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Some aspects of quasi-uniform box products

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Abstract. The quasi-uniform box product is a topology on the product of countably many copies of a quasi-uniform space that is finer than the Tychonov product topology but coarser than the uniform box product. In this paper, we present completeness, connectedness and some separation properties in quasi-uniform box products.

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1. Introduction

The quasi-uniform box product is a topology on the product of countably many copies of a quasi-uniform space that sits between the Tychonov product topology and the uniform box product. This topology was introduced in [13]. In this article, we showed that the quasi-uniform box product is generated by a quasi-uniformity called the constant quasi-uniformity whose symmetrised uniformity coincides with constant uniformity in the sense of Bell [1]. In [15], we introduced infinite games of two players, played in a quasi uniform space and used these games to show that the quasi-uniform box product of a Fort-Space is collectionwise normal, collectionwise Hausdorff and countably paracompact. In this article, we continue investigating this concept of quasi-uniform box products. In particular, we first show that the quasi-uniform box product is coarser than the uniform box product and finer than the Tychonov product topology. We then consider some results on quotient spaces associated with the quasi-uniform box product. Thereafter, we observe that the quasi-uniform box product is a particular case of the topology of quasi-uniform convergence. We then use this fact to adapt some results on completeness in the quasi-uniformity of uniform convergence to the framework of quasi-uniform box products. Furthermore, we observe that the quasi-uniform box product is a bitopological space in its own right and, thereafter, present some separation properties of a quasi uniform space that are preserved by its quasi-uniform box product. Finally, we present connectedness in quasi-uniform box products.

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2. Quasi-uniform box products

In this section, we present the quasi-uniform box product, a concept that generalises the uniform box product to the framework of quasi-uniform spaces. This concept was introduced in [13]. In this section, we show that the quasi-uniform box product is finer than the Tychonov product topology but coarser than the uniform box product. We also observe that the quasi-uniform box product is a particular case of the topology of quasi-uniform convergence.

Definition 2.1. [9] A quasi-uniformity $\mathcal{U}$ on a set X is a filter on $X \times X$ such that

(i) each member U of $\mathcal{U}$ contains the diagonal $\triangle = \{(x,x) : x \in X\}$ of X,

(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = V \circ V = \{(x,z) \in X \times X :$ there is $y \in X$ such that $(x,y) \in V, (y,z) \in V\}$.

The members $U \in \mathcal{U}$ are called *entourages* of $\mathcal{U}$ and the elements of X are called *points*. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*.

If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X. The quasi-uniformity $\mathcal{U}^{-1}$ is called the *conjugate* of $\mathcal{U}$. A quasi-uniformity that is equal to its conjugate is called a *uniformity*. The union of a quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ yields a subbase of the coarsest uniformity, denoted $\mathcal{U}^s$, finer than $\mathcal{U}$. If $U \in \mathcal{U}$, the elements of $\mathcal{U}^s$ are of the form $U \cap U^{-1}$ and are denoted by U^s. For $U \in \mathcal{U}$, $x \in X$ and $Z \subset X$, put $U(x) = \{y \in X : (x,y) \in U\}$ and $U(Z) = \bigcup\{U(z) : z \in Z\}$. A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets $\{U(x) : U \in \mathcal{U}\}$ is a base of neighbourhoods of any point $x \in X$.

A subset A of X belongs to $\tau(\mathcal{U})$ if and only if for each $x \in A$, there is an entourage $U \in \mathcal{U}$ such that $U(x) \subset A$. Thus for each $x \in X$ and $U \in \mathcal{U}$, $U(x)$ is a $\tau(\mathcal{U})$-neighbourhood of x. Note that $U(x)$ need not be $\tau(\mathcal{U})$-open in general. However, there is always a base $\mathcal{B}$ for $\mathcal{U}$ such that for each $B \in \mathcal{B}$ and $x \in X$, $B(x) \in \tau(\mathcal{U})$.

Definition 2.2. [13] Let $(X,\mathcal{U})$ be a quasi-uniform space and $\prod_{n\in\mathbb{N}} X$ be the product of countably many copies of X. Then $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$ is a filter base generating the quasi-uniformity on $\prod_{n\in\mathbb{N}} X$, where

$$\overline{U} = \left\{(x,y) \in \prod_{n\in\mathbb{N}} X \times \prod_{n\in\mathbb{N}} X : \ (x(n), y(n)) \in U \text{ whenever } n \in \mathbb{N}\right\}.$$

The quasi-uniformity $\overline{\mathcal{U}}$ is called *constant quasi-uniformity* on the product $\prod_{n\in\mathbb{N}} X$ and the pair $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is called *quasi-uniform box product.*

One can use the following facts to verify that $\overline{\mathcal{U}}$ is indeed a quasi-uniformity on $\prod_{n\in\mathbb{N}} X$:

(i) for all $U, V \in \mathcal{U}$, $\overline{U \cap V} = \overline{U} \cap \overline{V}$

(ii) for all $U, V \in \mathcal{U}$, $\overline{U \circ V} \supset \overline{U} \circ \overline{V}$

