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Study of nonlinear stochastic Cauchy problems in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

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Abstract. We use the framework of the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras of J-A. Marti to study some nonlinear stochastic Cauchy problems for a simple equation, namely the transport equation in basic form, with stochastic generalized processes. Until now such studies were made in Colombeau-type algebras.

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1. Introduction

To study some nonlinear stochastic Cauchy problems we reformulate them in the framework of the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras of J.-A. Marti [6, 7, 8], [2]. These algebras allow us to treat singular processes in stochastic analysis following the example of Colombeau algebras. In this article we use the notations and concepts of our previous paper, [5].

The plan of this article is as follows. This section is followed by Section 2, which introduces the definitions and properties for stochastic analysis, $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras and algebras of generalized stochastic processes. We refer the reader to [5] in which we make similar studies.

In Section 3, we examine the following Cauchy problems associated to a simple equation, namely the transport equation in basic form, formally written:

$$(P) : \frac{\partial U}{\partial t} = F(U) + W, \;\; U|_{\gamma} = f,$$

and

$$(P') : \frac{\partial U}{\partial t} = F(U)W, \;\; U|_{\gamma} = f,$$

with a smooth function F on the right-hand side. F can be non Lipschitz (in U) but F and all its derivatives have polynomial growth. γ is a monotonic curve with the equation $x = l(t)$, γ is not a characteristic curve, f is a generalized stochastic process on $\mathbb{R}$, W a generalized process on $\mathbb{R}^2$. That is, f and W are weakly measurable maps of some probability space (Ω, Σ, μ) with values in the distribution space $\mathcal{D}'(\mathbb{R})$, respectively $\mathcal{D}'(\mathbb{R}^2)$.

For ω fixed, ω in Ω, using regularizations and cutoff techniques, we define a well formulated problem $(P(\omega)_{gen})$ (resp. $(P'(\omega)_{gen})$) associated to problem

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(P) (resp. (P')) in a convenient algebra. We must use two parameters. The first parameter is used to regularize the data and the second one to replace the problem by a family of Lipschitz problems. We prove that problem (P) (resp. (P')) has a unique solution in some algebras of generalized stochastic processes.
Section 4, is devoted to a nonlinear stochastic Cauchy problem with the white noise as initial data

$$(P_1) : \frac{\partial U}{\partial t} = F(U), U|_{\gamma} = W,$$

where γ is the curve of equation $x = l(t)$, γ is not a characteristic curve, W is the white noise on $\mathbb{R}$. The function F is smooth, it can be non Lipschitz but F and all derivatives have polynomial growth. We study problem (P_1) as the previous ones and we examine the limiting behavior of the generalized solution.

2. Algebra of generalized stochastic processes

2.1. The presheaves of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

We refer the reader to the references [3], [4], [5], for the definition and the properties of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras, the notion of overgenerated rings, the relationship with distribution theory and the association process, the notion of algebra $\mathcal{A}(\Omega)$ stable under the family $(F_\eta)_\lambda$ [2], the definition of the generalized operator $\mathcal{F}$ associated to the family $(F_\eta)_\lambda$, the definition of the generalized second side restriction mapping $\mathcal{R}_g$ associated to the function g.

We use the same notations and the same notions as the references. All these elements of the theory of the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras are now well-known.

2.2. Algebras $\mathcal{A}_p(O), p \in \mathbb{N}^*$ and $\mathcal{A}(O)$

Take $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and Λ a set of indices, $\lambda \in \Lambda$. Take $p \in \mathbb{N}^*$. For any open set O in $\mathbb{R}^d \mathcal{E}(O) = \mathrm{C}^\infty(O)$, is endowed with the $\mathcal{P}_p(O)$ topology defined by the family of the seminorms

$$N^p_{K,l}(u_\lambda) = \sup_{|\alpha| \le l} N^p_{K,\alpha}(u_\lambda), \text{ with}$$
$$N^p_{K,\alpha}(u_\lambda) = \left\| D^\alpha u_\lambda(x) \right\|_{L^p(K)}, K \Subset O,$$

and $D^\alpha = \dfrac{\partial^{\alpha_1 + \ldots + \alpha_d}}{\partial z_1^{\alpha_1} \ldots \partial z_d^{\alpha_d}}$ for $z = (z_1, \ldots, z_d) \in O$, $l \in \mathbb{N}$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, $K \Subset O$ means that K is a compact subset of O. Let A be a subring of the ring $\mathbb{R}^\Lambda$. We consider a solid ideal I_A of A. Put

$$\mathcal{X}_p(O) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(O)]^\Lambda : \forall K \Subset O, \forall l \in \mathbb{N}, \left(N^p_{K,l}(u_\lambda)\right)_\lambda \in |A|\},$$

$$\mathcal{N}_p(O) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(O)]^\Lambda : \forall K \Subset O, \forall l \in \mathbb{N}, \left(N^p_{K,l}(u_\lambda)\right)_\lambda \in |I_A|\},$$

$$\mathcal{A}_p(O) = \mathcal{X}_p(O) / \mathcal{N}_p(O).$$

The generalized derivation $D^\alpha : u(=[u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provides $\mathcal{A}_p(O)$ with a differential algebraic structure.

For $p = +\infty$, $\mathcal{E}(O) = \mathrm{C}^\infty(O)$ is endowed with the $\mathcal{P}(O)$ topology defined by the family of the seminorms

$$P_{K,l}(u_\lambda) = \sup_{|\alpha|\le l} P_{K,\alpha}(u_\lambda), \text{ with}$$
$$P_{K,\alpha}(u_\lambda) = \|D^\alpha u_\lambda(x)\|_{L^\infty(K)}\,, \;\; K \Subset O$$

Put

$$\mathcal{X}(O) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(O)]^\Lambda : \forall K \Subset O, \forall l \in \mathbb{N},\; (P_{K,l}(u_\lambda))_\lambda \in |A|\},$$
$$\mathcal{N}(O) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(O)]^\Lambda : \forall K \Subset O, \forall l \in \mathbb{N},\; (P_{K,l}(u_\lambda))_\lambda \in |I_A|\}.$$

The generalized derivation $D^\alpha : u(=[u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provides $\mathcal{A}(O) = \mathcal{X}(O)/\mathcal{N}(O)$ with a differential algebraic structure.

