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Common e-soft fixed points of soft set-valued maps

Akbar Azam[1] and Mohammed Shehu Shagari[2][3]

Abstract. Recently, a new type of set-valued mapping called soft set-valued map was introduced in the literature as a generalization of the concepts of fuzzy set-valued and multi-valued mappings. The present article extends the new notion by establishing, among others, the idea of common e-soft fixed point of soft set-valued maps. From application point of view, one of our obtained results is employed to establish novel sufficient conditions for the existence of fuzzy number-valued solution of fuzzy Volterra integral equation. Non-trivial examples are further provided to support the hypotheses of our results. The presented idea herein includes several fixed point theorems on point-to-point and point-to-set valued mappings as consequences. A few of these special cases are highlighted and discussed.

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1. Introduction

Fixed point theory is one of the most active research fields in modern nonlinear functional analysis. In general, fixed point problem is of the form $Tx = x$, where T is a self-mapping on a non-empty set X. This problem can be reformulated as $g(x) = 0$, where $g(x) = x - Tx$. As simple as this problem statement is, finding its solution may be extremely difficult and sometimes it is not obtainable. The earliest affirmative response to this problem was presented by Banach [8] under some suitable conditions: when T is a contraction and X is equipped with a norm such that the corresponding topology yields completeness. So far, fixed point techniques have gained enormous applications in diverse areas such as biology, economics, chemistry, physics, engineering and so on. In 1969, Kannan[19] gave an analogue sort of contractive condition that demonstrated the existence of fixed point. The basic distinction between Banach fixed point theorem (BFT) and that of Kannan contraction is that continuity of contraction is not required in the later. Similar well-known improvements of the BFT were established by Chatterjea [11] and Edelstein [14]. In the last five decades, the above results have been generalized in different

[1]Department of Mathematics, COMSATS University, Chak Shahzad, Islamabad, 44000, Pakistan, e-mail: akbarazam@yahoo.com
[2]Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University Zaria, Nigeria, e-mail: ssmohammed@abu.edu.ng
[3]Corresponding author

directions. For a comprehensive survey on this subject, the interested reader may consult Rhoades [27], Smart [31] or Taskovic [33].

Along the line, the area of applied mathematics witnessed tremendous developments as a result of the introduction of soft set theory by Molodstov [22]. The method of handling problems in classical mathematics is the opposite of the technique of soft set theory. In conventional mathematics, to describe any system or object, we first construct its mathematical model and then attempt to obtain the exact solution. If the exact solution is too complicated, then we define the notion of approximate solution. On the other hand, in soft set theory, the initial description of an object takes an approximate nature with no restriction, and the notion of exact solution is not essential. In other words, to describe an object in soft set theory, any convenient parametrization tools which may be words, sentences, numbers, mappings, functions, to mention a few, may be used. Thereby, the theory becomes easier and more flexible in terms of applications in everyday life. In [22], Moldstov highlighted several directions for the applications of soft sets, such as smoothness of functions, game theory, Riemann-integration, operation research, probability and so on. Currently, the concept of soft set is gaining more than a handful of extensions and applications in different fields of studies. For example, see [10, 12, 15, 28] and references therein.

It is well-known that set-valued analysis has enormous applications in control theory, game theory, biomathematics, qualitative physics, viability theory, and so on. In this continuation, not long ago, Shagari and Azam [21, 30] studied the concept of soft set-valued maps and introduced the notions of e-soft fixed points and E-soft fixed points of maps whose range set is a family of soft sets. It is shown in [21] that every fuzzy mapping is a particular kind of soft set-valued map. Since every fuzzy mapping has its corresponding multifunction analogue (see [16, Theorem 2.2]), hence, the idea of e-soft fixed point theorems is a generalization of the concept of fuzzy fixed points and fixed points of multi-valued maps. In this manuscript, we extend the main result in [21]. In particular, the concept of common e-soft fixed point of soft set-valued maps is initiated, among others. From application point of view, one of our obtained results is employed to establish new sufficient conditions for the existence of fuzzy number-valued solution of fuzzy Volterra integral equation. Examples are supplied to validate the hypotheses of our results.

2. Preliminaries

In this section, we recall specific concepts of soft sets and soft set-valued maps from [21, 30, 22]. Let X be the universal set and E be the universe of discourse of all parameters related to the elements in X. In this case, each parameter is a word, sentence or function. Let $P(X)$ be the power set of X. Molodstov [22] defined the concept of soft set in the following manner.

Definition 2.1. [22] A pair (F, A) is called a soft set over X, where $A \subseteq E$ and F is a set- valued mapping $F : A \longrightarrow P(X)$. In this way, a soft set over X is a parameterized family of subsets of X.

Example 2.2. Suppose the soft set (F, E) describes the structures of certain number of men. Let the universal set of all men be

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$

and the universe of all parameters be represented by

$$E = \{e_1, e_2, e_3, e_4\} = \{\text{fat}, \text{tall}, \text{muscular}, \text{lanky}\}.$$

In this case, to define a soft set means to point out fat men, tall men, muscular men, and lanky men. Thus, we may define $F : E \longrightarrow P(X)$ by $F(e_1) = \{x_1, x_2, x_5\}$, $F(e_2) = \{x_2, x_4, x_5\}$, $F(e_3) = \{x_5\}, F(e_4) =$ empty. So, the soft set (F, E) is a parameterized family $\{F(e_i) : i = 1, 2, 3, 4\}$ of $P(X)$.

Shagari and Azam [21] initiated the idea of soft-valued maps and e-soft fixed points via the following preliminary concepts.

