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On Huang, Jaggi-Das, Khan and Abbas type results in the context of $\mathcal{F}$-metric spaces and applications to integral equations

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Abstract. The purpose of this paper is to present some fixed point results in F-complete F-metric spaces. Our results are generalizations of Banach contraction principle and many other ones in exiting literature. Also, some examples and an application to an integral equation are given to illustrate the usefulness of the obtained results.

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1. Introduction and Preliminaries

Fixed point theory plays a pivotal role in functional and nonlinear analysis. The Banach contraction principle is an important result of the fixed point theory. In recent years, various extensions of metric spaces have been introduced (see e.g. [2, 3, 4, 5, 7, 8, 14, 17, 16, 18, 19, 20] and references therein). The notion of an $\mathcal{F}$-metric space was firstly introduced and studied by Jleli and Samet in [12] (see e.g. [6, 10, 15] and references therein).

In this paper, we also prove and extend some results of Huang et al.[9], Jaggi et al.[11], Khan[13] and Abbas et al.[1] to the context of $\mathcal{F}$-metric spaces. We recall some of the basic definitions and results in the sequel.
Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbb{R}$ such that
$\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.
$\mathcal{F}_2$) For every sequence $\{t_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to+\infty} t_n = 0 \text{ if and only if } \lim_{n\to+\infty} f(t_n) = -\infty.$$

Definition 1.1. [12] Let X be a (nonempty) set. A function $D : X \times X \to [0, +\infty)$ is called an $\mathcal{F}$-metric on X if there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that for all $x, y \in X$ the following conditions hold:
(D_1) $D(x, y) = 0$ if and only if $x = y$.
(D_2) $D(x, y) = D(y, x)$.

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(D_3) For every $N \in \mathbb{N}, N \geq 2$ and for every $\{u_i\}_{i=1}^N \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x,y) > 0 \text{ implies } f(D(x,y)) \leq f(\sum_{i=1}^{N-1} D(u_i, u_{i+1})) + \alpha.$$

In this case, the pair (X, D) is called an $\mathcal{F}$-metric space.

Example 1.2. [12] Let $X = \mathbb{R}$ and $D : X \times X \to [0, +\infty)$ be defined as follows:

$$D(x,y) = \begin{cases} (x-y)^2 & (x,y) \in [0,3] \times [0,3], \\ |x-y| & \text{otherwise}, \end{cases}$$

and let $f(t) = ln(t)$ for all $t > 0$ and $\alpha = ln(3)$. Then, D is an $\mathcal{F}$-metric on X. Since $D(1,3) = 4 \geq D(1,2) + D(2,3) = 2$, Then D is not a metric on X.

Example 1.3. [12] Let $X = \mathbb{R}$ and $D : X \times X \to [0, +\infty)$ be defined as follows:

$$D(x,y) = \begin{cases} e^{|x-y|} & x \neq y, \\ 0 & x = y. \end{cases}$$

Then, D is an $\mathcal{F}$-metric on X. Since $D(1,3) = e^2 \geq D(1,2) + D(2,3) = 2e$, Then D is not a metric on X.

Definition 1.4. [12] Let (X, D) be an $\mathcal{F}$-metric space and $\{x_n\}$ be a sequence in X.
1) A sequence $\{x_n\}$ is called $\mathcal{F}$-convergent to $x \in X$, if $\lim_{n\to+\infty} D(x_n, x) = 0$.
2) A sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, if and only if $\lim_{n,m\to+\infty} D(x_n, x_m) = 0$.
3) An $\mathcal{F}$-metric space (X, D) is said to be $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to some element in X.

Lemma 1.5. *[15] Let (X, D) be an $\mathcal{F}$-metric space . Let $\{x_n\}$ be a sequence in (X, D) such that*

$$D(x_n, x_{n+1}) \leq \lambda D(x_{n-1}, x_n), \quad n \in \mathbb{N},$$

for λ, $0 \leq \lambda < 1$. Then $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence in (X, D).