Remark 2.1. The $N^2_{K,l}$ norms are bounded by the $P_{K,l}$ norms. We have $\mathcal{A}(O) \subset \mathcal{A}_2(O)$.

Remark 2.2. $\mathcal{A}_p(O)$ have properties similar to those of $\mathcal{A}(O)$.

2.3. Stochastic analysis

We refer the reader to [11], [12], [10] and [9], for construction of white noise and the relation between the white noise and Wiener process on $\mathbb{R}^d$.

Let (Ω, Σ, μ) be a probability space. A generalized stochastic process on $\mathbb{R}^d$ is a weakly measurable map

$$X : \Omega \to \mathcal{D}'\left(\mathbb{R}^d\right)$$

For any fixed test function $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$, the map $\Omega \to \mathbb{R};\;\; \omega \mapsto \langle X(\omega), \varphi\rangle$ is a random variable. The space of generalized stochastic processes is denoted by $\mathcal{D}'_\Omega\left(\mathbb{R}^d\right)$.

2.4. Algebras of generalized stochastic processes

Let O be an open set in $\mathbb{R}^d$, (Ω, Σ, μ) a probability space.

Definition 2.3. A mapping $U : \Omega \to \mathcal{A}(O)$ such that there is a representing function

$$u = R_U : \Lambda \times O \times \Omega \to \mathbb{R}$$

with the properties:
(i) for fixed $\lambda \in \Lambda$, the map $(x, \omega) \mapsto u(\lambda, x, \omega)$ is jointly measurable on $O \times \Omega$;
(ii) almost surely in $\omega \in \Omega$, the map $\lambda \mapsto u(\lambda, \cdot, \omega)$ belongs to $\mathcal{X}(O)$ and it is a representative of $U(\omega)$, i.e. almost surely in $\omega \in \Omega$, $(U(\omega)_\lambda)_\lambda = (u(\lambda, \cdot, \omega))_\lambda \in \mathcal{X}(O)$, is called a $\mathcal{A}(O)$-generalized stochastic processes on the probability space (Ω, Σ, μ).
The algebra of generalized stochastic processes is denoted by $\mathcal{A}^\Omega(O)$.

Definition 2.4. A map $U : \Omega \to \mathcal{A}_2(O)$ such that there is a representing function

$$u = R_U : \Lambda \times O \times \Omega \to \mathbb{R}$$

with the properties:
(i) for fixed $\lambda \in \Lambda$, the map $(x, \omega) \mapsto u(\lambda, x, \omega)$ is jointly measurable on $O \times \Omega$;
(ii) almost surely in $\omega \in \Omega$, the map $\lambda \mapsto u(\lambda, \cdot, \omega)$ belongs to $\mathcal{X}_2(O)$ and it is a representative of $U(\omega)$, i.e. almost surely in $\omega \in \Omega$, $(U(\omega)_\lambda)_\lambda = (u(\lambda, \cdot, \omega))_\lambda \in \mathcal{X}_2(O)$, is called a $\mathcal{A}_2(0)$-generalized stochastic processes on the probability space (Ω, Σ, μ).
The algebra of generalized stochastic processes is denoted by $\mathcal{A}_2^\Omega(O)$.

Remark 2.5. Let φ of the form $\varphi(x, y) = \chi(x)\chi(y)$ with $\chi \in \mathcal{D}(\mathbb{R})$ with the property

$$\int \chi(s)ds = 1; \int s^p \chi(s)ds = 0, 1 \leq p \leq 2.$$

Let $V \in \mathcal{D}'_\Omega\left(\mathbb{R}^d\right)$ be a generalized stochastic process. If $\lambda \in \Lambda$, then $V(\omega) * \varphi_\lambda$ is measurable with respect to $\omega \in \Omega$ and smooth with respect to $x \in \mathbb{R}^d$, hence jointly measurable. So, $(V(\omega) * \varphi_\lambda)_\lambda$ belongs to $\mathcal{X}(\mathbb{R}^d)$. Then

$$R_V(\lambda, x, \omega) = (V(\omega) * \varphi_\lambda)(x) = V(\omega)_\lambda(x)$$

qualifies as a representative for a random generalized function. Thus we have an imbedding $\tau : \mathcal{D}'_\Omega\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{A}^\Omega(\mathbb{R}^d)$.

3. Some nonlinear stochastic problems

3.1. A nonlinear stochastic problem with additive generalized stochastic process

Consider the Cauchy problem formally written:

$$(P) : \frac{\partial U}{\partial t} = F(U) + W, \ U|_\gamma = f, \tag{1}$$

where γ is a monotonic curve of equation $x = l(t)$, γ is not a characteristic curve, $f \in \mathcal{A}^\Omega(\mathbb{R})$, $W \in \mathcal{A}^\Omega(\mathbb{R}^2)$ is a $\mathcal{A}\left(\mathbb{R}^2\right)$-generalized stochastic process on a probability space (Ω, Σ, μ). F is smooth, it can be non Lipschitz but F and all derivatives have polynomial growth and $F(0) = 0$. We look for a solution $\left(U : \Omega \to \mathcal{A}_2\left(\mathbb{R}^2\right)\right) \in \mathcal{A}_2^\Omega(\mathbb{R}^2)$. (For example, $F(U) = -U - U^2$ or $F(U) = -U^2$.)

