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# A note on a generalization of an infinite decomposability result

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**Abstract.** A recent infinite decomposability result shows that, for any integer $m$, a random variable following the exponential distribution can be written as the sum of $m$ discontinuous random variables and another one following the exponential distribution, all of them independent. This note extends this result to the Gamma, Laplace and $n$-Laplace distributions, with a clear identification on the involved discontinuous distribution. We also discuss some properties of this new discontinuous distribution, proving that it is also infinitely decomposable.

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## 1. Introduction

The exponential distribution is one of the most useful lifetime distribution in probability and statistics. It depends on a positive scale parameter $\mu$ and has the probability density function (pdf) given by $f(x) = \mu e^{-\mu x}$ for $x \geq 0$. It is widely used to model waiting times in a various scenarios (Poisson process, queuing theory...). In this regard, we refer the reader to [1], and the references therein. Among its well-known desirable properties, the exponential distribution is infinitely divisible; if $X$ is a random variable following the exponential distribution with parameter $\mu$, then, for every natural number $n$, one can write

$$X = \sum_{i=1}^{n} X_i, \tag{1.1}$$

where $X_1, \ldots, X_n$ are independent and identically (i.i.d.) random variables following the gamma distribution with parameters $\mu$ and $1/n$. A similar but less demanding concept is given by the infinite decomposability; in (1.1), it only requires that $X_1, \ldots, X_n$ are independent.

In this regard, it is proved in [7] that the exponential distribution is infinite decomposable in the following sense: for any integer $m$, a random variable

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following the exponential distribution can be written as the sum of $m$ discontinuous random variables (with a discontinuity at 0) and another one following the exponential distribution, all of them independent. The involved discontinuous distribution, depending on two parameters: $\mu > 0$ and $\rho \in (0,1)$, can be defined as follows. We say that a random variable $X$ follows the distribution $\exp(\mu, \rho)$ if: $P(X = 0) = 1 - \rho$ and $P(X > x) = \rho e^{-\mu(1-\rho)x}$ for all $x \geq 0$. In other words, $\exp(\mu, \rho)$ has a jump at 0 with value $1 - \rho$, and other than that it looks like an exponential distribution. For more detail on the general notion of decomposable distribution, we may refer to [5] and [6]. In addition to [7], modern developments involving this notion in diverse theoretical contexts can be found in [3], [4] and [8].

In this note, we contribute to the subject by showing how the main result of [7] can be extended to the gamma, Laplace and $n$-Laplace distributions, with a clear identification of the involved discontinuous distribution. This discontinuous distribution is not listed in the literature, and has the originality to extend the distribution $\exp(\mu, \rho)$ by the use of a binomial structure. A complementary study is devoted to the main characteristics of this new discontinuous distribution, proving that it is also infinitely decomposable. To the best of our knowledge, these probabilistic results are new, and are of potential interest in queuing theory and the related applications (service disciplines, traffic engineering, ...).

The rest of the note is structured as follows. Section 2 presents some central definitions. The generalization of the main result in [7] can be found in Section 3. The infinite decomposability of the introduced discontinuous distribution is discussed in Section 4.

## 2. Definitions

First of all, some preliminary definitions are presented. Let $\alpha \in \{1, 2\}$, $\mu > 0$ and $n$ be an integer. We say that a random variable $X$ follows the distribution $\mathcal{D}(\alpha, \mu, n)$ if it has the following moment generating function:

$$M_X(t) = E(e^{tX}) = \left[\frac{\mu}{\mu - t^\alpha}\right]^n, \tag{2.1}$$

with $t$ such that $t^\alpha < \mu$.

