt{[x'(t)]^2+[y'(t)]^2},\ t\in[a,b].$$

Therefore

$$\begin{aligned}
M(f,\partial D) &\le \frac{\sqrt{2}}{2}\int_a^b |f(x(t),y(t))|\,|x(t)-y(t)|\sqrt{[x'(t)]^2+[y'(t)]^2}dt \\
&= \frac{\sqrt{2}}{2}\int_{\partial D}|f(x,y)|\,|x-y|\,d\ell = N(f,\partial D).
\end{aligned}$$

By using Hölder's inequality for arc-lenght integral we have

$$\begin{aligned}
&\int_{\partial D}|f(x,y)|\,|x-y|\,d\ell \\
&\le \begin{cases}
\|f\|_{\partial D,\infty}\int_{\partial D}|x-y|\,d\ell; \\[2ex]
\|f\|_{\partial D,p}\left(\int_{\partial D}|x-y|^q\,d\ell\right)^{1/q} \text{ where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\|f\|_{\partial D,1}\sup_{(x,y)\in\partial D}|x-y|,
\end{cases}
\end{aligned}$$

which proves the last part of (2.13). □

3. Examples for Rectangles

Let $a < b$ and $c < d$. Put $A = (a, c)$, $B = (b, c)$, $C = (b, d)$, $D = (a, d) \in \mathbb{R}^2$ the vertices of the rectangle $ABCD = [a, b] \times [c, d]$. Consider the counterclockwise segments

$$AB : \begin{cases} x = (1 - t)\, a + tb \\ \\ y = c \end{cases}, \; t \in [0, 1]$$

$$BC : \begin{cases} x = b \\ \\ y = (1 - t)\, c + td \end{cases}, \; t \in [0, 1]$$

$$CD : \begin{cases} x = (1 - t)\, b + ta \\ \\ y = d \end{cases}, \; t \in [0, 1]$$

and

$$DA : \begin{cases} x = a \\ \\ y = (1 - t)\, d + tc \end{cases}, \; t \in [0, 1]\,.$$

Therefore $\partial\,(ABCD) = AB \cup BC \cup CD \cup DA$.

We have

$$\begin{aligned} &\oint_{AB} [(x - y)\, f\,(x, y)\, dx + (x - y)\, f\,(x, y)\, dy] \\ &= (b - a) \int_0^1 ((1 - t)\, a + tb - c)\, f\,((1 - t)\, a + tb, c)\, dt \\ &= (b - a) \int_0^1 (t\,(b - a) + a - c)\, f\,((1 - t)\, a + tb, c)\, dt, \end{aligned}$$

$$\begin{aligned} &\oint_{BC} [(x - y)\, f\,(x, y)\, dx + (x - y)\, f\,(x, y)\, dy] \\ &= (d - c) \int_0^1 (b - (1 - t)\, c - td)\, f\,(b, (1 - t)\, c + td)\, dt \\ &= (d - c) \int_0^1 (b - c - t\,(d - c))\, f\,(b, (1 - t)\, c + td)\, dt, \end{aligned}$$

$$\begin{aligned}
&\oint_{CD} \left[(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy \right] \\
&= (a-b) \int_0^1 ((1-t)\, b + ta - d)\, f((1-t)\, b + ta, d)\, dt \\
&= (a-b) \int_0^1 (t(a-b) + b - d)\, f((1-t)\, b + ta, d)\, dt \\
&= (a-b) \int_0^1 ((1-t)(a-b) + b - d)\, f((1-t)\, a + tb, d)\, dt \\
&= (b-a) \int_0^1 (d - a - t(b-a))\, f((1-t)\, a + tb, d)\, dt
\end{aligned}$$