Thus U is a solution to problem (P) if and only if for each $\omega \in \Omega$, $U(\omega)$ is solution to the formally written problem

$$(P(\omega)) \begin{cases} \dfrac{\partial U(\omega)}{\partial t} = F(U(\omega)) + W(\omega), \\ U(\omega)|_\gamma = f(\omega). \end{cases}$$

3.2. A nonlinear stochastic problem with multiplicative generalized stochastic process

Consider the Cauchy problem formally written:

$$(P') : \frac{\partial U}{\partial t} = F(U)W, \, U|_{\gamma} = f, \tag{2}$$

where γ is a monotonic curve of equation $x = l(t)$, γ is not a characteristic curve, $f \in \mathcal{A}^{\Omega}(\mathbb{R})$, $W \in \mathcal{A}^{\Omega}(\mathbb{R}^2)$ is a $\mathcal{A}\left(\mathbb{R}^2\right)$-generalized stochastic process on a probability space (Ω, Σ, μ). F is smooth, it can be non Lipschitz, but F and all derivatives have polynomial growth and $F(0) = 0$. We look for a solution $\left(U : \Omega \to \mathcal{A}_2\left(\mathbb{R}^2\right)\right) \in \mathcal{A}_2^{\Omega}(\mathbb{R}^2)$. (For example, $F(U) = -U^2$).

Thus U is a solution to problem (P) if and only if for each $\omega \in \Omega$, $U(\omega)$ is solution to the formally written problem

$$P'(\omega)) \begin{cases} \dfrac{\partial U(\omega)}{\partial t} = F(U(\omega))W(\omega), \\ U(\omega)|_{\gamma} = f(\omega). \end{cases}$$

3.3. Cut off procedure

Take $(r_\eta)_\eta$ be in $\mathbb{R}_*^{(0,1]}$ such that $r_\eta > 0$ and $\lim\limits_{\eta \to 0} r_\eta = +\infty$. Set $E_\eta = [-r_\eta, r_\eta]$.

Set a family of smooth one-variable functions $(h_\eta)_\eta$ such that

$$\sup_{z \in I_\eta} |h_\eta(z)| = 1, \, h_\eta(z) = \begin{cases} 0, \text{ if } \; |z| \geq r_\eta \\ 1, \text{ if } \; -r_\eta + 1 \leq z \leq r_\eta - 1 \end{cases} \tag{3}$$

Suppose that $\dfrac{\partial^n h_\eta}{\partial z^n}$ is bounded on E_η for any integer n, $n > 0$. Set

$$\sup_{z \in E_\eta} \left| \frac{\partial^n h_\eta}{\partial z^n}(z) \right| = M_n.$$

Let $\phi_\eta(z) = z h_\eta(z)$. We approximate the function F by the family of functions $(F_\eta)_\eta$ defined by

$$F_\eta(z) = F(\phi_\eta(z)) = F(z h_\eta(z)).$$

Suppose that $F(0) = 0$. F is smooth, it can be non Lipschitz but F and all derivatives have polynomial growth. More precisely, we assume the existence of $p \in \mathbb{N}$ such that

$$\forall l \in \mathbb{N}, \exists c_l > 0, \sup_{z \in \mathbb{R}} \left| D^l F(z) \right| \leq c_l (1 + |z|)^p.$$

Then

$$\forall l \in \mathbb{N}, \exists \mu_l > 0, \sup_{z \in \mathbb{R}; |\alpha| \leq l} |D^\alpha F_\eta(z)| = \sup_{|z| \leq r_\eta; |\alpha| \leq l} |D^\alpha F(\phi_\eta(z))| \leq a_l (1 + r_\eta)^p.$$

Thus, according to [3], [4], $\mathcal{A}(\mathbb{R})$ is stable under the family $(F_\eta)_\eta$.

3.4. Construction of $\mathcal{A}(\mathbb{R}^2)$

Take $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ and

$$W(\omega)_{\varepsilon,\eta}(t,x) = \left(\phi_\eta(W(\omega) * \varphi_\varepsilon)\right)(t,x),$$

φ of the form $\varphi(t,x) = \chi(t)\chi(x)$, $\chi \in \mathcal{D}(\mathbb{R})$ having the property

$$\int \chi(s)ds = 1; \int s^p \chi(s)ds = 0, 1 \leq p \leq 2$$

and $(\chi_\varepsilon)_\varepsilon$ being a family of mollifiers such that $\varkappa_\varepsilon(x) = \frac{1}{\varepsilon}\chi(\frac{x}{\varepsilon})$, thus $\varphi_\varepsilon(t,x) = \varkappa_\varepsilon(t)\varkappa_\varepsilon(x)$. Take $f(\omega)_\varepsilon = f(\omega) * \chi_\varepsilon$.

We make the following assumptions to generate a convenient $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra adapted to our problem.

$$(H_1): \exists p > 0, \forall n \in \mathbb{N}, \exists a_n > 0, \sup_{z \in \mathbb{R}; |\alpha| \leq n} |D^\alpha F_\eta(z)| \leq a_n (1 + r_\eta)^p.$$

$$(H_2): \forall K \Subset \mathbb{R}^2, \forall n \in \mathbb{N}, \exists \rho_n > 0, P_{K,n}(W(\omega)_{\varepsilon,\eta}) \leq \rho_n (1 + r_\eta)^p.$$

$$(H_3) \begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{(0,1]} \\ (\varepsilon)_{\varepsilon,\eta}, (\eta)_{\varepsilon,\eta}, (r_\eta)_{\varepsilon,\eta}, (\exp(1 + r_\eta))_{\varepsilon,\eta}. \end{cases}$$

$$(H_4) \begin{cases} \mathcal{A}\left(\mathbb{R}^2\right) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2) \text{ is built on } \mathcal{C} \text{ with} \\ (\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K \Subset \mathbb{R}^2, l \in \mathbb{N}}\right). \end{cases}$$

$$\begin{aligned} (H_5)\, \mathcal{A}_2\left(\mathbb{R}^2\right) &= \mathcal{X}_2(\mathbb{R}^2)/\mathcal{N}_2(\mathbb{R}^2) \text{ is built on } \mathcal{C} \\ \text{with } (\mathcal{E}, \mathcal{P}') &= \left(\mathrm{C}^\infty(\mathbb{R}^2), \left(N^2_{K,l}\right)_{K \Subset \mathbb{R}^2, l \in \mathbb{N}}\right). \end{aligned}$$

3.5. Generalized differential problems associated to the formal ones

We give a meaning to the problems formally written as (P) and (P').