Also, it is clear that a $\tau(\overline{\mathcal{U}})$-neighbourhood of an arbitrary $x \in \prod_{n\in\mathbb{N}} X$ is of the form

$$\begin{aligned} \overline{U}(x) &= \{y \in \prod_{n\in\mathbb{N}} X : (x,y) \in \overline{U}\} \\ &= \{y \in \prod_{n\in\mathbb{N}} X : y(n) \in U(x(n)) \text{ for all } n \in \mathbb{N}\} \end{aligned}$$

Remark 2.3. If $(X,\mathcal{U})$ is a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. Then quasi-uniform space $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}^{-1}}\right)$ is again a quasi-uniform box product of $(X,\mathcal{U})$, where $\overline{\mathcal{U}^{-1}} = \{\overline{U^{-1}} : U \in \mathcal{U}\}$ is again a filter base generating a quasi-uniformity on $\prod_{n\in\mathbb{N}} X$. Thus the triple $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}, \overline{\mathcal{U}^{-1}}\right)$ is a bitopological space in the sense of Kelly [7]. Also, $\overline{\mathcal{U}^{-1}} \vee \overline{\mathcal{U}} = \overline{\mathcal{U}}^s$ is a filter base generating a uniformity on $\prod_{n\in\mathbb{N}} X$ and the pair $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}^s\right)$ is a uniform box product of the uniform space $(X,\mathcal{U})$ which corresponds to the uniform box product in the sense of Bell (see [1, Definition 3.2]).

We now present the supremum T_0-quasi-metric and show how it is related to the constant quasi-uniformity.

Proposition 2.4. *(*Supremum T_0-quasi-metric*) Suppose (X,d) is a T_0-quasi-metric space and $\prod_{n\in\mathbb{N}} X$ is the product of countably many copies of X. Then q, defined by*

$$q(x,y) = \sup\{d(x(n), y(n)) : n \in \mathbb{N}\},$$

is a T_0-quasi metric on $\prod_{n\in\mathbb{N}} X$.

Remark 2.5. Let (X,d) be a T_0-quasi-metric space and $\prod_{n\in\mathbb{N}} X$ be the product of countably many copies of X. Then q^{-1}, defined by $q^{-1}(x,y) = q(y,x)$, is a T_0-quasi-metric on $\prod_{n\in\mathbb{N}} X$. Furthermore, $q^s = \max\{q, q^{-1}\}$ is a metric on $\prod_{n\in\mathbb{N}} X$.

Theorem 2.6. *Suppose (X,d) is a T_0-quasi-metric space and $\prod_{n\in\mathbb{N}} X$ is the product of countably many copies of X. Then the topology generated by the T_0-quasi-metric q, defined by $q(x,y) = \sup\{d(x(n), y(n)) : n \in \mathbb{N}\}$, is finer than the Tychonov product topology on the product of (X, d^s).*

Proof. Suppose $x \in \prod_{n\in\mathbb{N}} X$. Let $U = \prod_{n\in\mathbb{N}} U_n$ be a basic $\tau(d^s)$-open set in the Tychonov product topology on $\prod_{n\in\mathbb{N}} X$ with $x \in U$. Then there exists a finite subset J_0 of $\mathbb{N}$ such that if $n \in \mathbb{N} \setminus J_0$, then $U_n = X$. If $n \in J_0$, then because U_n is a $\tau(d^s)$-open subset of X and $x(n) \in U_n$, there is some $0 \leq \epsilon_n < 1$ such that $B_{d^s}(x(n), \epsilon_n) \subseteq U_n$. Let $\epsilon = \min_{n \in J_0} \epsilon_n$. If $q^s(x, y) < \epsilon$, then $d^s(x(n), y(n)) < \epsilon$ for all $n \in \mathbb{N}$ and this implies that $d^s(x(n), y(n)) < \epsilon_n$ for all $n \in J_0$. It follows that $d(x(n), y(n)) < \epsilon_n$ for all $n \in J_0$. This implies that $y(n) \in B_d(x(n), \epsilon_n) \subseteq U_n$ for all $n \in J_0$. Also, if $n \in \mathbb{N} \setminus J_0$, then $U_n = X$ and it follows that $y(n) \in U_n$. Therefore, if $y \in B_q(x, \epsilon)$, then $y \in U$, and this implies that $B_q(x, \epsilon) \subseteq U$. It follows that the topology generated by q on $\prod_{n \in \mathbb{N}} X$ is finer than the Tychonov product topology on $\prod_{n \in \mathbb{N}} X$. $\square$

The supremum T_0-quasi-metric and the constant quasi-uniformity are related in the following way. Note that this was observed by Bell [1] in the framework of uniform box products.

Example 2.7. Suppose (X, d) is a T_0-quasi-metric space, and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where

$$U_n = \{(x, y) : d(x, y) < 2^{-n}\},$$

is a filter base generating a quasi-uniformity on X. Let $\overline{\mathcal{U}}$ be the constant quasi-uniformity on $\prod_{n \in \mathbb{N}} X$. Also, let q be the T_0-quasi-metric on $\prod_{n \in \mathbb{N}} X$. From q, we define the following quasi-uniformity on $\prod_{n \in \mathbb{N}} X$:

$$\mathcal{S} = \{S_n : n \in \mathbb{N}\}, \text{ where } S_n = \{(x, y) : q(x, y) < 2^{-n}\}.$$

We show that $\overline{\mathcal{U}}$ and $\mathcal{S}$ generate the same quasi-uniformity on $\prod_{n \in \mathbb{N}} X$.