Theorem 1.6. *[12] Let (X, D) be $\mathcal{F}$-complete $\mathcal{F}$-metric space and let $T : X \to X$ be a self-mapping satisfying*

$$D(Tx, Ty) \leq \lambda D(x, y), \tag{1.1}$$

for all $x, y \in X$ where $0 \leq \lambda < 1$, then T has a unique fixed point.

Huang et al. [9] proved the following fixed point result in the setting of b-metric spaces.

Theorem 1.7. *Let (X,d) be a b-complete b-metric space with parameter $s \geq 1$ and $T : X \to X$ be a self-mapping such that for all $x, y \in X$*

$$\begin{aligned} d(Tx,Ty) \leq \lambda_1 d(x,y) + \lambda_2 \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \lambda_3 \frac{d(x,Ty)d(y,Tx)}{1+d(x,y)} \\ + \lambda_4 \frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} + \lambda_5 \frac{d(y,Ty)d(y,Tx)}{1+d(x,y)}, \end{aligned} \tag{1.2}$$

where $\lambda_1 + \lambda_2 + \lambda_3 + s\lambda_4 + s\lambda_5 < 1$. Then T has a unique fixed point.

Jaggi et al. [11], proved the following results.

Theorem 1.8. *Let (X,d) be a complete metric space. If a map $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq \lambda_1 d(x,y) + \lambda_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \tag{1.3}$$

for all $x, y \in X$, where $0 \leq \lambda_1 + \lambda_2 < 1$, Then T has a unique fixed point.

Khan [13] proved the following fixed point result for complete metric spaces.

Theorem 1.9. *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping such that*

$$d(Tx,Ty) \leq \lambda \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}, \tag{1.4}$$

for all $x, y \in X$, where $0 \leq \lambda < 1$. Then T has a unique fixed point.

Definition 1.10. [1] Let T and S be self maps of a set X. Two self-mappings T and S are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tx = Sx$ for some $x \in X$, then $TSx = STx$.

Proposition 1.11. *[1] Let T and S be weakly compatible self maps of a set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

Abbas et al. [1] proved following common fixed point theorem in a normal cone metric space.

Theorem 1.12. *Let (X,d) be a cone metric space and P a normal cone with normal constant K. Suppose the mappings $f, g : X \to X$ satisfy*

$$d(fx,fy) \leq kd(gx,gy),$$

for all $x, y \in X$, where $k \in [0,1)$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

2. Main results

In this section, we prove several fixed point theorems for mappings defined on an $\mathcal{F}$-metric space.

Theorem 2.1. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and let T be a self-mapping on X satisfying*

$$D(Tx,Ty) \le \lambda_1 D(x,y) + \lambda_2 \frac{D(x,Tx)D(y,Ty)}{1+D(x,y)} + \lambda_3 \frac{D(x,Ty)D(y,Tx)}{1+D(x,y)}, \tag{2.1}$$

for all $x, y \in X$, where $0 \le \lambda_1 + \lambda_2 + \lambda_3 < 1$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We can define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. In case $x_m = x_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$, then it is clear that x_m is a fixed point of T. So assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.1), we have

$$\begin{aligned}
D(x_n, x_{n+1}) &= D(Tx_{n-1}, Tx_n) \\
&\le \lambda_1 (x_{n-1}, x_n) + \lambda_2 \frac{D(x_{n-1}, Tx_{n-1})D(x_n, Tx_n)}{1+D(x_{n-1}, x_n)} \\
&\quad + \lambda_3 \frac{D(x_{n-1}, Tx_n)D(x_n, Tx_{n-1})}{1+D(x_{n-1}, x_n)} \\
&= \lambda_1 D(x_{n-1}, x_n) + \lambda_2 \frac{D(x_{n-1}, x_n)D(x_n, x_{n+1})}{1+D(x_{n-1}, x_n)} \\
&\quad + \lambda_3 \frac{D(x_{n-1}, x_{n+1})D(x_n, x_n)}{1+D(x_{n-1}, x_n)},
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, we have $D(x_n, x_{n+1}) \le \lambda D(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, where $\lambda = \frac{\lambda_1}{1-\lambda_2} < 1$. By Lemma 1.5, $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there exists $x^* \in X$ such that $\lim_{n\to+\infty} D(x_n, x^*) = 0$. We shall prove that x^* is a fixed point of T. Suppose $D(Tx^*, x^*) > 0$. From D_3 for all $n \in \mathbb{N}$, we have