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed and bounded subsets of X. Denote by $[P(X)]^E$ the family of all soft sets over X under E. Then consider two soft sets (F, A) and (G, B), $(a, b) \in A \times B$. Assume that $F(a), G(b) \in CB(X)$. For $\epsilon > 0$, define $N^d(\epsilon, F(a))$, $S_{EX}^{(a,b)}(F, G)$ and $E^d_{(F_a,G_b)}$, respectively, as follows:

$$N^d(\epsilon, F(a)) = \{x \in X : d(x, y) < \epsilon, \text{ for some } y \in F(a)\}$$

$$E^d_{(F_a,G_b)} = \{\epsilon > 0 : F(a) \subseteq N^d(\epsilon, G(b)), \quad G(b) \subseteq N^d(\epsilon, F(a))\},$$

and

$$S_{EX}^{(a,b)}(F, G) = \inf E^d_{(F_a,G_b)},$$

Let $\mathbb{R}_+ = [0, \infty)$ and define a distance function $S^\infty_{EX} : [P(X)]^E \times [P(X)]^E \longrightarrow \mathbb{R}_+$ by

$$S^\infty_{EX}(F, G) = \sup_{(a,b)\in \overline{A}\times\overline{B}} S_{EX}^{(a,b)}(F, G), \text{ where}$$

$$\overline{A} \times \overline{B} = \{(a, b) \in A \times B : F(a), G(b) \in CB(X)\}.$$

Definition 2.3. [21] A mapping $T : X \longrightarrow [P(X)]^E$ is called a soft set-valued map. A point $x \in X$ is called an e-soft fixed point of T if $x \in (Tx)(e)$, for some $e \in E$. This is also written as $x \in Tx$, for short. If $DomTx = E$ and $x \in (Tx)(e)$ for all $e \in E$, then x is said to be an E-soft fixed point of T.

We shall denote the set of all E-soft fixed points of a soft set-valued map T by $E_{Fix(T)}$. The domain of T, written as $DomT$, is given as

$$DomT = \{x \in X : (Tx)(e) \subseteq X, \ e \in E\}.$$

Notice that if $T : X \longrightarrow [P(X)]^E$ is a soft set-valued map, then (Tx, E) is a soft set over X, for all $x \in X$. In the remaining part of this paper, if $T : X \longrightarrow [P(X)]^E$ is a soft set-valued map, then the set $(Tx)(e)$ shall be written as $(T_e x)$.

Several examples of soft set-valued maps have been provided in [21, 30]. However, we give an additional example as follows.

Example 2.4. Let $X = \{6,7,8\}$ and $E = \{1,2\}$. For all $x \in X$, define $T : X \longrightarrow [P(X)]^E$ as follows:

$$(T_e x) = \begin{cases} \{6,8\}, & \text{if } e = 1 \\ \{7,8\}, & \text{if } e = 2. \end{cases}$$

Then T is a soft set-valued map. Notice that $6 \in (T_e 6)$ for $e = 1$ and $7 \in (T_e 7)$ for $e = 2$; hence, 6 and 7 are e-soft fixed points of T. But, $7 \notin (T_e 7)$ and $6 \notin (T_e 6)$ for $e = 1$ and $e = 2$, respectively. It follows that 6 and 7 are not E-soft fixed points of T. On the other hand, $8 \in (T_e 8)$ for all $e \in E$; thus, the set of all E-soft fixed points of T is given by $E_{Fix(T)} = \{8\}$. The map T can be represented as in Figure 1. Notice that in Figure 1, the dots represent other subsets of X.

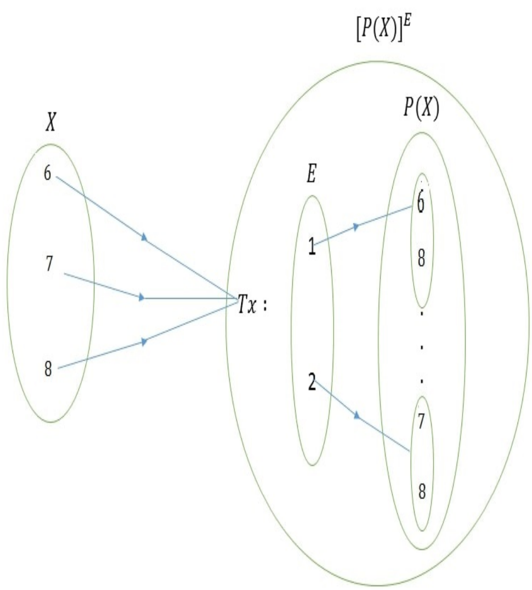

Figure 1: Graphical representation of the soft set-valued map in Example 2.4

3. Main results

We start this section with the following definitions.

Definition 3.1. Let $S, T : X \longrightarrow [P(X)]^E$ be soft set-valued maps.

(*i*) A point x is called an e-soft coincidence point of S and T if $(Sx)(a(x)) = (Tx)(a(x))$ for some $a(x) = e \in E$. A point $y \in X$ such that $y \in (Sx)(a(x)) = (Tx)(a(x))$ is said to be an e-soft point of coincidence of S and T.

(*ii*) A point $x \in X$ is known as a common e-soft fixed point of S and T if $x \in (Sx)(a(x)) \cap (Tx)(a(x))$, for some $a(x) \in E$. Similarly, x is called a common E-soft fixed point of S and T if $DomSx = DomTx = E$ and $x \in (Sx)(a(x)) \cap (Tx)(a(x))$ for all $a(x) = e \in E$.

Remark 3.2. Note that "$a(x)$" is a notation (and not a function of x) representing an element in the parameter set E with regards to an element x in the universal set X. Whenever there is danger of confusion, we write $a(x) = e \in E$.