F_η is defined above. Let $\mathcal{F}$ be the generalized operator associated to F via the family $(h_\eta)_\eta$. $\mathcal{R}_l$ is the generalized second-size mapping associated with l [5].

3.5.1. Generalized differential problem associated to (P)

For ω fixed, the problem associated to $(P(\omega))$ can be written as the well-formulated problem

$$(P(\omega)_{gen}) \begin{cases} \dfrac{\partial U(\omega)}{\partial t} = \mathcal{F}(U(\omega)) + [W(\omega)_{\varepsilon,\eta}], \\ \mathcal{R}_l(U(\omega)) = [f(\omega)_\varepsilon], \end{cases}$$

then

$$\begin{cases} \dfrac{\partial U(\omega)}{\partial t} = [F_\eta(U(\omega))] + [W(\omega)_{\varepsilon,\eta}]\,, \\ U(\omega)|_\gamma = [f(\omega) * \chi_\varepsilon]\,. \end{cases}$$

In terms of representatives, and thanks to the stability and restriction hypothesis, if we find $U(\omega)_{\varepsilon,\eta} \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$(P(\omega)_{(\varepsilon,\eta)}) \begin{cases} \dfrac{\partial U(\omega)_{\varepsilon,\eta}}{\partial t}(t,x) = F_\eta(U(\omega)_{\varepsilon,\eta}(t,x)) + W(\omega)_{\varepsilon,\eta}(t,x), \\ U(\omega)_{\varepsilon,\eta}(t,l(t)) = f(\omega)_\varepsilon(t) = (f(\omega) * \chi_\varepsilon)(t)\,, \end{cases}$$

and if we prove that $(U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{X}_2(\mathbb{R}^2)$, thus $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ is a solution of $P(\omega)_{gen}$.

Let $V(\omega) = [V(\omega)_{\varepsilon,\eta}]$ be another solution to $P(\omega)_{gen}$.
If $(V(\omega)_{\varepsilon,\eta} - U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{N}(\mathbb{R}^2)$ the solution to $P_{gen}(\omega)$ is unique.

Remark 3.1. Dependence on some regularizing family. The problem $P(\omega)_{gen}$ itself, so a solution of it, a priori depends on the family of cutoff functions and, in the case of irregular data, on the family of mollifiers [4].

Remark 3.2. $F(U)$ is such that

$$F(U) : \Omega \to \mathcal{A}\left(\mathbb{R}^2\right),\ \omega \mapsto [F_\eta(U(\omega)_{\varepsilon,\eta})]$$

and

$$F_\eta(U(\omega)_{\varepsilon,\eta}) : \mathbb{R}^2 \to \mathbb{R},\ (t,x) \mapsto F_\eta(U(\omega)_{\varepsilon,\eta}(t,x)).$$

Moreover

$$R_U = u : \Lambda \times \mathbb{R}^2 \times \Omega \to \mathbb{R}; (\lambda,(t,x),\omega) \mapsto U(\omega)_{\varepsilon,\eta}(t,x) = u(\lambda,t,x,\omega)\,,$$

with $\lambda = (\varepsilon,\eta)$.

3.5.2. Generalized differential problem associated to (P')

For ω fixed, the problem associated to $(P'(\omega))$ can be written as the well-formulated problem

$$(P'_{gen}(\omega)) \begin{cases} \dfrac{\partial U(\omega)}{\partial t} = \mathcal{F}(U(\omega))\,[W(\omega)_{\varepsilon,\eta}]\,, \\ \mathcal{R}_l(U(\omega)) = [f(\omega)_\varepsilon]\,, \end{cases}$$

then

$$\begin{cases} \dfrac{\partial U(\omega)}{\partial t} = [F_\eta(U(\omega))]\,[W(\omega)_{\varepsilon,\eta}]\,, \\ U(\omega)|_\gamma = [f(\omega) * \chi_\varepsilon]\,. \end{cases}$$

In terms of representatives, and thanks to the stability and restriction hypothesis, if we find $U(\omega)_{\varepsilon,\eta} \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$(P'(\omega)_{(\varepsilon,\eta)}) \begin{cases} \dfrac{\partial U(\omega)_{\varepsilon,\eta}}{\partial t}(t,x) = F_\eta(U(\omega)_{\varepsilon,\eta}(t,x))W(\omega)_{\varepsilon,\eta}(t,x), \\ U(\omega)_{\varepsilon,\eta}(t,l(t)) = f(\omega)_\varepsilon(t) = (f(\omega) * \chi_\varepsilon)(t)\,, \end{cases}$$

and if we prove that $(U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{X}_2(\mathbb{R}^2)$, thus $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ is a solution of $(P'(\omega)_{gen})$.

3.6. Generalized problems

3.6.1. Solution to the parametrized regular problems

Fix ω, consider the regularized problems $(P(\omega)_{(\varepsilon,\eta)})$ and $(P'(\omega)_{(\varepsilon,\eta)})$.

Under assumptions (H_1), (H_2) and the assumptions

$$(H_{\varepsilon,\eta})\begin{cases} \text{a)} & l \in \mathrm{C}^{\infty}(\mathbb{R}),\ l' > 0,\ l(\mathbb{R}) = \mathbb{R}, \\ \text{b)} & F_\eta \in \mathrm{C}^{\infty}(\mathbb{R},\mathbb{R}), \\ \text{c)} & f\left(\omega\right)_{\varepsilon} \in \mathrm{C}^{\infty}(\mathbb{R}), \end{cases}$$

one can prove that $(P\left(\omega\right)_{(\varepsilon,\eta)})$ admits a unique smooth solution $U(\omega)_{\varepsilon,\eta}$ such that

$$U(\omega)_{\varepsilon,\eta}(t,x) = f\left(\omega\right)_{\varepsilon}(l^{-1}(x)) + \int_{l^{-1}(x)}^{t} \left(F_\eta(U(\omega)_{\varepsilon,\eta}(\zeta,x)) + W(\omega)_{\varepsilon,\eta}(\zeta,x)\right) d\zeta$$

and $(P'\left(\omega\right)_{(\varepsilon,\eta)})$ admits a unique smooth solution $U(\omega)_{\varepsilon,\eta}$ such that

$$U(\omega)_{\varepsilon,\eta}(t,x) = f\left(\omega\right)_{\varepsilon}(l^{-1}(x)) + \int_{l^{-1}(x)}^{t} \left(F_\eta(U(\omega)_{\varepsilon,\eta}(\zeta,x))W(\omega)_{\varepsilon,\eta}(\zeta,x)\right) d\zeta,$$