$$f(D(Tx^*, x^*)) \le f(D(Tx^*, Tx_n) + D(Tx_n, x^*)) + \alpha.$$

Using (2.1) and $(\mathcal{F}_1)$, we obtain

$$\begin{aligned}
f(D(Tx^*, x^*) &\le f(\lambda_1 D(x^*, x_n) + \lambda_2 \frac{D(x^*, Tx^*)D(x_n, Tx_n)}{1+D(x^*, x_n)} \\
&\quad + \lambda_3 \frac{D(x^*, Tx_n)D(x_n, Tx^*)}{1+D(x^*, x_n)} + D(Tx_n, x^*)) + \alpha.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{n\to+\infty} (\lambda_1 D(x^*, x_n) &+ \lambda_2 \frac{D(x^*, Tx^*)D(x_n, Tx_n)}{1+D(x^*, x_n)} \\
&+ \lambda_3 \frac{D(x^*, Tx_n)D(x_n, Tx^*)}{1+D(x^*, x_n)} + D(Tx_n, x^*)) = 0,
\end{aligned}$$

from $(\mathcal{F}_2)$ we have

$$\lim_{n\to+\infty} f(\lambda_1 D(x^*,x_n)+\lambda_2\frac{D(x^*,Tx^*)D(x_n,Tx_n)}{1+D(x^*,x_n)}$$

$$+\lambda_3\frac{D(x^*,Tx_n)D(x_n,Tx^*)}{1+D(x^*,x_n)}+D(Tx_n,x^*))+\alpha=-\infty,$$

which is a contradiction. Then, we have $D(Tx^*,x^*)=0$, that is $Tx^*=x^*$. Finally, we shall show that the fixed point is unique. To this end, we assume that there exists another fixed point z^* and $D(x^*,z^*)>0$. From (2.1), we have

$$\begin{aligned}
D(x^*,z^*) =&D(Tx^*,Tz^*)\\
\le&\lambda_1 D(x^*,z^*)+\lambda_2\frac{D(x^*,Tx^*)D(z^*,Tz^*)}{1+D(x^*,z^*)}+\lambda_3\frac{D(x^*,Tz^*)D(z^*,Tx^*)}{1+D(x^*,z^*)}\\
\le&(\lambda_1+\lambda_3)D(x^*,z^*)\\
&< D(x^*,z^*),
\end{aligned}$$

which is a contradiction and hence $x^*=z^*$. □

If $\lambda_2=\lambda_3=0$, Theorem 2.1 reduces to the Banach contraction principle in an $\mathcal{F}$-metric space.

Corollary 2.2. *Let (X,D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $T:X\to X$ be a self-mapping satisfying*

$$D(Tx,Ty)\le\lambda D(x,y) \tag{2.2}$$

for all $x,y\in X$, where $0\le\lambda<1$. Then T has a unique fixed point.

Example 2.3. Let $X=[0,+\infty)$ be endowed with the $\mathcal{F}$-metric given in Example 1.3. Define $T:X\to X$ by

$$Tx=\begin{cases}\frac{9}{8}x & x\in[0,\frac{1}{2}],\\ 0 & x\notin[0,\frac{1}{2}].\end{cases}$$

Set $\lambda_1=\frac{3}{4},\lambda_2=\frac{7}{8}$ and $\lambda_3=0$. Hence, all the conditions of Theorem 2.1 are satisfied and T has a unique fixed point in X.