Theorem 3.3. *Let (X, d) be a complete metric space, $g : X \longrightarrow X$ be a surjection with $gx = \bar{x}$, E be the parameter set and $S, T : X \longrightarrow [P(X)]^E$ any two soft set-valued maps. Suppose that for each $x \in X$ there exist $a(x) = e \in DomSx$ and $a(x) = e \in DomTx$ such that $(Sx)(a(x)), (Tx)(a(x)) \in CB(X)$. If there exists $\rho \in (0, 1)$ such that for all $x, y \in X$,*

$$S_{EX}^{(a(x),a(y))}(Sx, Ty) \leq \rho d(\bar{x}, \bar{y}), \tag{3.1}$$

then there exists $\bar{u} \in X$ such that $\bar{u} \in (Su)(a(u)) \cap (Tu)(a(u))$, for some $a(u) = e \in E$.

Proof. Let x_0 be an arbitrary but fixed element of X, then by the hypotheses, there exists $a(x_0) \in DomS(x_0)$ such that $(Sx_0)(a(x_0))$ is a nonempty closed and bounded subset of X. Let $gx_1 = \bar{x}_1 \in (Sx_0)(a(x_0))$, it follows that $(Sx_0)(a(x_0)) \in CB(X)$. Hence, for this $\bar{x}_1$, we can find $\bar{x}_2 \in X$ such that $\bar{x}_2 \in (Sx_1)(a(x_1))$. Without loss of generality, assume that $\bar{x}_n \in (Sx_{n-1})(a(x_{n-1}))$, for all $n \in \mathbb{N}$. We shall show that $\bar{x}_{n+1} \in (Tx_n)(a(x_n))$, for all $n \in \mathbb{N}$. If there exists $n^* \in \mathbb{N}$ such that $\bar{x}_{n^*+1} = \bar{x}_{n^*} \in (Sx_{n^*}) \cap (Tx_{n^*})(a(x_{n^*}))$, for some $a(x_{n^*}) \in E$, then $\bar{u} = \bar{x}_{n^*}$ is a common e-soft fixed of S and T. So assume that $\bar{x}_{n+1} \neq \bar{x}_n$ for all $n \in \mathbb{N}$. Setting $\bar{x} = \bar{x}_0$ and $\bar{y} = \bar{x}_1$ in (3.1) with $\lambda = \sqrt{\rho}$, and $\kappa = \lambda d(\bar{x}_0, \bar{x}_1)$, we have

$$S_{EX}^{(a(x_0),a(x_1))}(Sx_0, Tx_1) \leq \rho d(\bar{x}_0, \bar{x}_1) < \kappa.$$

Then $\kappa \in E^d_{((Sx_0)(a(x_0)),(Tx_1)(a(x_1)))}$. This means

$$(Sx_0)(a(x_0)) \subseteq N^d(\kappa, (Tx_1)(a(x_1))) \text{ and } (Tx_1)(a(x_1)) \subseteq N^d(\kappa, (Sx_0)(a(x_0))).$$

This implies that $\bar{x}_1 \in N^d(\kappa, (Tx_1)(a(x_1)))$ and hence, there exists some $\bar{x}_2 \in (Tx_1)(a(x_1))$ such that

$$d(\bar{x}_1, \bar{x}_2) < \kappa. \tag{3.2}$$

Again, take $\bar{x} = \bar{x}_1$ and $\bar{y} = \bar{x}_2$ in (3.9) with $\lambda = \sqrt{\rho}$ and $\kappa^2 = \lambda d(\bar{x}_1, \bar{x}_2)$ to have

$$S_{EX}^{(a(x_1),a(x_2))}(Sx_1, Tx_2) \quad \leq \quad \rho d(\bar{x}_1, \bar{x}_2) < \kappa^2.$$

It follows that $\kappa^2 \in E^d_{((Sx_1)(a(x_1)),(Tx_2)(a(x_2)))}$. In other words,

$$(Sx_1)(a(x_1)) \subseteq N^d\left(\kappa^2, (Tx_2)(a(x_2))\right) \text{ and } (Tx_2)(a(x_2)) \subseteq N^d\left(\kappa^2, (Sx_1)(a(x_1))\right).$$

By our assumption, it follows that $\bar{x}_2 \in N^d\left(\kappa^2, (Tx_2)(a(x_2))\right)$. Thus, there exists $\bar{x}_3 \in (Tx_2)(a(x_2))$ such that

$$d(\bar{x}_2, \bar{x}_3) < \kappa^2 \leq \lambda^2 d(\bar{x}_0, \bar{x}_1). \tag{3.3}$$

Continuing this process repeatedly, for $\bar{x}_n \in (Sx_{n-1})(a(x_{n-1}))$, we can find $\bar{x}_{n+1} \in (Tx_n)(a(x_n))$ such that

$$d(\bar{x}_n, \bar{x}_{n+1}) < \kappa^n \leq \lambda^n d(\bar{x}_0, \bar{x}_1), \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

From (3.4), by the triangle inequality, for all $\xi \geq 1$, we have

$$\begin{aligned} d(\bar{x}_n, \bar{x}_{n+\xi}) &\leq d(\bar{x}_n, \bar{x}_{n+1}) + d(\bar{x}_{n+1}, \bar{x}_{n+2}) \\ &\quad + \cdots d(\bar{x}_{n+\xi-1}, \bar{x}_{n+\xi}) \\ &\leq \sum_{j=n}^{n+\xi-1} \lambda^j d(\bar{x}_0, \bar{x}_1) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