Theorem 3.3. *Under assumptions $(H_{\varepsilon,\eta})$, (H_1) and (H_2), problem $(P\left(\omega\right)_{(\varepsilon,\eta)})$ (resp. $(P'\left(\omega\right)_{(\varepsilon,\eta)})$)has a unique solution, $U(\omega)_{\varepsilon,\eta}$, in $C^{\infty}(\mathbb{R}^2)$.*

See [1].

3.6.2. Solution to the problems

Theorem 3.4. *Suppose that $U(\omega)_{\varepsilon,\eta}$ is the solution to problem $(P\left(\omega\right)_{(\varepsilon,\eta)})$ (resp. $(P'\left(\omega\right)_{(\varepsilon,\eta)})$) then problem $(P(\omega)_{gen})$ (resp. $(P'\left(\omega\right)_{gen})$) has a unique solution $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ in $\mathcal{A}\left(\mathbb{R}^2\right)$.*

$U(\omega)$ is the solution to $(P\left(\omega\right)_{(\varepsilon,\eta)})$ (resp. $(P'\left(\omega\right)_{(\varepsilon,\eta)})$) if $\left(U(\omega)_{\varepsilon,\eta}\right)_{\varepsilon,\eta} \in \mathcal{X}(\mathbb{R}^2)$, that is

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \left(P_{K,l}(U(\omega)_{\varepsilon,\eta})\right)_{\varepsilon,\eta} \in A.$$

The proof follows the same steps as the existence results which can be found in [3], replacing $u_{\varepsilon,\eta}$ by $U(\omega)_{\varepsilon,\eta}$ and $F_\eta(x,y,u_{\varepsilon,\eta}(x,y))$ by $F_\eta(U(\omega)_{\varepsilon,\eta}(x,y)) + W(\omega)_{\varepsilon,\eta}(x,y)$ (resp. $F_\eta(U(\omega)_{\varepsilon,\eta}(x,y))W(\omega)_{\varepsilon,\eta}(x,y)$). An induction process on the order of the successive derivatives shows that $\left(U(\omega)_{\varepsilon,\eta}\right)_{\varepsilon,\eta}$ belongs to $\mathcal{X}(\mathbb{R}^2)$. For the uniqueness, the Gronwall lemma is an essential tool.

Theorem 3.5. *Suppose that $U(\omega)_{\varepsilon,\eta}$ is the solution to problem $(P\left(\omega\right)_{(\varepsilon,\eta)})$ (resp. $(P'\left(\omega\right)_{(\varepsilon,\eta)})$) then problem $(P(\omega)_{gen})$ (resp. $(P'\left(\omega\right)_{gen})$) has a unique solution $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$.*

Proof. $U(\omega)$ is the solution to $(P(\omega)_{(\varepsilon,\eta)})$ if $(U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{X}_2(\mathbb{R}^2)$. We must prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \left(N^2_{K,l}(U(\omega)_{\varepsilon,\eta})\right)_{\varepsilon,\eta} \in A.$$

However

$$\|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_{L^2(K)} \leq (\mu(K))^{1/2} \|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_{L^\infty(K)}$$

and, as $(U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{X}(\mathbb{R}^2)$, we have $\left(\|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_\infty\right)_{\varepsilon,\eta} \in A$. Then

$$(\|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_{L^2(K)})_{\varepsilon,\eta} = \left(N^2_{K,l}(U(\omega)_{\varepsilon,\eta})\right)_{\varepsilon,\eta} \in A.$$

So $U(\omega) \in \mathcal{A}_2\left(\mathbb{R}^2\right)$ and it is the solution to problem $(P(\omega)_{gen})$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$. Set

$$U : \Omega \to \mathcal{A}_2\left(\mathbb{R}^2\right), \omega \mapsto U(\omega).$$

Thus $U \in \mathcal{A}_2^\Omega(\mathbb{R}^2)$. □

Theorem 3.6. *The mapping U is the solution to problem (P) (resp. (P')) and it is almost surely unique in $\mathcal{A}_2^\Omega(\mathbb{R}^2)$.*

Proof. Since $U(\omega)$ is the unique solution to problem $(P(\omega)_{gen})$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$ then almost surely in $\omega \in \Omega$, the map $\lambda \mapsto R_U(\lambda, (\cdot,\cdot), \omega) = U(\omega)_\lambda$, $(\lambda = (\varepsilon,\eta))$, belongs to $\mathcal{X}_2(\mathbb{R}^2)$ and it is a representative of $U(\omega)$ (i.e. $U(\omega) = [U(\omega)_\lambda]$). For fixed $\lambda = (\varepsilon,\eta) \in \Lambda$, the map

$$((x,y),\omega) \mapsto R_U(\lambda,(x,y),\omega) = U(\omega)_\lambda(x,y) = u_\lambda((x,y),\omega)$$

is jointly measurable on $\mathbb{R}^2 \times \Omega$. Then U is the solution to problem (P) almost surely unique in $\mathcal{A}_2^\Omega(\mathbb{R}^2)$. □

3.7. A special case

Consider the Cauchy problem formally written as

$$(S) : \frac{\partial U}{\partial t} = W, U|_\gamma = f, \tag{2}$$

where γ is a monotonic curve of equation $x = l(t)$, γ is not a characteristic curve, $f \in \mathcal{A}^\Omega(\mathbb{R})$, $W \in \mathcal{A}^\Omega(\mathbb{R}^2)$ is a $\mathcal{A}\left(\mathbb{R}^2\right)$-generalized stochastic process on a probability space (Ω,Σ,μ).