Theorem 2.4. *Let (X,D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a self-mapping on X satisfying*

$$D(Tx,Ty)\le\lambda_1 D(x,y)+\lambda_2\frac{D(x,Tx)D(y,Ty)}{D(x,y)+D(x,Ty)+D(y,Tx)}, \tag{2.3}$$

for all $x,y\in X$ where $\lambda_1+\lambda_2<1$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We can define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for each $n \geq 0$. Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.3), we have

$$\begin{aligned} D(x_n, x_{n+1}) =& D(Tx_{n-1}, Tx_n) \\ \leq& \lambda_1 D(x_{n-1}, x_n) \\ &+ \lambda_2 \frac{D(x_{n-1}, Tx_{n-1}) D(x_n, Tx_n)}{D(x_{n-1}, x_n) + D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})} \\ =& \lambda_1 D(x_{n-1}, x_n) + \lambda_2 D(x_n, x_{n+1}). \end{aligned}$$

for all $n \in \mathbb{N}$. Then, we get $D(x_n, x_{n+1}) \leq \lambda D(x_{n-1}, x_n)$, where $\lambda = \frac{\lambda_1}{1-\lambda_2} < 1$. Applying Lemma 1.5, $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence. Since X is $\mathcal{F}$-complete, then there exists $x^* \in X$ such that $\lim_{n\to+\infty} D(x_n, x^*) = 0$. Now, we show that x^* is a fixed point of T. Suppose $D(Tx^*, x^*) > 0$. From D_3 we get

$$f(D(Tx^*, x^*)) \leq f(D(Tx^*, Tx_n) + D(Tx_n, x^*)) + \alpha,$$

for all $n \in \mathbb{N}$. Using (2.3) and $\mathcal{F}_1$, we obtain

$$\begin{aligned} f(D(Tx^*, x^*)) \leq& f(\lambda_1 D(x^*, x_n) + \lambda_2 \frac{D(x^*, Tx^*) D(x_n, Tx_n)}{D(x^*, x_n) + D(x^*, Tx_n) + D(x_n, Tx^*)} \\ &+ D(Tx_n, x^*)) + \alpha. \end{aligned}$$

Since

$$\lim_{n\to+\infty} (\lambda_1 D(x^*, x_n) + \lambda_2 \frac{D(x^*, Tx^*) D(x_n, Tx_n)}{D(x^*, x_n) + D(x^*, Tx_n) + D(x_n, Tx^*)} + D(Tx_n, x^*)) = 0,$$

from $(\mathcal{F}_2)$, we have

$$\lim_{n\to+\infty} f(\alpha D(x^*, x_n) + \lambda_2 \frac{D(x^*, Tx^*) D(x_n, Tx_n)}{D(x^*, x_n) + D(x^*, Tx_n) + D(x_n, Tx^*)} + D(Tx_n, x^*)) + \alpha = -\infty,$$

which is a contradiction. Then, we have $D(Tx^*, x^*) = 0$, that is, $Tx^* = x^*$. We show that the fixed point is unique. Assume on the contrary that $Tz^* = z^*$, $D(x^*, z^*) > 0$. From (2.3), we have

$$\begin{aligned} D(x^*, z^*) &= D(Tx^*, Tz^*) \\ &\leq \lambda_1 D(x^*, z^*) + \lambda_2 \frac{D(x^*, Tx^*) D(z^*, Tz^*)}{D(x^*, z^*) + D(x^*, Tz^*) + D(z^*, Tx^*)} \\ &\leq \lambda_1 D(x^*, z^*) \\ &< D(x^*, z^*), \end{aligned}$$

which is a contradiction and hence $x^* = z^*$. □

Theorem 2.5. *Let (X, D) be $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a self-mapping on X satisfying*

$$D(Tx, Ty) \leq \lambda_1 D(x, y) + \lambda_2 \frac{D(x, Tx) D(x, Ty) + D(y, Ty) D(y, Tx)}{D(x, Ty) + D(y, Tx)}, \tag{2.4}$$

for all $x, y \in X$, where $\lambda_1 + \lambda_2 < 1$. Then T has a unique fixed point.