This implies that $\{\bar{x}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $\bar{u} \in X$ such that

$$\lim_{n\longrightarrow\infty} d(\bar{x}_n, u) = 0. \tag{3.5}$$

Now, by the triangle inequality and using the contractive condition (3.1), we have

$$\begin{aligned} d(\bar{u}, (Su)(a(u))) &\leq d(\bar{u}, \bar{x}_{n+1}) + d(\bar{x}_{n+1}, (Su)(a(u))) \\ &\leq d(\bar{u}, \bar{x}_{n+1}) + d((Tx_n)(a(x_n)), (Su)(a(u))) \\ &\leq d(\bar{u}, \bar{x}_{n+1}) + \inf E^d_{(Tx_n, Su)} \\ &= d(\bar{u}, \bar{x}_{n+1}) + S_{EX}^{(a(x_n), a(u))}(Tx_n, Su) \\ &\leq d(\bar{u}, \bar{x}_{n+1}) + \rho d(\bar{x}_n, \bar{u}). \end{aligned} \tag{3.6}$$

Letting $n \longrightarrow \infty$ in (3.6), gives $d(\bar{u}, (Su)(a(u))) \leq 0$. Therefore, $\bar{u} \in (Su)(a(u))$. By similar steps, using

$$d(\bar{u}, (Tu)(a(u))) \leq d(\bar{u}, \bar{x}_n) + d(\bar{x}_n, (Tu)(a(u))),$$

we can show that $\bar{u} \in (Tu)(a(u))$. Consequently, $\bar{u} \in (Su)(a(u)) \cap (Tu)(a(u))$. □

Example 3.4. Let $E = [0, 1]$ and $X = \mathbb{R}_+$ be endowed with the usual metric. Then (X, d) is a complete metric space. For each $x \in X$ and $e = a(x) \in E$, define two soft set-valued maps $S, T : X \longrightarrow [P(X)]^E$ by

$$(Sx)(a(x)) = \begin{cases} [0, 8x], & \text{if } 0 \leq a(x) \leq \frac{1}{10} \\ \left(\frac{1}{2}, 4x\right), & \text{if } \frac{1}{10} < a(x) \leq \frac{1}{6} \\ \emptyset, & \text{if } \frac{1}{6} < a(x) \leq 1, \end{cases}$$

$$(Tx)(a(x)) = \begin{cases} (0, 6x), & \text{if } 0 < a(x) < \frac{1}{25} \\ [0, 8x], & \text{if } \frac{1}{25} \leq a(x) \leq \frac{1}{13} \\ \left(\frac{1}{3}, 8x\right], & \text{if } \frac{1}{13} < a(x) \leq 1. \end{cases}$$

Define $g : X \longrightarrow X$ by $gx = 9x = \bar{x}$. Clearly, the mapping g is surjective. Then for each $x \in X$ there exist $a(x) \in \left[0, \frac{1}{10}\right]$ and $a(x) \in \left[\frac{1}{25}, \frac{1}{13}\right]$ such that

$$(Sx)(a(x)) = [0, 8x] \in CB(X) \text{ and } (Tx)(a(x)) = [0, 8x] \in CB(X).$$

Thus, for two soft sets (Sx, E) and (Ty, E), we have the following two cases:

Case I : If $x = y$, then

$$E^d_{(Sx,Ty)} = [0, \infty).$$

Case II : If $x \neq y$, then

$$E^d_{(Sx,Ty)} = [8x - 8y, \infty).$$

Hence, from case II (there is nothing to show in case I), we have

$$\begin{aligned} S^{(a(x),a(y))}_{EX}(Sx, Ty) &\leq S^{\infty}_{EX}(Sx, Ty) \\ &= |8x - 8y| \\ &\leq \frac{8}{9}|9x - 9y| \\ &\leq \frac{8}{9}|\bar{x} - \bar{y}| \\ &\leq \rho d(\bar{x}, \bar{y}), \ \ (\text{where } \rho \in (0,1))\,. \end{aligned}$$

Consequently, all conditions of Theorem 3.3 are satisfied to find $g0 = \bar{0} \in (S0)(e) \cap (T0)(e)$ for some $a(x) = e \in E$.

Corollary 3.5. *Let (X, d) be a complete metric space, $I_X : X \longrightarrow X$ be the identity mapping on X, E be the parameter set and $S, T : X \longrightarrow [P(X)]^E$ be any two soft set-valued maps. Suppose that for each $x \in X$ there exist $a(x) \in DomSx$ and $a(x) \in DomTx$ such that $(Sx)(a(x)), (Tx)(a(x)) \in CB(X)$. If there exists $\rho \in (0, 1)$ such that for all $x, y \in X$,*

$$S^{(a(x),a(y))}_{EX}(Sx, Ty) \leq \rho d(x, y), \tag{3.7}$$

then there exists $u \in X$ such that $u \in (Su)(a(u)) \cap (Tu)(a(u))$, for some $a(u) \in E$.

Corollary 3.6. *[21, Thrm 3.1] Let (X, d) be a complete metric space, $g : X \longrightarrow X$ be a surjection with $gx = \bar{x}$, E be the parameter set and $S : X \longrightarrow [P(X)]^E$ a soft set-valued map. Suppose that for each $x \in X$ there exist $a(x) \in DomSx$ such that $(Sx)(a(x)) \in CB(X)$. If there exists $\rho \in (0, 1)$ such that for all $x, y \in X$,*

$$S^{(a(x),a(y))}_{EX}(Sx, Sy) \leq \rho d(\bar{x}, \bar{y}), \tag{3.8}$$

then there exists $\bar{u} \in X$ such that $\bar{u} \in (Su)(a(u))$, for some $a(u) \in E$.