This problem coincides with problem (P) for $F = 0$ and with problem (P') for $F = 1$. Problem (S) admits a solution U. $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$ is defined, with the previous notations, by

$$U(\omega)_{\varepsilon,\eta}(t,x) = f(\omega)_\varepsilon\left(l^{-1}(x)\right) + \int_{l^{-1}(x)}^t W(\omega)_{\varepsilon,\eta}(\zeta,x)d\zeta.$$

4. A nonlinear stochastic Cauchy problem with the white noise as data

Consider the Cauchy problems formally written:

$$(P_1) : \frac{\partial U}{\partial t} = F(U), \, U|_{\gamma} = W, \tag{4}$$

and

$$(P_2) : \frac{\partial V}{\partial t} = 0, \, V|_{\gamma} = W, \tag{5}$$

where γ is a monotonic curve of equation $x = l(t)$, γ is not a characteristic curve, $W \in \mathcal{A}^{\Omega}(\mathbb{R})$ is the white noise on $\mathbb{R}$. F is smooth, it can be non Lipschitz but F and all its derivatives have polynomial growth. We look for a solution $\left(U : \Omega \to \mathcal{A}\left(\mathbb{R}^2\right)\right) \in \mathcal{A}^{\Omega}(\mathbb{R}^2)$ and $\left(V : \Omega \to \mathcal{A}\left(\mathbb{R}^2\right)\right) \in \mathcal{A}^{\Omega}(\mathbb{R}^2)$.

U is a solution to problem (P_1) if and only if for every $\omega \in \Omega$, $U(\omega)$ is a solution to the formal problem

$$(P_1(\omega)) : \frac{\partial U(\omega)}{\partial t} = F(U(\omega)), \, U(\omega)|_{\gamma} = W(\omega).$$

V is a solution to problem (P_2) if and only if, for any $\omega \in \Omega$, $V(\omega)$ is a solution to the formally problem

$$(P_2(\omega)) : \frac{\partial V(\omega)}{\partial t} = 0, \, V(\omega)|_{\gamma} = W(\omega).$$

We make the same hypotheses and we take the same spaces $\mathcal{A}\left(\mathbb{R}^2\right)$ and $\mathcal{A}_2\left(\mathbb{R}^2\right)$ built for problems (P) and (P').

4.1. A generalized differential problem associated to the formal one

For ω fixed, the problem associated to $(P_1(\omega))$ can be written as the well-formulated problem

$$(P_{1gen}(\omega)) \begin{cases} \dfrac{\partial U(\omega)}{\partial t} = \mathcal{F}(U(\omega)), \\ \mathcal{R}_l\left(U(\omega)\right) = \left[W(\omega)_{\varepsilon}\right], \end{cases}$$

then

$$\begin{cases} \dfrac{\partial U(\omega)}{\partial t} = \left[F_{\eta}(U(\omega))\right], \\ U(\omega)|_{\gamma} = \left[W\left(\omega\right) * \chi_{\varepsilon}\right]. \end{cases}$$

The problem associated to $(P_2(\omega))$ can be written as the well-formulated problem

$$(P_{2gen}(\omega)) : \frac{\partial V(\omega)}{\partial t} = 0, \mathcal{R}_l\left(V(\omega)\right) = \left[W(\omega)_{\varepsilon}\right],$$

so

$$\frac{\partial V(\omega)}{\partial t} = 0, \, V(\omega)|_{\gamma} = \left[W\left(\omega\right) * \chi_{\varepsilon}\right].$$

In terms of representatives, and thanks to the stability and restriction hypothesis, if we find $U(\omega)_{\varepsilon,\eta} \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$(P_1(\omega)_{(\varepsilon,\eta)}) \begin{cases} \dfrac{\partial U(\omega)_{\varepsilon,\eta}}{\partial t}(t,x) = F_\eta(U(\omega)_{\varepsilon,\eta}(t,x)), \\ U(\omega)_{\varepsilon,\eta}(t,l(t)) = (W(\omega) * \chi_\varepsilon)(t) \end{cases}$$

and if we prove that $\left(U(\omega)_{\varepsilon,\eta}\right)_{\varepsilon,\eta} \in \mathcal{X}_2(\mathbb{R}^2)$, then $U(\omega) = \left[U(\omega)_{\varepsilon,\eta}\right]$ is a solution of $(P_1(\omega)_{gen})$.

As $\dfrac{\partial V(\omega)}{\partial t} = 0$, we have

$$V(\omega)_{\varepsilon,\eta}(t,x) = W(\omega) * \chi_\varepsilon(l^{-1}(x)).$$

4.2. Generalized problem

4.2.1. Solution to the parametrized regular problem

For ω fixed consider the family of regularized problems $\left(P_1(\omega)_{(\varepsilon,\eta)}\right)$. We must prove that $\left(P_1(\omega)_{(\varepsilon,\eta)}\right)$ has a unique smooth solution under the following assumptions

$$(H_{\varepsilon,\eta}) \begin{cases} \text{a)} & l \in \mathrm{C}^\infty(\mathbb{R}),\ l' > 0,\ l(\mathbb{R}) = \mathbb{R}, \\ \text{b)} & F_\eta \in \mathrm{C}^\infty(\mathbb{R},\mathbb{R}), \\ \text{c)} & W(\omega)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}). \end{cases}$$

$$(H_1) : \exists p > 0, \forall l \in \mathbb{N}, \exists a_l > 0, \sup_{z\in\mathbb{R};|\alpha|\le l} |D^\alpha F_\eta(z)| \le a_l(1+r_\eta)^p.$$

one can prove that $(P_1(\omega)_{(\varepsilon,\eta)})$ admits a unique smooth solution $U(\omega)_{\varepsilon,\eta}$ such that

$$U(\omega)_{\varepsilon,\eta}(t,x) = (W(\omega) * \chi_\varepsilon)(l^{-1}(x)) + \int_{l^{-1}(x)}^t F_\eta(U(\omega)_{\varepsilon,\eta}(\zeta,x))d\zeta$$