Proof. The proof is similar to Theorem 2.4, therefore we omit it. □

Corollary 2.6. *If $\lambda_2 = 0$, Theorem 2.4 and Theorem 2.5, reduced to Banach contraction principle.*

Example 2.7. Let $X = [0, +\infty)$ be endowed with the $\mathcal{F}$-metric given in Example 1.2. Define $T : X \to X$ by $Tx = \frac{1}{e^{x+1}}$. We discuss two possible cases.
Case 1) If $(x, y) \in [0, 3] \times [0, 3]$,

$$D(Tx, Ty) = (\frac{1}{e^{x+1}} - \frac{1}{e^{y+1}})^2.$$

By the Mean Value Theorem, there exists a real number c between x and y, such that

$$\begin{aligned} D(Tx, Ty) &= (-\frac{1}{e^{c+1}})^2 |x - y|^2 \\ &\leq \frac{1}{e}|x - y|^2 \\ &= \frac{1}{e} D(x, y) \end{aligned}$$

Case 2) If $(x, y) \notin [0, 3] \times [0, 3]$, we have

$$\begin{aligned} D(Tx, Ty) &= (\frac{1}{e^{x+1}} - \frac{1}{e^{y+1}})^2 \\ &\leq |\frac{1}{e^{x+1}} - \frac{1}{e^{y+1}}|. \end{aligned}$$

By the Mean Value Theorem, there exists a real number c between x and y, such that

$$\begin{aligned} D(Tx, Ty) &\leq | - \frac{1}{e^{c+1}}||x - y| \\ &\leq \frac{1}{e}|x - y| \\ &= \frac{1}{e} D(x, y). \end{aligned}$$

Therefore, we deduce that

$$D(Tx, Ty) \leq \frac{1}{e} D(x, y)$$

Hence for $\lambda_1 = \frac{1}{e}$ and $\lambda_2 = 0$, all the conditions of Theorem 2.4 and Theorem 2.5 are satisfied and hence T has a unique fixed point in X.

Theorem 2.8. *Let (X, D) be an $\mathcal{F}$-metric space and $T, S : X \to X$ be self-mappings on X which satisfy*

$$D(Tx, Ty) \leq \lambda D(Sx, Sy), \tag{2.5}$$

for $x, y \in X$, where $0 < \lambda < 1$. If $T(X) \subseteq S(X)$ and $S(X)$ is an $\mathcal{F}$-complete subspace of X, then T and S have a unique point of coincidence in X. Also, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $T(X) \subseteq S(X)$, we choose a point x_1 in X such that $Tx_0 = Sx_1$. Inductively, we can define a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using (2.5), we obtain

$$\begin{aligned} D(Sx_{n+1}, Sx_n) &= D(Tx_n, Tx_{n-1}) \\ &\leq \lambda D(Sx_n, Sx_{n-1}), \end{aligned}$$

for all $n \in \mathbb{N}$, where $\lambda < 1$. By Lemma 1.5, $\{Sx_n\}$ is a Cauchy sequence. By the completeness of $S(X)$ there is some $p \in X$ such that

$$lim_{n \to +\infty} Sx_n = Sp. \tag{2.6}$$

Using (D_3), we have

$$f(D(Tp, Sp)) \leq f(D(Sx_n, Sp) + D(Sx_n, Tp)) + \alpha, \quad n \in \mathbb{N}.$$