Suppose $(x, y) \in S_n$. Then $\sup\{d(x(i), y(i)) : i \in \mathbb{N}\} < 2^{-n}$. Thus for all $i \in \mathbb{N}$, $d(x(i), y(i)) < 2^{-n}$. Hence $(x, y) \in \overline{U}_n$ and so $S_n \subseteq \overline{U}_n$. Suppose $(x, y) \in \overline{U}_{n+1}$. Then for all $i \in \mathbb{N}$, $d(x(i), y(i)) < 2^{-(n+1)}$. Thus

$$q(x, y) = \sup\{d(x(i), y(i)) : i \in \mathbb{N}\} \leq 2^{-(n+1)} < 2^{-n}$$

so that $(x, y) \in S_n$. Therefore, $\overline{U}_{n+1} \subseteq S_n$.

Remark 2.8. From Theorem 2.6 and Example 2.7, we conclude that the quasi-uniform box product is finer than the Tychonov product topology. Also, from Remark 2.3, we see that the quasi-uniform box product is coarser than the uniform box product.

We now turn our attention to the quasi-uniformity of uniform convergence and show how it is related to the quasi-uniform box product.

Definition 2.9. [9] Let (X, τ) be a topological space and $(Y, \mathcal{V})$ be a quasi-uniform space. Furthermore, let $\mathcal{D}$ be a family of maps from X to Y. If $\mathcal{A}$ is a family of subsets of X, we denote by $\mathcal{V}_{\mathcal{A}}$ the quasi-uniformity on $\mathcal{D}$ which has, as subbase, the family of all relations of the form

$$(A, U) = \{(f, g) \in \mathcal{D} \times \mathcal{D} : (f(x), g(x)) \in U \text{ whenever } x \in A\}$$

whenever $A \in \mathcal{A}$ and $U \in \mathcal{V}$. The quasi-uniformity $\mathcal{V}_{\mathcal{A}}$ is called the *quasi-uniformity of uniform convergence* of $\mathcal{A}$.

Note that if $X = \mathbb{N}$ in the definition above, then the quasi-uniformity of uniform convergence is just the constant quasi-uniformity.

Example 2.10. [9] We note the following:

(i) If $\mathcal{A} = \{X\}$, $\mathcal{V}_{\mathcal{A}}$ is called the *quasi-uniformity of uniform convergence* and is denoted by $\mathcal{V}_X$.

(ii) If $\mathcal{A} = \{K \subseteq X : K \text{ is a compact subset of } (X, \tau)\}$, $\mathcal{V}_{\mathcal{A}}$ is called the *quasi-uniformity of compact convergence* and is denoted by $\mathcal{V}_K$.

(iii) If $\mathcal{A} = \{F \subseteq X : F \text{ is a finite subset of } X\}$, then $\mathcal{V}_{\mathcal{A}}$ is called the *quasi-uniformity of pointwise convergence* and is denoted by $\mathcal{V}_p$. Note that on the product space Y^X, $\mathcal{V}_p$ agrees with the product quasi-uniformity.

In this paper, we are going to present results on function spaces with the quasi-uniformity of uniform convergence that can be adapted to quasi-uniform box products.

Let us first look at quotient spaces in quasi-uniform box products.

3. Quotient spaces

In [16], Williams introduced the concept of the nabla product on the box product to prove that the countable box product of compact spaces is paracompact if and and only if its nabla product is paracompact. However, this approach is not useful on uniform box products. Therefore, in [2], Bell defined an equivalence relation on the uniform box product in order to study connections between the nabla product on the uniform box product and this new equivalence relation. In this section we are going to extend the results of Bell [2] to quasi-uniform spaces.

The following definition was first introduced in [16] and it does not depend on the uniformity. Therefore, we keep it in the context of quasi-uniform spaces.

Definition 3.1. (compare [2, Definition 5.3]) Suppose $(X, \mathcal{U})$ is a quasi-uniform space and $x, y \in \prod_{n\in\mathbb{N}} X$. Then $x \approx y$ if and only if the set $\{n \in \mathbb{N} : x(n) \neq y(n)\}$ is finite.

Remark 3.2. Obviously, the relation $\approx$ is an equivalence relation on $\prod_{n\in\mathbb{N}} X$ and its quotient space is called the *nabla product* ([16, Definition 5.3]). Furthermore, if $x \approx y$, then we say x and y are *mod-finite* equivalent (see [2]). This equivalence relation was used in [2] on uniform box products.

The following definition is an extension of [16, Definition 3.1] to quasi-uniform spaces.

Definition 3.3. (compare [2, Definition 5.3]) Suppose $(X, \mathcal{U})$ is a quasi-uniform space and $x, y \in \prod_{n\in\mathbb{N}} X$. Then we define the relation $\sim$ by

$$x \sim y \text{ if and only if the set } \{n \in \mathbb{N} : (x(n), y(n)) \notin U \cap U^{-1}\} \text{ is finite}$$

whenever $U \in \mathcal{U}$.