Proof. Put $S = T$ in Theorem 3.3. □

Corollary 3.7. *Let (X, d) be a complete metric space, $g : X \longrightarrow X$ be a surjection with $gx = \bar{x}$, E be the parameter set and $S, T : X \longrightarrow [P(X)]^E$ any two soft set-valued maps. Suppose that for each $x \in X$ there exist $a(x) \in DomSx$ and $a(x) \in DomTx$ such that $(Sx)(a(x)), (Tx)(a(x)) \in CB(X)$. If there exists $\rho \in (0, 1)$ such that for all $x, y \in X$,*

$$S_{EX}^{(a(x),a(y))}(Sx, Ty) \le \rho\left(\frac{d(\bar{x}, \bar{y})}{1 + d(\bar{x}, \bar{y})}\right), \tag{3.9}$$

then there exists $\bar{u} \in X$ such that $\bar{u} \in (Su)(a(u)) \cap (Tu)(a(u))$, for some $a(u) \in E$.

Proof. Since

$$\begin{aligned} S_{EX}^{(a(x),a(y))}(Sx, Ty) &\le \rho\left(\frac{d(\bar{x}, \bar{y})}{1 + d(\bar{x}, \bar{y})}\right) \\ &\le \rho d(\bar{x}, \bar{y}), \end{aligned}$$

the conclusion follows from Theorem 3.3. □

Corollary 3.8. *Let (X, d) be a complete metric space, $g : X \longrightarrow X$ be a surjection with $gx = \bar{x}$, E be the parameter set and $S, T : X \longrightarrow [P(X)]^E$ be soft set-valued maps. Suppose for each $x \in X$, there exists $a(x) \in DomSx$ and $a(x) \in DomTx$ such that $(Sx)(a(x)), (Tx)(a(x)) \in CB(X)$. If there exists $\rho \in (0, 1)$ such that for all $x, y \in X$,*

$$S_{EX}^{\infty}(Sx, Ty) \le \rho d(\bar{x}, \bar{y})), \tag{3.10}$$

then there exists $\bar{u} \in X$ such that $\bar{u} \in (Su)(a(u)) \cap (Tu)(a(u))$, for some $a(u) \in E$.

Proof. As

$$S_{EX}^{(a(x),a(y))}(Sx, Ty) \le S_{EX}^{\infty}(Sx, Ty),$$

the proof follows directly by applying Theorem 3.3. □

Corollary 3.9. *[21] Let (X, d) be a complete metric space, $g : X \longrightarrow X$ be a surjection with $gx = \bar{x}$, E be the parameter set and $S : X \longrightarrow [P(X)]^E$ a soft set-valued map. Suppose for $x \in X$, there exists $a(x) \in DomSx$ such that $(Sx)(a(x)) \in CB(X)$. If there exists $\beta \in (0, 1)$ such that for all $x, y \in X$,*

$$S_{EX}^{\infty}(Sx, Sy) \le \beta d(\bar{x}, \bar{y})), \tag{3.11}$$

then there exists $\bar{u} \in X$ such that $\bar{u} \in (Au)(a(u))$, for some $a(u) \in E$.

4. Fixed points of fuzzy and multivalued mappings

As a generalization of the notion of crisp sets, fuzzy sets was introduced by Zadeh [35]. Since then, to use this concept, many authors have progressively extended the theory and its applications to other branches of sciences, social sciences and engineering. In 1981, Heilpern [17] used the idea of fuzzy set to initiate a class of fuzzy set-valued maps and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem of Nadler [23]. Subsequently, several authors have studied the existence of fixed points of fuzzy set-valued maps, for example, Al-Mazrooei etal [2], Abu-Donia [1], Azam et al [5, 6, 4], Bose and Sahani [9], Qiu and Shu [26], and so on. For convenience, we recall some specific concepts of fuzzy sets and fuzzy mappings as follows.

Definition 4.1. [35] Let X be a universal set. A fuzzy set in X is a function with domain X and values in $[0,1]=I$. If A is a fuzzy set in X, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \text{ if } \alpha \in (0,1]$$
$$[A]_0 = \overline{\{x \in X : A(x) > 0\}},$$

where by $\overline{M}$, we mean the closure of the crisp set M. Denote by I^X, the collection of all fuzzy sets in X.

Let (X,d) be a metric space. For $[A]_\alpha, [B]_\alpha \in CB(X)$, the function $H : CB(X) \times CB(X) \longrightarrow \mathbb{R}_+$, defined by

$$H([A]_\alpha, [B]_\alpha) = \max\left\{ \sup_{x\in[A]_\alpha} d(x,[B]_\alpha), \sup_{x\in[B]_\alpha} d(x,[A]_\alpha) \right\}$$

is called the Hausdorff-Pompeiu metric on $CB(X)$ induced by the metric d, where

$$d(x,[A]_\alpha) = \inf_{y\in[A]_\alpha} d(x,y).$$

Definition 4.2. [17] A fuzzy set A in a metric linear space X is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V and $\sup_{x\in V} A(x) = 1$.

We denote the collection of all approximate quantities in X by $W(X)$. If there exists an $\alpha \in [0,1]$ such that $[A]_\alpha, [B]_\alpha \in W(X)$, then define

$$p_\alpha(A,B) = \inf_{x\in[A]_\alpha, y\in[B]_\alpha} d(x,y).$$

$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha).$$
$$p(A,B) = \sup_\alpha p_\alpha(A,B)$$
$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

Definition 4.3. [17] Let X be an arbitrary set and Y be a metric space. A mapping $\digamma : X \longrightarrow I^Y$ is called a fuzzy mapping. A fuzzy mapping $\digamma$ is a fuzzy subset of $X \times Y$. The function $\digamma(x)(y)$ is the degree of membership of y in $\digamma(x)$. An element u in X is said to be a fuzzy fixed point of $\digamma$ if there exists an $\alpha \in I$ such that $u \in [\digamma u]_\alpha$.