Applying (2.5) and $(\mathcal{F}_1)$, we get

$$\begin{aligned} f(D(Tp, Sp)) &= f(D(Sx_n, Sp) + D(Sx_n, Tp)) + \alpha \\ &= f(D(Sx_n, Sp) + D(Tx_{n-1}, Tp)) + \alpha \\ &\leq f(D(Sx_n, Sp) + \lambda D(Sx_{n-1}, Sp)) + \alpha, \end{aligned}$$

for all $n \in \mathbb{N}$. On the other hand, using $(\mathcal{F}_2)$ and (2.6), we obtain

$$\lim_{n \to +\infty} f(D(Sx_n, Sp) + \lambda D(Sx_{n-1}, Sp)) + \alpha = -\infty.$$

This is a contradiction, unless $D(Tp, Sp) = 0$, i.e. $Tp = Sp$ and p is a coincidence point of T and S. For uniqueness, assume that there exists another point $q \in X$ such that $Tq = Sq$ with $p \neq q$. Using (2.5), we have $D(Sp, Sq) = D(Tp, Tq) \leq \lambda D(Sp, Sq)$, a contradiction. Applying Proposition 1.11, T and S have a unique common fixed point. □

Corollary 2.9. *If $S = I$ (the identity mapping on X), we obtain the Banach contraction principle.*

Example 2.10. Let $X = [0, +\infty)$ be endowed with the $\mathcal{F}$-metric given in Example 1.2. Define $T, S : X \to X$ by $Tx = \frac{1}{e^{x+3}}$ and $Sx = \frac{x}{e}$. For $\lambda = \frac{1}{e}$, all conditions of Theorem 2.8 are satisfied and then T and S have a unique point of coincidence in X.

3. Application to integral equation

Let $X = C[a, b]$ be the set of all real continuous functions on $[a, b]$ equipped with the $\mathcal{F}$-metric

$$D(u, v) = ||u - v||_\infty.$$

It is well known that (X, D) is an $\mathcal{F}-$complete $\mathcal{F}-$metric space with $f(t) = lnt$ and $\alpha = 0$. We consider the integral equation:

$$u(t) = \int_a^b k(t, s, u(s))ds, \tag{3.1}$$

where $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$. Let $T : X \to X$ be a mapping defined by:

$$Tu(t) = \int_0^l k(t, s, u(s)), \quad u \in X, t, s \in [a, b].$$

Theorem 3.1. *Assume that the following conditions are satisfied:*
(1) $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous;*
(2) for all $u, v \in X$ *and* $t, s \in [a, b]$*, we have*

$$|k(t, s, u(s))) - k(t, s, v(s))|^2 \leq G(t, s)ln(\frac{|u(s) - v(s)|^2}{4} + 1).$$

where $G : [a, b] \times [a, b] \to \mathbb{R}$ *is continuous function and for all* $t, s \in [a, b]$*, we have:*

$$\int_a^b G(t, s)^2 ds \leq \frac{1}{b - a}.$$

Then, the integral equation (3.1) has a solution in X*.*

Proof. Let $u, v \in X$. Using condition (2) and the Cauchy Schwarz inequality, we have

$$\begin{aligned}
|Tu(t) - Tv(t)|^2 &= \Big(\int_a^b |k(t, s, u(s)) - k(t, s, v(s))ds|\Big)^2 \\
&\leq \int_a^b 1^2 ds \int_a^b |k(t, s, u(s)) - k(t, s, v(s))|^2 ds \\
&\leq (b - a) \int_a^b G(t, s)ln(\frac{|u(s) - v(s)|^2}{4} + 1)ds \\
&\leq (b - a) \int_a^b G(t, s)ln(\frac{D(u, v)^2}{4} + 1)ds \\
&\leq (b - a)\Big(\int_a^b G(t, s)ds\Big)ln(\frac{D(u, v)^2}{4} + 1) \\
&< ln(\frac{D(u, v)^2}{4} + 1) \\
&\leq \frac{D(u, v)^2}{4}
\end{aligned}$$

So, we get

$$D(Tu, Tv) \leq \frac{D(u, v)}{2}.$$

Hence for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 0$, all the conditions of Theorem 2.4 and Theorem 2.5 are satisfied and hence T has a unique fixed point in X. □

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