The following result can be compared to [2, Proposition 5.4] but it has some variations. Therefore, we need to prove it.

Lemma 3.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the relation $\sim$ in Definition 3.3, is an equivalence relation on $\prod_{n\in\mathbb{N}}X$ and we call the relation $\sim$ the* quasi-uniform equivalence *on $\prod_{n\in\mathbb{N}}X$.*

Proof. Let $x, y, z \in \prod_{n\in\mathbb{N}}X$. We have that $x \sim x$ since the set

$$\{n \in \mathbb{N} : (x(n), x(n)) \notin U \cap U^{-1}\} = \emptyset \text{ is finite}$$

whenever $U \in \mathcal{U}$.

Suppose $x \sim y$. Then observe that

$$\{n \in \mathbb{N} : (x(n), y(n)) \in U \cap U^{-1}\} = \{n \in \mathbb{N} : (y(n), x(n)) \in U \cap U^{-1}\}$$

whenever $U \in \mathcal{U}$. It follows by taking the complement that

$$\{n \in \mathbb{N} : (y(n), x(n)) \notin U \cap U^{-1}\} = \{n \in \mathbb{N} : (x(n), y(n)) \notin U \cap U^{-1}\} \text{ is finite.}$$

Hence $y \sim x$.

If $x \sim y$ and $y \sim z$, then for any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$. We have

$$\{n \in \mathbb{N} : (x(n), y(n)) \in V \cap V^{-1}\} \cap \{n \in \mathbb{N} : (y(n), z(n)) \in V \cap V^{-1}\}$$
$$\subseteq \{n \in \mathbb{N} : (x(n), z(n)) \in (V \cap V^{-1}) \circ (V \cap V^{-1})\}.$$

Moreover, since $(V \cap V^{-1}) \circ (V \cap V^{-1}) \subseteq V^2 \cap V^{-2}$. It follows that

$$\{n \in \mathbb{N} : (x(n), y(n)) \in V \cap V^{-1}\} \cap \{n \in \mathbb{N} : (y(n), z(n)) \in V \cap V^{-1}\}$$
$$\subseteq \{n \in \mathbb{N} : x(n), z(n)) \in V^2 \cap V^{-2}\} \subseteq \{n \in \mathbb{N} : x(n), z(n)) \in U \cap U^{-1}\}.$$

By taking the complement, we have

$$\{n \in \mathbb{N} : (x(n), z(n)) \notin U \cap U^{-1}\}$$
$$\subseteq \{n \in \mathbb{N} : (x(n), y(n)) \notin V \cap V^{-1}\} \cup \{n \in \mathbb{N} : (y(n), z(n)) \notin V \cap V^{-1}\} \text{ is finite.}$$

Thus $x \sim z$. Therefore, $\sim$ is an equivalence relation. □

Remark 3.5. Observe that if $U = U^{-1}$ whenever $U \in \mathcal{U}$, then $\mathcal{U}$ is a uniformity on X. Therefore, the equivalence relation $\sim$ in Definition 3.3 coincides with the uniform equivalence relation in [2, Definition 5.3].

Lemma 3.6. *Suppose $(X,\mathcal{U})$ is a quasi-uniform space and $x, y \in \prod_{n\in\mathbb{N}}X$. If x is mod-finite equivalent to y, then x is equivalent to y with respect to $\sim$.*

Proof. Suppose that x is mod-finite equivalent to y i.e. $x \approx y$. If $U \in \mathcal{U}$, then the set

$$\{n \in \mathbb{N} : (x(n), y(n)) \notin U \cap U^{-1}\} \subseteq \{n \in \mathbb{N} : x(n) \neq y(n)\} \text{ is finite.}$$

It follows that x is quasi-uniform equivalent to y. □

Definition 3.7. Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $G\subseteq\prod_{n\in\mathbb{N}}X$ and G is $\tau(\overline{\mathcal{U}})$-open, then we define:

$$[G]_{\approx}:=\left\{y\in\prod\nolimits_{n\in\mathbb{N}}X:\text{ there exists}\,x\in G\,\text{such that}\,x\approx y\right\}$$

and

$$[G]_{\sim}:=\left\{y\in\prod\nolimits_{n\in\mathbb{N}}X:\text{ there exists}\,x\in G\,\text{such that}\,x\sim y\right\}.$$

Proposition 3.8. *(compare [2, Proposition 5.7]) Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If G is $\tau(\overline{\mathcal{U}})$-open, then $[G]_{\approx}=[G]_{\sim}$. Moreover, $[G]_{\sim}$ is $\tau(\overline{\mathcal{U}})$-open.*

Proof. Suppose G is $\tau(\overline{\mathcal{U}})$-open. Then we need to prove that $[G]_{\approx}=[G]_{\sim}$.