Thus defined, $\mathcal{D}(\alpha, \mu, n)$ is composed of four well-referenced distributions:

- $\mathcal{D}(1, \mu, 1)$ corresponds to the exponential distribution with parameter $\mu$, i.e., with the pdf: $f(x) = \mu e^{-\mu x}$, $x \geq 0$,
- $\mathcal{D}(1, \mu, n)$ corresponds to the gamma distribution with parameters $\mu$ and $n$, i.e., with the pdf: $f(x) = [(n-1)!]^{-1} x^{n-1} \mu^n e^{-\mu x}$, $x \geq 0$,
- $\mathcal{D}(2, \mu, 1)$ corresponds to the Laplace distribution with parameter $\beta = \mu^{1/2}$, i.e., with the pdf: $f(x) = 2^{-1} \beta e^{-\beta |x|}$, $x \in \mathbb{R}$,

- $\mathcal{D}(2,\mu,n)$ corresponds to the $n$-Laplace distribution with parameters $\beta = \mu^{1/2}$ and $n$, i.e., with the pdf:
  $f(x) = \left\{[(n-1)!]^{-2}\sum_{i=0}^{n-1}\binom{n-1}{i}(2n-i-2)!2^{-2n+i+1}\beta^i|x|^i\right\}\beta e^{-\beta|x|}$, $x \in \mathbb{R}$.

Thanks to (2.1), it is immediate that a sum of $n$ i.i.d. random variables following the distribution $\mathcal{D}(\alpha,\mu,1)$ follows the distribution $\mathcal{D}(\alpha,\mu,n)$.

Now, let $\rho \in (0,1)$, along with the above notations. We say that a random variable $X$ follows the distribution $\mathcal{E}(\alpha,\mu,\rho,n)$ if it can be written as

$$X = Y_N, \tag{2.2}$$

where $N$ denotes a random variable following the binomial distribution with parameters $n$ and $\rho$, i.e., with the probability mass function given by

$$P(N=k) = \frac{n!}{k!(n-k)!}\rho^k(1-\rho)^{n-k}, \quad k = 0,\dots,n,$$

and $Y_0,\dots,Y_n$ denote $n+1$ random variables independent of $N$ such that $P(Y_0 = 0) = 1$ and $Y_i$ follows the distribution $\mathcal{D}(\alpha,\mu(1-\rho),i)$ for $i = 1,\dots,n$.

One can remark that the distribution $\mathcal{E}(\alpha,\mu,\rho,n)$ is discontinuous; we have $P(X=0) = P(N=0) = (1-\rho)^n \neq 0$, whereas $P(X=x) = 0$ for any $x \neq 0$. Also, $\mathcal{E}(1,\mu,\rho,1)$ corresponds to the distribution $\exp(\mu,\rho)$ introduced by [7].

As an important characterization, the cumulative distribution function (cdf) of the distribution $\mathcal{E}(\alpha,\mu,\rho,n)$ is given by

$$F_X(x) = P(X \le x) = (1-\rho)^n \mathbf{1}_{\{x\ge 0\}} + \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}\rho^k(1-\rho)^{n-k}F_{Y_k}(x),$$

where $\mathbf{1}_A$ denotes the indicator function such that $\mathbf{1}_A = 1$ if $x \in A$ and 0 elsewhere, and $F_{Y_k}(x)$ denotes the cdf of $Y_k$.

To the best of our knowledge, this discontinuous distribution has not received a special treatment in the literature, which is one aim of this paper. In particular, in the next part of the study, we show how it naturally appears in several infinite decomposable results involving standard distributions (exponential, gamma, Laplace and $n$-Laplace distributions).

## 3. Main results

First of all, we present the moment generating function of the distribution $\mathcal{E}(\alpha,\mu,\rho,n)$, which will be crucial in the subsequent proofs.

**Lemma 3.1.** *The moment generating function of a random variable $X$ following the distribution $\mathcal{E}(\alpha,\mu,\rho,n)$ is given by*

$$M_X(t) = \left[\frac{(1-\rho)(\mu - t^\alpha)}{\mu(1-\rho) - t^\alpha}\right]^n,$$

*for $t$ such that $t^\alpha < \mu(1-\rho)$.*

*Proof.* We use the representation of $X$ given by (2.2). Since $Y_0, \ldots, Y_n$ and $N$ are independent, by using (2.1), we get

$$\begin{aligned} M_X(t) &= E(e^{tX}) = E[E(e^{tY_N} \mid N)] = \sum_{k=0}^{n} M_{Y_k}(t) P(N = k) \\ &= \sum_{k=0}^{n} \left[ \frac{\mu(1-\rho)}{\mu(1-\rho) - t^\alpha} \right]^k \frac{n!}{k!(n-k)!} \rho^k (1-\rho)^{n-k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left[ \frac{\mu\rho(1-\rho)}{\mu(1-\rho) - t^\alpha} \right]^k (1-\rho)^{n-k}. \end{aligned}$$