We recall that a set-valued mapping $\Theta : X \longrightarrow CB(X)$ is called a multi-valued map. A point $x \in X$ is said to be a fixed point of Θ if $x \in \Theta x$. In 1969, Nadler [23] first gave a generalization of the Banach contraction principle for multivalued map by using the Hausdorff metric. Since then, a number of generalizations in various frames of Nadler's fixed point theorem have been investigated by several authors; see, for example, [7, 20, 24] and the references therein.

In this section, as an application of Theorem 3.3, we deduce the conclusions of some common fixed point theorems of Heilpern [17] and Nadler [23] type. Our main aim here is to further illustrate the connections between fuzzy mappings, multi-valued mappings and the concepts of soft set-valued maps. First, recall that in [21], it is shown that every fuzzy mapping $\digamma : X \longrightarrow I^X$ can be considered as a soft set-valued map $T_\digamma : X \longrightarrow [P(X)]^{E=[0,1]}$, defined by

$$T_\digamma(x)(e) = \{t \in X : (\digamma x)(t) \geq e\}.$$

Similarly, $X \longmapsto P(X)$ is embedding by $x \longrightarrow \{x\}$ and $P(X) \longrightarrow I^X$ is embedding by $M \longrightarrow \chi_M$, for every subset M of $P(X)$; where χ_M is the characteristic function of the crisp set M. In the like manner, $I^X \longrightarrow [P(X)]^{[0,1]}$ is embedding by $U \longrightarrow \Upsilon_U$, for every U in I^X; where

$$\Upsilon_U(e) = U_e = \{t \in X : U(t) \geq e\}.$$

Theorem 4.4. *Let X be a complete linear metric space and $\digamma_1, \digamma_2 : X \longrightarrow W(X)$ be fuzzy mappings . If there exists $\rho \in (0,1)$ such that*

$$d_\infty(\digamma_1 x, \digamma_2 y) \leq \rho d(x,y), \quad \text{for each } x, y \in X,$$

then there exists $u \in X$ such that $\chi_{\{u\}} \subset \digamma_1(u)$ and $\chi_{\{u\}} \subset \digamma_2(u)$.

Proof. Consider two soft set-valued maps $\Omega_{\digamma_1}, \Omega_{\digamma_2} : X \longrightarrow [P(X)]^{E=[0,1]}$, defined by

$$\Omega_{\digamma_1}x(e) = \{t \in X : (\digamma_1 x)(t) \geq e\} = [\digamma_1 x]_e.$$

and

$$\Omega_{\digamma_2}x(e) = \{t \in X : (\digamma_2 x)(t) \geq e\} = [\digamma_2 x]_e.$$

Then,

$$\begin{aligned} S_{EX}^{(1,1)}(\Omega_{\digamma_1}x, \Omega_{\digamma_2}y) &= \inf E^d_{((\digamma_1 x)(1),(\digamma_2 y)(1))} \\ &= H([\digamma_1 x]_1, [\digamma_2 y]_1) \\ &\leq d_\infty(\digamma_1 x, \digamma_2 y) \\ &\leq \rho d(x,y). \end{aligned}$$

Hence, Theorem 3.3 can be applied with $g = I_X$, the identity mapping on X, to find $u \in X$ such that $u \in (\Omega_{F_1} u)(1) = [F_1 u]_1$ and $u \in (\Omega_{F_2} u)(1) = [F_2 u]_1$. It follows that $\chi_{\{u\}} \subset F_1(u)$ and $\chi_{\{u\}} \subset F_2(u)$. □

Corollary 4.5. *[17] Let X be a complete linear metric space and $F : X \longrightarrow W(X)$ be a fuzzy mapping. If there exists $\beta \in (0,1)$ such that*

$$d_\infty(Fx, Fy) \leq \rho d(x,y), \quad \text{for each } x, y \in X,$$

then there exists $u \in X$ such that $\chi_{\{u\}} \subset F(u)$.

Proof. Put $F_1 = F_2 = F$ in Theorem 4.4. □

Theorem 4.6. *Let (X,d) be a complete metric space and*

$$\Theta, \Lambda : X \longrightarrow CB(X)$$

be multi-valued mappings satisfying the following conditions: there exists $\rho \in (0,1)$ such that

$$H(\Theta_x, \Lambda_y) \leq \rho d(x,y).$$

Then there exists $u \in X$ such that $u \in \Theta u \cap \Lambda u$.

Proof. Let $E = \{e_1, e_2\}$ and consider two soft set-valued maps

$$T_1, T_2 : X \longrightarrow [P(X)]^{\{e_1,e_2\}},$$

defined by

$$T_1 x(e) = \begin{cases} X, & \text{if } e = e_2 \\ \Theta x, & \text{if } \ e = e_1. \end{cases}$$

and

$$T_2 x(e) = \begin{cases} \Lambda x, & \text{if } e = e_2 \\ X & \text{if } \ e = e_1. \end{cases}$$

Then,

$$\begin{aligned} S_{EX}^{(e_1(x),e_2(y))}(T_1 x, T_2 y) &= \inf E^d_{((T_1 x)(e_1(x)),(T_2 y)(e_2(y)))} \\ &= \inf E^d_{(\Theta x, \Lambda y)} \\ &= H(\Theta x, \Lambda y) \\ &\leq \rho d(x,y). \end{aligned}$$

Therefore, by Theorem 3.3 there exits $u \in X$ such that

$$u \in T_1 u(e_1(u)) = \Theta u \quad \text{and } u \in T_2 u(e_2(u)) = \Lambda u.$$

□

Corollary 4.7. *[23] Let (X, d) be a complete metric space and*

$$\Theta : X \longrightarrow CB(X)$$

be a multi-valued mapping satisfying the following conditions: there exists $\rho \in (0, 1)$ such that

$$H(\Theta_x, \Theta_y) \leq \rho d(x, y).$$

Then there exists $u \in X$ such that $u \in \Theta u$.