Suppose $y\in[G]_{\sim}$. Then there exists $x\in G$ such that $x\sim y$. It follows that there exists $U\in\mathcal{U}$ such that $\overline{U}(x)\subseteq G$ since G is $\tau(\overline{\mathcal{U}})$-open. Consider a point $z\in\prod_{n\in\mathbb{N}}X$ such that

$$z(n)=\begin{cases} x(n) & \text{if } (x(n),y(n))\notin U\cap U^{-1}\\ y(n) & \text{elsewhere.}\end{cases}$$

It follows that $(x(n),z(n))\in U\cap U^{-1}$ whenever $n\in\mathbb{N}$. Therefore, $z(n)\in U(x(n))$ whenever $n\in\mathbb{N}$ and so $z\in\overline{U}(x)\subseteq G$. For every $V\in\mathcal{U}$, we have that $\{n\in\mathbb{N}:(x(n),y(n))\notin V\cap V^{-1}\}$ is finite since $x\sim y$. This implies that the set $\{n\in\mathbb{N}:z(n)=x(n)\neq y(n)\}$ is finite. Hence $z\approx y$. Thus $y\in[G]_{\approx}$ since $z\in G$ and $z\approx y$. Therefore, $[G]_{\sim}\subseteq[G]_{\approx}$. Furthermore, we have that $[G]_{\approx}\subseteq[G]_{\sim}$ from Lemma 3.6 and so $[G]_{\approx}=[G]_{\sim}$.

Secondly, we prove that $[G]_{\approx}$ is $\tau(\overline{\mathcal{U}})$-open whenever G is $\tau(\overline{\mathcal{U}})$-open. Let $y\in[G]_{\approx}$. We need to prove that there exists $U\in\mathcal{U}$ such that $\overline{U}(y)\subseteq[G]_{\approx}$.

Since $y\in[G]_{\approx}$, there exists $x\in G$ such the set $\{n\in\mathbb{N}:x(n)\neq y(n)\}$ is finite. Hence there exists $m\in\mathbb{N}$ such that whenever $n>m$, we have $x(n)=y(n)$. As $x\in G$, a $\tau(\overline{\mathcal{U}})$-open set, there exists $U\in\mathcal{U}$ such that $\overline{U}(x)\subseteq G$. Let $u\in\overline{U}(y)$. This implies that $(y,u)\in\overline{U}$ and so $(y(n),u(n))\in U$ for all $n\in\mathbb{N}$. Consider a point $v\in\prod_{n\in\mathbb{N}}X$ such that

$$v(n)=\begin{cases} u(n) & \text{if } n>m;\\ x(n) & \text{if } n\leq m.\end{cases}$$

If $n\leq m$, then $v(n)=x(n)$ and $(x(n),v(n))\in U$. If $n>m$, then $v(n)=u(n)$ and $x(n)=y(n)$. Thus $(x(n),v(n))=(y(n),u(n))\in U$. This implies that the set $\{n\in\mathbb{N}:v(n)\neq u(n)\}$ is finite. Hence $v\approx u$ and $(x(n),v(n))\in U$ whenever $n\in\mathbb{N}$. Furthermore, $v\in\overline{U}(x)\subseteq G$. Hence $u\in[G]_{\approx}$ since $v\in G$ and $v\approx u$. Therefore, $\overline{U}(y)\subseteq[G]_{\approx}$. $\square$

Proposition 3.9. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then for all $x \in \prod_{n\in\mathbb{N}} X$, we have that $[x]_\sim$ is $\tau(\overline{\mathcal{U}})$-closed and $\tau(\overline{\mathcal{U}}^{-1})$-closed. Furthermore, $cl_{\tau(\overline{\mathcal{U}}^s)}[x]_\approx = [x]_\sim$.*

Proof. We are going to prove that $[x]_\sim$ is $\tau(\overline{\mathcal{U}})$-closed and the proof that $[x]_\sim$ is $\tau(\overline{\mathcal{U}}^{-1})$-closed will follow by similar arguments.

Suppose $y \notin [x]_\sim$. Then y is not quasi-uniformly equivalent to x. This implies that there exists $U \in \mathcal{U}$ such that the set

$$\{n \in \mathbb{N} : (x(n), y(n)) \notin U \cap U^{-1}\}$$

is infinite. It follows that there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$ since $U \in \mathcal{U}$. We are going to show that $(\overline{V} \cap \overline{V^{-1}})(y) \cap [x]_\sim = \emptyset$. Let $z \in (\overline{V} \cap \overline{V^{-1}})(y)$. Then $(y, z) \in \overline{V} \cap \overline{V^{-1}}$ and so $(z, y) \in \overline{V} \cap \overline{V^{-1}}$. This implies that $(z(n), y(n)) \in V \cap V^{-1}$ whenever $n \in \mathbb{N}$. If $m \in \mathbb{N}$ is such that $(x(m), z(m)) \in V \cap V^{-1}$, then $(x(m), y(m)) \in V^2 \subseteq U$. Thus

$$\{m \in \mathbb{N} : (x(m), z(m)) \in V \cap V^{-1}\} \subseteq \{m \in \mathbb{N} : (x(m), y(m)) \in U \cap U^{-1}\}.$$

Hence

$$\{m \in \mathbb{N} : (x(m), y(m)) \notin U \cap U^{-1}\} \subseteq \{m \in \mathbb{N} : (x(m), z(m)) \notin V \cap V^{-1}\}.$$

Therefore, the set $\{n \in \mathbb{N} : (x(n), z(n)) \notin V \cap V^{-1}\}$ is infinite since the set $\{n \in \mathbb{N} : (x(n), y(n)) \notin U \cap U^{-1}\}$ is infinite. Hence $z \notin [x]_\sim$.