It follows from the standard binomial formula and some algebra that

$$M_X(t) = \left[ \frac{\mu\rho(1-\rho)}{\mu(1-\rho) - t^\alpha} + (1-\rho) \right]^n = \left[ \frac{(1-\rho)(\mu - t^\alpha)}{\mu(1-\rho) - t^\alpha} \right]^n.$$

This ends the proof of Lemma 3.1. □

From Lemma 3.1, several distributional and mathematical properties of the distribution $\mathcal{E}(\alpha, \mu, \rho, n)$ can be determined. In particular, a sum of $n$ i.i.d. random variables following the distribution $\mathcal{E}(\alpha, \mu, \rho, 1)$, follows the distribution $\mathcal{E}(\alpha, \mu, \rho, n)$. Also, the $s$-th ordinary moment of $X$ can be obtained as $\mu'_s = E(X^s) = M(t)^{(s)} \mid_{t=0}$.

The following result emphasizes the link between the distributions $\mathcal{D}(\alpha, \mu, n)$ and $\mathcal{E}(\alpha, \mu, \rho, n)$.

**Proposition 3.2.** *Let $X$ be a random variable following the distribution $\mathcal{E}(\alpha, \mu, \rho, n)$ and $Y$ a random variable following the distribution $\mathcal{D}(\alpha, \mu, n)$, with $X$ and $Y$ independent. Then, $Z = X + Y$ follows the distribution $\mathcal{D}(\alpha, \mu(1-\rho), n)$.*

*Proof.* By using the independence of $X$ and $Y$, and the expression of their respective moment generating functions, owing to Lemma 3.1 and (2.1), for $t$ such that $t^\alpha < \mu(1-\rho) < \mu$, we obtain

$$\begin{aligned} M_Z(t) &= M_X(t) M_Y(t) = \left[ \frac{(1-\rho)(\mu - t^\alpha)}{\mu(1-\rho) - t^\alpha} \right]^n \left[ \frac{\mu}{\mu - t^\alpha} \right]^n \\ &= \left[ \frac{\mu(1-\rho)}{\mu(1-\rho) - t^\alpha} \right]^n. \end{aligned}$$

Then, we recognize the moment generating function of a random variable following the distribution $\mathcal{D}(\alpha, \mu(1-\rho), n)$. This ends the proof of Proposition 3.2. □

We are now in position to state the first main result of the paper, which mainly follows from Proposition 3.2.

**Proposition 3.3.** *Let $\theta > 0$ and $X$ be random variable following the distribution $\mathcal{D}(\alpha, \theta, n)$. Then, for every integer $m$, we can write*

$$X = \sum_{i=1}^{m+1} X_i,$$

*where $X_i$ follows the distribution $\mathcal{E}\left(\alpha, \theta/(1-\rho)^i, \rho, n\right)$ for $i = 1, \ldots, m$, and $X_{m+1}$ follows the distribution $\mathcal{D}\left(\alpha, \theta/(1-\rho)^m, n\right)$, with $X_1, \ldots, X_m, X_{m+1}$ independent.*

*Proof.* The proof follows immediately by induction and Proposition 3.2; one can notice that the initialization step, i.e., the case $m = 1$, immediately follows from Proposition 3.2 with $\mu = \theta/(1-\rho)$. □

By applying Proposition 3.3 with $\alpha = 1$ and $n = 1$, we rediscover the main result in [7].

As a simple application, we can mention the M/M/1 queue with inter-arrival times modeled by a random variable following the exponential distribution $\exp(\lambda)$ and service times modeled by a random variable following the exponential distribution $\exp(\mu)$. Then, the waiting time in the queue has a jump at zero and it is modeled by a random variable following the distribution $\exp(\mu, \rho)$, where $\rho = \lambda/\mu$, and the total time in the system (waiting time plus service time) is modeled by a random variable following the distribution $\exp(\mu(1-\rho))$, which is a particular case of Proposition 3.2.