Proof. Put $\Theta = \Lambda$ in Theorem 4.6. □

5. Application to fuzzy Volterra integral equations

Fuzzy Volterra integral equations have been studied extensively due to their applications in diverse fields such as medical diagnosis, predator-prey model and in modeling of dynamical behaviour of several physical processes. The notion of fuzzy integral equations was introduced by Kaleva [18] and Seikkala [29]. Several authors applied different fixed point theorems such as the classical Banach contraction principle [32], Shauder fixed point theorem [3], and some fixed point theorems on partially ordered spaces [34]. In the recent time, the above techniques have been extended in various ways. Thus, in this section, we study such existence theorems of fuzzy number-valued Volterra integral equations by using the ideas of fuzzy integrals presented by Puri and Ralescu [25]. Our investigation offers, for the first time, an existence theorem for fuzzy number-valued Volterral integral equations in which the idea of soft set-valued maps is utilized. In what follows, we give some notations and preliminary concepts which are needed in the sequel. For these basic concepts, we follow [13, 18, 25, 32].

Let $\mathcal{P}_C(\mathbb{R}^n)$ denotes the family of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Addition and multiplication in $\mathcal{P}_C(\mathbb{R}^n)$ are defined as usual, i.e. for $A, B \in \mathcal{P}_C(\mathbb{R}^n)$ and scalar $\lambda \in \mathbb{R}$, we have $A + B = \{a + b : a \in A \text{ and } b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$. The space of fuzzy numbers (see [13]), denoted by $\mathcal{F}^n$, is the set of functions $u : \mathbb{R} \longrightarrow [0, 1]$ satisfying the following properties:

(i) u is normal, that is, there exists $t_0 \in \mathbb{R}$ such that $u(t_0) = 1$;

(ii) u is fuzzy convex, that is, $u(\lambda t_1 + (1 - \lambda)t_2) \geq \min\{u(t_1), u(t_2)\}$, for all $t_1, t_2 \in \mathbb{R}$;

(iii) u is upper semicontinuous, that is, $[u]^\alpha$ is closed for all $\alpha \in [0, 1]$;

(iv) $[u]^0 = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$, where $\overline{M}$ denotes the closure of M.

For $\alpha \in (0, 1]$, the α-level set of u in $\mathcal{F}^n$ is defined as

$$[u]^\alpha = \{t \in \mathbb{R} : u(t) > \alpha\}.$$

For $u, v \in \mathcal{F}^n$ and $\lambda \in \mathbb{R}$, we define the addition $u + v$ and multiplication λu as:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \text{ and } [\lambda u]^\alpha = \lambda [u]^\alpha.$$

Every real number can be embedded in $\mathcal{F}^n$ via the rule $\lambda \longrightarrow \hat{\lambda}(t)$, where

$$\hat{\lambda}(t) = \begin{cases} 1, & \text{for } t = \lambda \\ 0, & \text{otherwise.} \end{cases}$$

Let $d^* : \mathcal{F}^n \times \mathcal{F}^n \longrightarrow [0, \infty)$ be defined by

$$d^*(u, v) = \sup_{0 \leq \alpha \leq 1} H\left([u]^\alpha, [v]^\alpha\right),$$

where H is the Hausdorff distance in $\mathcal{P}_C(\mathbb{R}^n)$. Then d^* is a metric on $\mathcal{F}^n$ and $(\mathcal{F}^n, d^*)$ is a complete metric space (see [18, 25]). Also, for every $u, v \in \mathcal{F}^n$, we have

$$d^*(u + w, v + w) = d^*(u, v) \text{ and } d^*(\lambda u, \lambda v) = \lambda d^*(u, v),$$

for all $\lambda \in \mathbb{R}$. By using $H(A + B, C + D) \leq H(A, C) + H(B, D)$, it can be verified directly that

$$d^*(u + v, w + z) \leq d^*(u, w) + d^*(v, z).$$

Definition 5.1. [18] A mapping $T : [0, 1] \longrightarrow \mathcal{F}^n$ is said to be integrably bounded if there exists an integrable function f such that $\|x\| \leq f(t)$ for all $x \in T(t)$.

Definition 5.2. [18] We say that a mapping $T : [0, 1] \longrightarrow \mathcal{F}^n$ is strongly measurable if for all $\alpha \in [0, 1]$, the set-valued mapping $T_\alpha : [0, 1] \longrightarrow \mathcal{P}_C(\mathbb{R}^n)$ defined by

$$T_\alpha(t) = [T(t)]^\alpha,$$

is (Lebesgue) measurable, when $\mathcal{P}_C(\mathbb{R}^n)$ is equipped with the Haudorff metric.

We recall that a function f defined on some collection X of nonempty sets is said to be a selection for X if it belongs to the direct product of X.

Definition 5.3. [25] The integral of a fuzzy number-valued mapping $T : [0, 1] \longrightarrow \mathcal{F}^n$ is defined levelwise by

$$\begin{aligned} \left[\int_{[0,1]} T(t)dt\right]^\alpha &= \int_{[0,1]} T_\alpha(t)dt \\ &= \left\{ \int_{[0,1]} f(t)dt : f : [0,1] \longrightarrow \mathbb{R}^n \right. \\ &\qquad \left. \text{is a measurable selection for } T_\alpha \text{ for } \alpha \in (0, 1] \right\}. \end{aligned}$$

A strongly measurable and integrably bounded mapping $T : [0, 1] \longrightarrow \mathcal{F}^n$ is said to be integrable over $[0, 1]$ if $\int_{[0,1]} T(t)dt \in \mathcal{F}^n$.