Theorem 4.1. *Under assumptions $(H_{\varepsilon,\eta})$, (H_1), problem $(P_1(\omega)_{(\varepsilon,\eta)})$ has a unique solution, $U(\omega)_{\varepsilon,\eta}$, in $C^\infty(\mathbb{R}^2)$.*

4.2.2. Solution to (P_1)

Theorem 4.2. *Suppose that $U(\omega)_{\varepsilon,\eta}$ is the solution to problem $(P_1(\omega)_{(\varepsilon,\eta)})$ then problem $(P_1(\omega)_{gen})$ has a unique solution $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ in $\mathcal{A}\left(\mathbb{R}^2\right)$.*

The proof follows the same steps as the existence results which can be found in [3] (replacing $u_{\varepsilon,\eta}$ by $U(\omega)_{\varepsilon,\eta}$ and $F_\eta(x,y,u_{\varepsilon,\eta}(x,y))$ by $F_\eta(U(\omega)_{\varepsilon,\eta}(x,y))$).

Theorem 4.3. *Suppose that $U(\omega)_{\varepsilon,\eta}$ is the solution to problem $(P_1(\omega)_{(\varepsilon,\eta)})$ then problem $(P_1(\omega)_{gen})$ has a unique solution $U(\omega) = [U(\omega)_{\varepsilon,\eta}]$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$.*

Proof. $U(\omega)$ is the solution to $(P_1(\omega)_{(\varepsilon,\eta)})$ if $(U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{X}_2(\mathbb{R}^2)$. We must prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \left(N^2_{K,l}(U(\omega)_{\varepsilon,\eta})\right)_{\varepsilon,\eta} \in A.$$

However

$$\|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_{L^2(K)} \leq (\mu(K))^{1/2} \|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_{L^\infty(K)}$$

and, as $(U(\omega)_{\varepsilon,\eta})_{\varepsilon,\eta} \in \mathcal{X}(\mathbb{R}^2)$, we have $\left(\|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_\infty\right)_{\varepsilon,\eta} \in A$. Then

$$(\|D^\alpha(U(\omega)_{\varepsilon,\eta})\|_{L^2(K)})_{\varepsilon,\eta} = \left(N^2_{K,l}(U(\omega)_{\varepsilon,\eta})\right)_{\varepsilon,\eta} \in A.$$

Thus $U(\omega) \in \mathcal{A}_2\left(\mathbb{R}^2\right)$ and it is the solution to problem $(P_1(\omega)_{gen})$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$. Set

$$U : \Omega \to \mathcal{A}_2\left(\mathbb{R}^2\right), \omega \mapsto U(\omega).$$

Thus $U \in \mathcal{A}_2^\Omega(\mathbb{R}^2)$. □

Theorem 4.4. *The mapping* U *is the solution to problem* (P_1) *and it is almost surely unique in* $\mathcal{A}_2^\Omega(\mathbb{R}^2)$.

Proof. Since $U(\omega)$ is the unique solution to problem $(P_1(\omega)_{gen})$ in $\mathcal{A}_2\left(\mathbb{R}^2\right)$ thus almost surely in $\omega \in \Omega$, the map $\lambda \mapsto R_U(\lambda, (\cdot,\cdot), \omega) = U(\omega)_\lambda$, $(\lambda = (\varepsilon, \eta))$, belongs to $\mathcal{X}_2(\mathbb{R}^2)$ and it is a representative of $U(\omega)$ (i.e. $U(\omega) = [U(\omega)_\lambda]$). For fixed $\lambda = (\varepsilon, \eta) \in \Lambda$, the map

$$((x,y), \omega) \mapsto R_U(\lambda, (x,y), \omega) = U(\omega)_\lambda(x,y)$$

is jointly measurable on $\mathbb{R}^2 \times \Omega$. Then U is the solution to problem (P_1) almost surely unique in $\mathcal{A}_2^\Omega(\mathbb{R}^2)$. □

4.3. Limiting behavior of the solution

See [11], [12]. Take $W_\varepsilon = (W(\omega) * \chi_\varepsilon)$. We have $E(W_\varepsilon) = 0$ and $V(W_\varepsilon) = \sigma_\varepsilon^2 = \|\chi_\varepsilon\|^2_{L^2(\mathbb{R})}$. Then the variance of W_ε tends to infinity as ε tends to 0. That implies

Theorem 4.5. *There is a subsequence* $\varepsilon_k \to 0$ *such that* μ*-almost surely in* $\omega \in \Omega$,

$$\lim_{k\to 0} |R_V((\varepsilon_k, \eta), (t,x), \omega)| = \lim_{k \to 0} |V(\omega)_{\varepsilon_k,\eta}(t,x)| = \infty$$

for almost all $(x,y) \in \mathbb{R}^2$.

Proof. See [11] Corollary 1 and [12]. □

Suppose that $\lim_{|z|\to\infty} F(z) = L$. Define the function $M : \mathbb{R}^2 \to \mathbb{R}$ by $M(t,x) = tL$.

Theorem 4.6. *Under the assumptions above, every subsequence of $\varepsilon \to 0$ has a subsequence $\varepsilon_k \to 0$ such that for any compact set $K \Subset \mathbb{R}^2$*

$$\lim_{k\to 0} \left\| R_U\left(\left(\varepsilon_k,\eta\right),\left(\cdot,\cdot\right),\omega\right) - R_V\left(\left(\varepsilon_k,\eta\right),\left(\cdot,\cdot\right),\omega\right) - M\right\|_{L^1(K)} = 0$$

μ-almost surely.

That is

$$\lim_{k\to 0} \left\| U(\omega)_{\varepsilon_k,\eta} - V(\omega)_{\varepsilon_k,\eta} - M\right\|_{L^1(K)} = 0$$

μ-almost surely.