Since $x \approx y$ implies $x \sim y$, we have $[x]_\approx \subset [x]_\sim$ and so $cl_{\tau(\overline{\mathcal{U}}^s)}[x]_\approx \subset [x]_\sim$ since $[x]_\sim$ is $\tau(\overline{\mathcal{U}})$-closed and $\tau(\overline{\mathcal{U}}^{-1})$-closed. Suppose $y \in [x]_\sim$ and $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Define $z \in \prod_{n\in\mathbb{N}} X$ by

$$z(n) = \begin{cases} x(n) & \text{if } (x(n), y(n)) \in V \cap V^{-1} \\ y(n) & \text{else.} \end{cases}$$

Since $y \sim x$, $\{n : (x(n), y(n)) \notin V \cap V^{-1}\}$ is finite. Therefore, $\{n : (x(n) \neq z(n)\}$ is finite and so $z \approx x$. But $z \in (\overline{V} \cap \overline{V^{-1}})(y) \subseteq (\overline{U} \cap \overline{U^{-1}})(y)$ so that $(\overline{U} \cap \overline{U^{-1}})(y) \cap [x]_\approx \neq \emptyset$. Therefore, $y \in cl_{\tau(\overline{\mathcal{U}}^s)}[x]_\approx$. $\square$

Proposition 3.10. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\sim$ be the quasi-uniform equivalence relation on its quasi-uniform box product $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$. Then $\widetilde{\mathcal{U}}$ is a quasi-uniformity on the quotient space $\prod_{n\in\mathbb{N}} X/\sim =: \widetilde{\prod_{n\in\mathbb{N}}} X$, where*

$$\widetilde{\mathcal{U}} = \{\widetilde{U} : U \in \mathcal{U}\} \text{ and } \widetilde{U} = \{([x]_\sim, [y]_\sim) : (x, y) \in \overline{U}\}$$

whenever $U \in \mathcal{U}$.

4. Completeness in quasi-uniform box products

In this section, we discuss completeness in quasi-uniform box products. These results generalise the result on completeness in uniform box products in [16] to the quasi-uniform setting. In [13], we considered C-completeness and D-Completeness in quasi-uniform box products. In this section, we present other forms of completeness in quasi-uniform box products. The results presented in this section were initially observed by [11] and [12] for general function spaces and the quasi-uniformity of uniform convergence. We adapt these results to quasi-uniform box products.

We begin by recalling the following definitions:

Definition 4.1. [12] A net $(x_\lambda)_{\lambda\in\Lambda}$ in a quasi-uniform space $(X,\mathcal{U})$ is said to converge to a point $x \in X$ iff for each $U \in \mathcal{U}$, x_λ is eventually in $U(x)$.

Definition 4.2. [12] A net $(x_\lambda)_{\lambda\in\Lambda}$ in a quasi-uniform space $(X,\mathcal{U})$ is said to be Cauchy iff for each $U \in \mathcal{U}$ there exists an $x \in X$ such that x_λ is eventually in $U(x)$.

Definition 4.3. A quasi-uniform space $(X,\mathcal{U})$ is said to be complete if every Cauchy net in X converges to a point in X.

Theorem 4.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X,\mathcal{U})$ is complete and $(x_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $\prod_{n\in\mathbb{N}} X$, then there is $(y(n))_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X$ such that $(x_\lambda(n))_{\lambda\in\Lambda}$ converges to $y(n)$ for all $n \in \mathbb{N}$.*

Proof. Since $(x_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $\prod_{n\in\mathbb{N}} X$, then for each $U \in \mathcal{U}$, there exists $(x(n))_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X$ such that for each $n \in \mathbb{N}$, $x_\lambda(n)$ is eventually in $U(x(n))$. This shows that for each $n \in \mathbb{N}$, $(x_\lambda(n))_{\lambda\in\Lambda}$ is a Cauchy net in $(X,\mathcal{U})$ and since $(X,\mathcal{U})$ is complete, $(x_\lambda(n))_{\lambda\in\Lambda}$ converges to a point in $(X,\mathcal{U})$ which we call $y(n)$. □

Theorem 4.5. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X,\mathcal{U})$ is complete and Hausdorff, then $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is complete.*