and

$$\begin{aligned}
&\oint_{DA} \left[(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy \right] \\
&= (c-d) \int_0^1 (a - (1-t)\, d - tc)\, f(a, (1-t)\, d + tc)\, dt \\
&= (c-d) \int_0^1 (a - td - (1-t)\, c)\, f(a, (1-t)\, c + td)\, dt \\
&= (d-c) \int_0^1 (t(d-c) + c - a)\, f(a, (1-t)\, c + td)\, dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\oint_{\partial(ABCD)} \left[(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy \right] \\
&= (b-a) \int_0^1 (t(b-a) + a - c)\, f((1-t)\, a + tb, c)\, dt \\
&+ (b-a) \int_0^1 (d - a - t(b-a))\, f((1-t)\, a + tb, d)\, dt \\
&+ (d-c) \int_0^1 (b - c - t(d-c))\, f(b, (1-t)\, c + td)\, dt \\
&+ (d-c) \int_0^1 (t(d-c) + c - a)\, f(a, (1-t)\, c + td)\, dt.
\end{aligned}$$

If we make the change of variable $(1-t)\, a + tb = x$, then $dx = (b-a)\, dt$, $t = \frac{x-a}{b-a}$. Also for the change of variable $(1-t)\, c + td = y$, we have $dy =$

$(d-c)\,dt$ and $t = \frac{y-c}{d-c}$. Therefore

$$\oint_{\partial(ABCD)} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy] = \frac{1}{2}\int_a^b [(x-c) f(x,c) + (d-x) f(x,d)]\,dx + \frac{1}{2}\int_c^d [(b-y) f(b,y) + (y-a) f(a,y)]\,dy. \tag{3.1}$$

If $[a,b] = [c,d]$, then by (3.1) we get

$$\oint_{\partial(ABCD)} [(x-y) f(x,y)\,dx + (x-y) f(x,y)\,dy] = \frac{1}{2}\int_a^b [(x-a) f(x,a) + (b-x) f(x,b)]\,dx + \frac{1}{2}\int_a^b [(b-y) f(b,y) + (y-a) f(a,y)]\,dy. \tag{3.2}$$

Observe that for $q \geq 1$

$$\int_a^b \int_a^b |x-y|^q\,dxdy = \int_a^b \frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1}\,dx = \frac{2(b-a)^{q+2}}{(q+1)(q+2)}$$

and, in particular,

$$\int_a^b \int_a^b |x-y|\,dxdy = \frac{(b-a)^3}{3}.$$

Also,

$$\sup_{(x,y)\in[a,b]\times[c,d]} |x-y| = b-a.$$

By making use of Theorem 5 we can state:

Proposition 8. Assume that the function $f : [a,b]^2 \to \mathbb{C}$ has continuous partial derivatives on the domain $[a,b]^2$. Then

$$\begin{aligned}(3.3)\quad &\left|\frac{1}{2}\int_a^b [(x-a) f(x,a) + (b-x) f(x,b)]\, dx \right.\\ &\left. +\frac{1}{2}\int_a^b [(b-y) f(b,y) + (y-a) f(a,y)]\, dy - \int_a^b\int_a^b f(x,y)\, dxdy\right| \\ &\le \frac{1}{2}\begin{cases} \frac{(b-a)^3}{3}\left\|\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right\|_{[a,b]^2,\infty}; \\ \\ \frac{2^{1/q}(b-a)^{1+2/q}}{(q+1)^{1/q}(q+2)^{1/q}}\left\|\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right\|_{[a,b]^2,p} \\ \text{where } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ (b-a)\left\|\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right\|_{[a,b]^2,1}. \end{cases}\end{aligned}$$

We also have

$$\begin{aligned}\int_a^b\int_c^d (x-y)^2\, dxdy &= \int_a^b \left(\int_c^d (y-x)^2\, dy\right) dx \\ &= \frac{1}{3}\int_a^b \left[(x-c)^3-(x-d)^3\right] dx \\ &= \frac{1}{12}\left[(b-c)^4-(a-c)^4-(d-b)^4+(d-a)^4\right].\end{aligned}$$

We have:

Proposition 9. Assume that the function $f:[a,b]\times[c,d]\to\mathbb{C}$ has continuous partial derivatives on the domain $[a,b]\times[c,d]$. Then

$$\begin{aligned}(3.4)\quad &\left|\frac{1}{2}\int_a^b [(x-c) f(x,c) + (d-x) f(x,d)]\, dx \right.\\ &\left. +\frac{1}{2}\int_c^d [(b-y) f(b,y) + (y-a) f(a,y)]\, dy - \int_a^b\int_c^d f(x,y)\, dxdy\right| \\ &\le \frac{\sqrt{3}}{12}\left\|\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right\|_{[a,b]\times[c,d],2}\left[(b-c)^4-(a-c)^4-(d-b)^4+(d-a)^4\right]^{1/2}.\end{aligned}$$

The proof follows by the inequality (2.6) for $p=q=2$ and $D=[a,b]\times[c,d]$.

By utilising Corollary 6 we also have:

Proposition 10. Assume that the function $f:[a,b]\times[c,d]\to\mathbb{C}$ has continuous partial derivatives on the domain $[a,b]\times[c,d]$ and there exists a constant $L>0$ such that

$$(3.5)\qquad \left|\frac{\partial f(x,y)}{\partial x}-\frac{\partial f(x,y)}{\partial y}\right| \le L|x-y| \text{ for } (x,y)\in[a,b]\times[c,d],$$

then we have

$$\left|\frac{1}{2}\int_a^b [(x-c) f(x,c) + (d-x) f(x,d)]\, dx \right. \tag{3.6}$$
$$\left. +\frac{1}{2}\int_c^d [(b-y) f(b,y) + (y-a) f(a,y)]\, dy - \int_a^b \int_c^d f(x,y)\, dxdy \right|$$
$$\leq \frac{1}{24} L \left[(b-c)^4 - (a-c)^4 - (d-b)^4 + (d-a)^4\right].$$

4. Examples for Disks

We consider the closed disk $D(O,R)$ centered in $O(0,0)$ and of radius $R > 0$. This is parametrized by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad r \in [0,R], \ \theta \in [0,2\pi]$$

and the circle $\mathcal{C}(O,R)$ is parametrized by

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}, \quad \theta \in [0,2\pi].$$

Observe that, if $f : D(O,R) \to \mathbb{R}$, then

$$\oint_{\mathcal{C}(O,R)} [(x-y) f(x,y)\, dx + (x-y) f(x,y)\, dy]$$
$$= -\int_0^{2\pi} R(R\cos\theta - R\sin\theta)\sin\theta f(R\cos\theta, R\sin\theta)\, d\theta$$
$$+ \int_0^{2\pi} R(R\cos\theta - R\sin\theta)\cos\theta f(R\cos\theta, R\sin\theta)\, d\theta$$
$$= R^2 \int_0^{2\pi} f(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2\, d\theta.$$

Also, we have

$$\int\int_{D(O,R)} f(x,y)\, dxdy = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, rdrd\theta$$

and

$$\int\int_{D(O,R)} (x-y)^2\, dxdy = \int_0^R \int_0^{2\pi} (R\cos\theta - R\sin\theta)^2\, rdrd\theta$$
$$= \frac{1}{2}R^4 \int_0^{2\pi} (\cos\theta - \sin\theta)^2\, d\theta$$
$$= \frac{1}{2}R^4 \int_0^{2\pi} (1 - 2\sin\theta\cos\theta)\, d\theta = \pi R^4.$$

Proposition 11. Assume that the function $f : D(O,R) \to \mathbb{C}$ has continuous partial derivatives on the domain $D(O,R)$ and there exists a constant $L > 0$ such that

$$\left| \frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y} \right| \le L|x-y| \text{ for } (x,y) \in D(O,R), \tag{4.1}$$

then

$$\left| \frac{1}{2} R^2 \int_0^{2\pi} f(R\cos\theta, R\sin\theta)(\cos\theta - \sin\theta)^2 d\theta \right. \\ \left. - \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r dr d\theta \right| \le \frac{1}{2} L\pi R^4. \tag{4.2}$$

The proof follows by Corollary 6 for $D = D(O,R)$.

Similar results may be obtained by employing the other inequalities above. The details are left to the interested reader.

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