As another example, in actuarial sciences, the claim-size random variable may be modelled with a discontinuity at zero (see [2, Example 2.3, p. 40]), so our results, at least theoretically, could be of interest in this context, though the discontinuities of the claim-size variable typically occur at some limit $L > 0$, above which the insurer only pays the policy limit $L$.

**Discussion.** We would like to mention that an alternative proof for Proposition 3.3 is possible; we can prove it by using the result in [7], some existing distributional results on $\mathcal{D}(\alpha, \theta, n)$ and further developments.

For instance, for the case $\alpha = 1$ and any integer $n$ corresponding to the gamma distribution case, the main lines of an alternative proofs are as follows. For any random variable $X$ following the distribution $\mathcal{D}(1, \theta, n)$, we can write $X = \sum_{i=1}^{n} U_i$, where $U_1, \ldots, U_n$ are $n$ i.i.d. random variables following the distribution $\mathcal{D}(1, \theta, 1)$. Then, by applying the infinite decomposability result of [7], we can write $U_i = \sum_{j=1}^{m+1} V_{j,i}$, where $V_{j,i}$ follows the distribution $\mathcal{E}\left(1, \theta/(1-\rho)^j, \rho, 1\right)$ for $j = 1, \ldots, m$, and $V_{m+1,i}$ follows the distribution $\mathcal{D}\left(1, \theta/(1-\rho)^m, 1\right)$, with $V_{1,i}, \ldots, V_{m,i}, V_{m+1,i}$ independent. Then, we have $X = \sum_{j=1}^{m+1} W_j$, where $W_j = \sum_{i=1}^{n} V_{j,i}$. We end by showing that $W_j$ follows the distribution $\mathcal{E}\left(1, \theta/(1-\rho)^j, \rho, n\right)$ for $j = 1, \ldots, m$, and $W_{m+1}$ follows the distribution $\mathcal{D}\left(1, \theta/(1-\rho)^m, n\right)$.

Also, for the case $\alpha = 2$ and any integer $n$ corresponding to the $n$-Laplace distribution case (including the former Laplace distribution for $n = 1$), a similar approach can be investigated. Indeed, for any random variable $X$ following the distribution $\mathcal{D}(2, \theta, n)$, we can write $X = \sum_{i=1}^{n}(A_i - B_i)$, where

$A_1, \ldots, A_n, B_1, \ldots, B_n$ are $2n$ i.i.d. random variables following the distribution $\mathcal{D}(1, \theta^{1/2}, 1)$. By applying the result in [7] on $A_i$ and $B_i$ for $i = 1, \ldots, n$ separately, with more efforts and developments, we are able to prove the desired result.

Thus, the main interest of Proposition 3.3 is to be elegant, and to identify the involved discontinuous distribution from the beginning, with a clear identification, revealing its binomial compounding structure.

## 4. On the infinite decomposability of $\mathcal{E}(\alpha, \mu, \rho, n)$

An interesting new fact is that $\mathcal{E}(\alpha, \mu, \rho, n)$ is also infinite decomposable. That claim is proved in this section.

**Proposition 4.1.** *Let $X$ be a random variable following the distribution $\mathcal{E}(\alpha, \mu_1, \rho_1, n)$, $Y$ a random variable following the distribution $\mathcal{E}(\alpha, \mu_2, \rho_2, n)$, with $\mu_1 = \mu_2(1 - \rho_2)$, and $X$ and $Y$ independent. Then, $Z = X + Y$ follows the distribution $\mathcal{E}(\alpha, \mu_2, \rho_1 + \rho_2 - \rho_1\rho_2, n)$.*