Proof. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a Cauchy net in $\prod_{n\in\mathbb{N}} X$. Then we need to show that $(x_\lambda)_{\lambda\in\Lambda}$ converges quasi-uniformly to a point in $\prod_{n\in\mathbb{N}} X$. Let $(y(n))_{n\in\mathbb{N}}$ be defined as in Theorem 4.4 above. Since $(x_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $\prod_{n\in\mathbb{N}} X$, then for each $U \in \mathcal{U}$, there exists $(x(n))_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X$ such that for all $n \in \mathbb{N}$, $x_\lambda(n)$ is eventually in $U(x(n))$. We claim that $(x(n))_{n\in\mathbb{N}} = (y(n))_{n\in\mathbb{N}}$. Suppose to the contrary that $(x(n))_{n\in\mathbb{N}} \neq (y(n))_{n\in\mathbb{N}}$. Then there exists $n_0 \in \mathbb{N}$ such that $x(n_0) \neq y(n_0)$. Since $(X,\mathcal{U})$ is Hausdorff, there exists $V \in \mathcal{U}$ such that $U(x(n_0)) \cap V(y(n_0)) = \emptyset$. By Theorem 4.4, $x_\lambda(n_0)$ is eventually in $V(y(n_0))$ which contradicts the fact that $x_\lambda(n_0)$ is eventually in $U(x(n_0))$. Therefore,

$(x_\lambda)_{\lambda\in\Lambda}$ converges to $(y(n))_{n\in\mathbb{N}}$ quasi-uniformly and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is complete. $\square$

Remark 4.6. Notice that the proof of Theorem 4.5 relies on the Hausdorff property of the factor space. Therefore, the result cannot hold if the factor space is not Hausdorff.

In [11], Künzi and Romaguera studied the completeness of the quasi-uniformity of uniform convergence with the aim of obtaining an appropriate quasi-uniform generalisation of the classical result that if X is a topological space and $(Y,\mathcal{U})$ is a complete uniform space, then the uniformity of uniform convergence is complete. Künzi and Romaguera [11] observed that the notions of quasi-uniform completeness based on the convergence of several types of stable filters provide satisfactory results and that the other known quasi-uniform completeness appear more intractable in this setting. Since the constant quasi-uniformity is a particular case of the quasi-uniformity of uniform convergence, we adapt the results of Künzi and Romaguera [11] to the framework of quasi-uniform box products. We first recall the following definitions:

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is called:

(i) a *$\mathcal{U}$-stable filter* if for each $U\in\mathcal{U}$, $\bigcap\{U(F) : F\in\mathcal{F}\}\in\mathcal{F}$ [3].

(ii) a *Cauchy filter* if for each $U\in\mathcal{U}$, there is $x\in X$ such that $U(x)\in\mathcal{F}$ [6].

(iii) a left *K-Cauchy filter* if for each $U\in\mathcal{U}$ there is $F\in\mathcal{F}$ such that $U(x)\in\mathcal{F}$ for all $x\in F$ [14].

(iv) a right *K-Cauchy filter* if it is left K-Cauchy on $(X,\mathcal{U}^{-1})$[14].

(v) a *$\mathcal{U}^s$-Cauchy filter* if it is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$ [3].

Let $(\mathcal{F},\mathcal{G})$ be an ordered pair of filters on X. Then the pair $(\mathcal{F},\mathcal{G})$ is called:

(vi) a *Cauchy filter pair* if $(\mathcal{F},\mathcal{G})\to 0$, where $(\mathcal{F},\mathcal{G})\to 0$ provided that for each $U\in\mathcal{U}$ there exists $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that $F\times G\subseteq U$ [4].

(vii) a *stable pair of filters* if $\mathcal{G}$ is $\mathcal{U}$-stable and $\mathcal{F}$ is $\mathcal{U}^{-1}$-stable [4].

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then

(i) $(X,\mathcal{U})$ is *convergent complete* provided that each Cauchy filter is $\tau(\mathcal{U})$-convergent [3].

(ii) $(X,\mathcal{U})$ is *left (right) K-complete* provided that each left (right) K-Cauchy filter is $\tau(\mathcal{U})$ convergent [14].

(iii) $(X,\mathcal{U})$ is *half complete* provided that each $\mathcal{U}^s$-Cauchy filter is $\tau(\mathcal{U})$-convergent [4].

(iv) $(X, \mathcal{U})$ is *bicomplete* provided that the uniform space $(X, \mathcal{U}^s)$ is complete [3].

(v) $(X, \mathcal{U})$ is called *C-complete* provided that each Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ converges, that is, $\mathcal{G}$ is $\tau(\mathcal{U})$-convergent to x and $\mathcal{F}$ is $\tau(\mathcal{U}^{-1})$-convergent to x [5].

(vi) $(X, \mathcal{U})$ is *D-complete* provided that if $(\mathcal{F}, \mathcal{G}) \to 0$, then the filter $\mathcal{G}$ is $\tau(\mathcal{U})$-convergent [5].

(vii) $(X, \mathcal{U})$ is strongly *D-complete* provided that if $(\mathcal{F}, \mathcal{G}) \to 0$, then the filter $\mathcal{F}$ has a $\tau(\mathcal{U})$-cluster point [8].

(viii) $(X, \mathcal{U})$ is *S-complete* provided that each stable Cauchy pair of filters $(\mathcal{F}, \mathcal{G})$ converges to a point $x \in X$, that is, $\mathcal{G}$ is $\tau(\mathcal{U})$-convergent to x and $\mathcal{F}$ is $\tau(\mathcal{U}^{-1})$-convergent to x [4].

(ix) $(X, \mathcal{U})$ is *U-complete* provided that each stable Cauchy pair of ultrafilters is convergent to a point $x \in X$ [4].