Proof. Take $\lambda = (\varepsilon,\eta)$. We have

$$\frac{\partial\left(U(\omega)_\lambda - V(\omega)_\lambda - M\right)}{\partial t} = \frac{\partial\left(U(\omega)_\lambda\right)}{\partial t} - \frac{\partial\left(V(\omega)_\lambda + M\right)}{\partial t}$$

and

$$\begin{aligned}
&\frac{\partial\left(U(\omega)_\lambda\right)}{\partial t} - \frac{\partial\left(V(\omega)_\lambda + M\right)}{\partial t}\\
&= F(U(\omega)_\lambda) - L\\
&= \left(F(U(\omega)_\lambda) - F(V(\omega)_\lambda + M)\right) + \left(F(V(\omega)_\lambda + M) - L\right)\\
&= \left(U(\omega)_\lambda - V(\omega)_\lambda - M\right)\int_0^1 \frac{\partial F}{\partial z}\left(U(\omega)_\lambda + \sigma\left(U(\omega)_\lambda - V(\omega)_\lambda - M\right)\right) d\sigma\\
&+ \left(F(V(\omega)_\lambda + M) - L\right).
\end{aligned}$$

So

$$\begin{aligned}
&\left\|\frac{\partial\left(U(\omega)_\lambda - V(\omega)_\lambda - M\right)}{\partial t}\right\|_{L^1(K)}\\
&\le \left\|U(\omega)_\lambda - V(\omega)_\lambda - M\right\|_{L^1(K)} \left\|\frac{\partial F}{\partial z}\right\|_{L^\infty(\mathbb{R})} + \left\|F(V(\omega)_\lambda + M) - L\right\|_{L^1(K)}
\end{aligned}$$

By Theorem 4.5, there is a subsequence $\varepsilon_k \to 0$ such that μ-almost surely in $\omega \in \Omega$ almost everywhere $\left((t,x)\in\mathbb{R}^2\right)$, $\lim\limits_{k\to 0} \left|V(\omega)_{\varepsilon_k,\eta}(t,x)\right| = \infty$.
As $\lim\limits_{|z|\to\infty} F(z) = L$, we deduce that

$$\lim_{k\to 0} \left\|F(V(\omega)_\lambda + M) - L\right\|_{L^1(K)} = 0$$

almost everywhere.

Hence by Lebesgue's theorem and Gronwall's lemma the assertion follows. □

Theorem 4.7. *Let $V \in \mathcal{D}'_\Omega\left(\mathbb{R}^2\right)$ be the distributional solution to the free equation (P_2). Then the representative $U(\omega)_{\varepsilon,\eta}$ of the generalized solution to the nonlinear problem (P_1) converges to $V+M$ with respect to the strong topology of $\mathcal{D}'\left(\mathbb{R}^2\right)$, in probability as $\varepsilon \to 0$.*

Proof. Let q be one of the defining seminorms of the strong topology of $\mathcal{D}'\left(\mathbb{R}^2\right)$. According to Theorem 4.6, every subsequence of $\varepsilon \to 0$ has a subsequence $\varepsilon_k \to 0$ such that for any compact set $K \Subset \mathbb{R}^2$

$$q(R_U((\varepsilon_k, \eta), (\cdot, \cdot), \omega) - R_V((\varepsilon_k, \eta), (\cdot, \cdot), \omega) - M) \to 0$$

almost surely. This is equivalent to convergence in probability. □

References

[1] Delcroix, A., Dévoué, V., and Marti, J.-A. Generalized solutions of singular differential problems. Relationship with classical solutions. *J. Math. Anal. Appl. 353*, 1 (2009), 386–402.

[2] Delcroix, A., Dévoué, V., and Marti, J.-A. Well-posed problems in algebras of generalized functions. *Appl. Anal. 90*, 11 (2011), 1747–1761.

[3] Dévoué, V. Generalized solutions to a non Lipschitz-Cauchy problem. *J. Appl. Anal. 15*, 1 (2009), 1–32.

[4] Dévoué, V. Generalized solutions to a non-Lipschitz Goursat problem. *Differ. Equ. Appl. 1*, 2 (2009), 153–178.

[5] Dévoué, V. Some nonlinear stochastic cauchy problems with stochastic generalized processes. *International Journal of Analysis 2016* (2016), 11 pages.

[6] Marti, J.-A. Fundamental structures and asymptotic microlocalization in sheaves of generalized functions. *Integral Transform. Spec. Funct. 6*, 1-4 (1998), 223–228. Generalized functions—linear and nonlinear problems (Novi Sad, 1996).

[7] Marti, J.-A. $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-sheaf structures and applications. In *Nonlinear theory of generalized functions (Vienna, 1997)*, vol. 401 of *Chapman & Hall/CRC Res. Notes Math.* Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 175–186.

[8] Marti, J.-A. Multiparametric algebras and characteristic cauchy problem. In *Non-linear algebraic analysis and applications* (2004), Proceeding of the International Conference on Generalized functions (ICGF 2000), Cambridge Sci. Publ. Ltd., Cambridge, pp. 181–192.

[9] Nedeljkov, M., and Rajter, D. Nonlinear stochastic wave equation with Colombeau generalized stochastic processes. *Math. Models Methods Appl. Sci. 12*, 5 (2002), 665–688.

[10] Nedeljkov, M., and Rajter, D. Nonlinear stochastic wave equation with colombeau stochastic generalized processes. *Math. Models Methods Appl. Sci. 12*, 5 (2002), 665–688.

[11] Oberguggenberger, M., and Russo, F. Nonlinear stochastic wave equations. In *Generalized functions—linear and nonlinear problems (Novi Sad, 1996)*, vol. 6. 1998, pp. 71–83.

[12] Oberguggenberger, M., and Russo, F. Singular limiting behavior in nonlinear stochastic wave equations. In *Stochastic analysis and mathematical physics*, vol. 50 of *Progr. Probab.* Birkhäuser Boston, Boston, MA, 2001, pp. 87–99.

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