*Proof.* By using the independence of $X$ and $Y$ and Lemma 3.1, for $t$ such that $t^\alpha < \mu_1(1 - \rho_1)$, we get

$$\begin{aligned} M_Z(t) = M_X(t)M_Y(t) &= \left[\frac{(1-\rho_1)(\mu_1 - t^\alpha)}{\mu_1(1-\rho_1) - t^\alpha}\right]^n \left[\frac{(1-\rho_2)(\mu_2 - t^\alpha)}{\mu_2(1-\rho_2) - t^\alpha}\right]^n \\ &= \left[\frac{(1-\rho_1)(\mu_2(1-\rho_2) - t^\alpha)}{\mu_2(1-\rho_2)(1-\rho_1) - t^\alpha} \times \frac{(1-\rho_2)(\mu_2 - t^\alpha)}{\mu_2(1-\rho_2) - t^\alpha}\right]^n \\ &= \left[\frac{(1-\rho_1)(1-\rho_2)(\mu_2 - t^\alpha)}{\mu_2(1-\rho_1)(1-\rho_2) - t^\alpha}\right]^n. \end{aligned}$$

We recognize the moment generating function of a random variable following the distribution $\mathcal{E}(\alpha, \mu_2, \rho_1 + \rho_2 - \rho_1\rho_2, n)$. The proof of Proposition 4.1 is completed. $\square$

The previous proposition and the induction principle allow us to claim the following result.

**Corollary 4.2.** *Let $X_1, \ldots, X_m$ be independent random variables such that $X_i$ follows the distribution $\mathcal{E}(\alpha, \mu_i, \rho_i, n)$ for $i = 1, \ldots, m$. If $\mu_i = \mu_{i+1}(1 - \rho_{i+1})$ for $i = 1, \ldots, m-1$, then*

$$X = \sum_{i=1}^{m} X_i$$

*follows the distribution $\mathcal{E}\left(\alpha, \mu_m, 1 - \prod_{i=1}^{m}(1 - \rho_i), n\right)$.*

Now, we are in position to state the second main result of the paper, dealing with the infinite decomposability of the distribution.

**Proposition 4.3.** *Let $\theta > 0$ and $X$ be a random variable following the distribution $\mathcal{E}(\alpha, \theta, \sigma, n)$. Then, for every integer $m$, we can write*

$$X = \sum_{i=1}^{m} X_i,$$

*where $X_i$ follows the distribution $\mathcal{E}\left(\alpha, \theta(1-\sigma)^{1-i/m}, 1-(1-\sigma)^{1/m}, n\right)$ with $X_1, \ldots, X_m$ independent.*

*Proof.* The proof follows from Corollary 4.2 applied with $\mu_i = \theta(1-\rho)^{m-i}$ and $\rho_i = \rho = 1-(1-\sigma)^{1/m}$ for $i = 1, \ldots, m$ (so $\mu_i = \theta(1-\sigma)^{1-i/m}$), satisfying the desired relation, i.e., $\mu_i = \mu_{i+1}(1-\rho_{i+1})$. $\square$

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## References

[1] Allen, A. O. *Probability, statistics, and queueing theory.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. With computer science applications, Computer Science and Applied Mathematics.

[2] Bahnemann, D. *Distributions for Actuaries.* CAS Monograph series, number 2, Casualty Actuarial Society, Arlington, Va., 2015.

[3] Hoffmann-Jø rgensen, J. r., Kagan, A. M., Pitt, L. D., and Shepp, L. A. Strong decomposition of random variables. *J. Theoret. Probab. 20*, 2 (2007), 211–220.

[4] Jurek, Z. J. Background driving distribution functions and series representation for log-gamma selfdecomposable random variables. *arXiv e-prints* (Apr. 2019), arXiv:1904.04160.

[5] Lamperti, J. W. *Probability*, second ed. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, 1996. A survey of the mathematical theory, A Wiley-Interscience Publication.

[6] Linnik, J. V., and Ostrovs'kiĭ, V. *Decomposition of random variables and vectors.* American Mathematical Society, Providence, R. I., 1977. Translated from the Russian, Translations of Mathematical Monographs, Vol. 48.

[7] Palacios, J. The exponential distribution as the sum of discontinuous distributions. *Bulletin of Computational Applied Mathematics 1*, 1 (2013), 7–10.

[8] Sreehari, M., Sandhya, E., and Rajasekharan, K. E. A note on the infinite divisibility of discrete distributions having almost lack of memory property. *J. Indian Statist. Assoc. 53*, 1-2 (2015), 109–122.

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