Definition 5.4. [18] Let $S, T : [0,1] \longrightarrow \mathcal{F}^n$ be integrable and $\lambda \in \mathbb{R}$. Then

(i) $\int (S+T) = \int S + \int T$

(ii) $\int \lambda T = \lambda \int T$

(iii) $d^*(S,T)$ is integrable

(i v) $d^*(\int S, \int T) \leq \int d^*(S,T)$.

Next, we prove an existence theorem for a fuzzy number-valued Volterra integral equation, given by

$$x(t) = \sigma \int_a^t L(t,s)x(s)ds + h(t),\ \sigma > 1. \tag{5.1}$$

Theorem 5.5. *Consider* (5.1). *Assume that* $L : [a,b] \times [a,b] \longrightarrow \mathbb{R}^n$ *and* $h : [a,b] \longrightarrow \mathcal{F}^n$ *are continuous functions. If* $\|L(t,s)\| \leq \frac{1}{(b-a)}$ *for all* $t, s \in [a,b]$, *then* (5.1) *has at least one fuzzy number-valued solution.*

Proof. Let $E = (0, \infty)$ and $X = C([a,b], \mathcal{F}^n)$ be the space of all continuous fuzzy number-valued functions defined on $[a,b]$. We metricize X by setting

$$d^*(x,y) = \sup_{0 \leq \alpha \leq 1} H([x]^\alpha, [y]^\alpha),$$

for all $x, y \in X$, where H is the Hausdorff metric defined on $\mathcal{P}_C(\mathbb{R}^n)$. Then d^* is a metric on X and (X, d^*) is a complete metric space (see [18, 25]). Moreover, for $x \in X$, take

$$\pi_x(t) = \sigma \int_a^t L(t,s)x(s)ds + h(t).$$

Then, define a soft set-valued map $T : X \longrightarrow [P(X)]^{(0,\infty)}$ by

$$(T_e x) = \begin{cases} \{x \in X : x(t) < \pi_x(t)\}, & \text{if } 0 < e < 5 \\ \{x \in X : x(t) = \pi_x(t)\}, & \text{if } 5 \leq e \leq 10 \\ \emptyset, & \text{if } 10 < e < \infty. \end{cases}$$

Thus, for $x \in X$, there exists $a(x) = e \in [5, 10]$ such that

$$(T_e x) = \{\pi_x(t)\} \in CB(X).$$

Further, take a number $\sigma \in (1, \infty)$ such that

$$S_{EX}^{\infty}(Tx, Ty) \leq \frac{1}{\sigma^2} d^*(\pi_x(t), \pi_y(t)),$$

where

$$S_{EX}^{\infty}(Tx, Ty) = \sup_{(a(x),a(y)) \in E \times E} S_{EX}^{(a(x),a(y))}(Tx, Ty)$$

and d^* is as defined earlier. Then, for all $x, y \in X$, we have

$$\begin{aligned}
&S_{EX}^{(a(x),a(y))}(Tx,Ty)\\
&\leq \sup_{(a(x),a(y))\in E\times E} S_{EX}^{(a(x),a(y))}(Tx,Ty)\\
&\leq \frac{1}{\sigma^2}d^*(\pi_x(t),\pi_y(t))\\
&\leq \frac{1}{\sigma^2}d^*\left(\sigma\int_a^t L(t,s)x(s)ds+h(t),\sigma\int_a^t L(t,s)y(s)ds+h(t)\right)\\
&\leq \frac{1}{\sigma^2}d^*\left(\sigma\int_a^t L(t,s)x(s)ds,\sigma\int_a^t L(t,s)y(s)ds\right)\\
&\leq \left(\frac{1}{\sigma}\right)d^*\left(\int_a^t L(t,s)x(s)ds,\int_a^t L(t,s)y(s)ds\right)\\
&\leq \frac{1}{\sigma(b-a)}\max\left\{\int_a^t d^*(x(s),y(s))ds : t\in[a,b]\right\}\\
&\leq \frac{1}{\sigma}d^*(x(s),y(s)) = \kappa d^*(x,y),
\end{aligned}$$

where $\kappa = \frac{1}{\sigma} \in (0,1)$. Therefore, all the hypotheses of Corollary 3.5 are satisfied with $S = T$. Consequently, the fuzzy Volterra integral equation (5.1) has at least one fuzzy number-valued solution in X. $\square$

Example 5.6. Consider the fuzzy number-valued Volterra integral equation

$$x(t) = 126\int_8^t \sin(t^2)\cos(s^3)x(s)ds + h(t), \tag{5.2}$$

where $h : [8,9] \longrightarrow \mathcal{F}^1$ is given by

$$h(t)(x) = \begin{cases} \frac{tx}{5}, & \text{if } x\in\left[0,\left(\frac{1}{t^2}\right)\right]\\ \frac{1}{13}, & \text{if } x\in\left[\left(\frac{1}{t^2}\right),1-\left(\frac{1}{t^2}\right)\right]\\ 0, & \text{elsewhere.}\end{cases} \tag{5.3}$$

Then

$$[h(t)]^\alpha = \left[\left(\frac{\alpha}{t^2}\right),1-\left(\frac{\alpha}{t^2}\right)\right] \text{ for } 0<\alpha\leq 1$$

and $[h(t)]^0 = [0,1]$. Obviously, the fuzzy number-valued function $h : [8,9] \longrightarrow \mathcal{F}^1$ defined by (5.3) is continuous. Further, from (5.2), we see that

$$\|K(t,s)\| = \left\|\sin(t^2)\cos(s^3)\right\| \leq \frac{1}{b-a},$$

where $a = 8$ and $b = 9$. Hence, by Theorem 5.5, there exists at least one fuzzy number-valued solution of problem 5.2.

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