Suppose $(X, \mathcal{U})$ is a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. Then following [11], we have that:

(i) if $\mathcal{F}$ is a filter on $\prod_{n\in\mathbb{N}} X$, $F \in \mathcal{F}$, $n \in \mathbb{N}$ and $F_n = \{x(n) : (x(n))_{n\in\mathbb{N}} \in F\}$, then $\mathcal{F}_n = \{F_n : F \in \mathcal{F}\}$ is a filter on X.

(ii) If $\mathcal{F}$ is a filter on X, $F \in \mathcal{F}$ and $j \in F$, we denote by $(x_j(n))_{n\in\mathbb{N}}$ a sequence such that $x_j(n) = j$ for all $n \in \mathbb{N}$. Suppose $B(F) = \{(x_j(n))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X : j \in F\}$. Then $B(\mathcal{F}) = \{B(F) : F \in \mathcal{F}\}$ is a filterbase on $\prod_{n\in\mathbb{N}} X$.

Theorem 4.7. *Suppose $(X, \mathcal{U})$ is a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. Then*

(i) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is half complete if and only if $(X, \mathcal{U})$ is half complete.

(ii) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is bicomplete if and only if $(X, \mathcal{U})$ is bicomplete.

(iii) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is right K-complete if and only if $(X, \mathcal{U})$ is right K-complete.

(iv) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is S-complete if and only if $(X, \mathcal{U})$ is S-complete.

(v) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is U-complete if and only if $(X, \mathcal{U})$ is U-complete.

Proof.

(i) Suppose $(X,\mathcal{U})$ is half complete. Let $\mathcal{F}$ be a $\overline{\mathcal{U}}^s$-Cauchy filter on $\prod_{n\in\mathbb{N}} X$. Then for each $\overline{U} \in \overline{\mathcal{U}}$, there is $F \in \mathcal{F}$ such that $F \times F \subseteq \overline{U}$. Fix $n \in \mathbb{N}$. Then $\mathcal{F}_n$ is a $\mathcal{U}^s$-Cauchy filter on X. Since $(X,\mathcal{U})$ is half complete, $\mathcal{F}_n$ is $\tau(\mathcal{U})$-convergent to a point $x_0(n) \in X$. Thus, we have defined $x_0 \in \prod_{n\in\mathbb{N}} X$ and we need to show that $\mathcal{F}$ converges to x_0 with respect to $\tau(\overline{\mathcal{U}})$. Given $U \in \mathcal{U}$, choose $V \in \mathcal{U}$ with $V^2 \subseteq U$. Then there is $F \in \mathcal{F}$ such that $F \times F \subseteq \overline{V}$. Now let $n \in \mathbb{N}$. Then there is some $G \in \mathcal{F}$ with $G_n \subseteq V(x_0(n))$. Thus for each $x \in F$ and $y \in F \cap G$, we get $(y(n), x(n)) \in V$. Since $y(n) \in V(x_0(n))$ implies that $(x_0(n), y(n)) \in V$, we have $(x_0(n), x(n)) \in V^2 \subseteq U$. Therefore, $x \in \overline{U}(x_0)$ and so $\mathcal{F}$ is $\tau(\overline{\mathcal{U}})$-convergent to x_0. Conversely, suppose that $\mathcal{F}$ is a $\mathcal{U}^s$-Cauchy filter on X. Then for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $F \times F \subseteq U$. Thus, $B(F) \times B(F) \subseteq \overline{U}$. Therefore, $B(\mathcal{F})$ is a $\overline{\mathcal{U}}^s$ Cauchy filter base on $\prod_{n\in\mathbb{N}} X$ so that it is convergent to $x_0 \in \prod_{n\in\mathbb{N}} X$ with respect to $\tau(\overline{\mathcal{U}})$. Fix $n_0 \in \mathbb{N}$ and let $x_0(n_0) = j_0$. It follows that $\mathcal{F}$ is $\tau(\mathcal{U})$-convergent to j_0.

(ii) The necessity for (ii) follows similarly to (i). We now prove the sufficiency. Since $(X,\mathcal{U}^s)$ is a complete uniform space, it follows that the uniform box product $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}^s\right)$ is complete. Therefore $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is bicomplete.

(iii) The necessity for (iii) follows similarly to (i). We now prove the sufficiency of (iii). Suppose $(x_\alpha)_{\alpha\in\Lambda}$ is a right K-Cauchy net in $\prod_{n\in\mathbb{N}} X$. Then for each $\overline{U} \in \overline{\mathcal{U}}$, there is $\lambda \in \Lambda$ such that $(x_\alpha, x_\beta) \in \overline{U}$ whenever $\lambda \le \beta \le \alpha$. Fix $n \in \mathbb{N}$. Then it follows that $(x_\alpha(n))_{\alpha\in\Lambda}$ is a right K-Cauchy net in $(X,\mathcal{U})$ and so it is $\tau(\mathcal{U})$-convergent to a point $x(n) \in X$. Thus we have defined $x \in \prod_{n\in\mathbb{N}} X$ and we need to show that $(x_\alpha)_{\alpha\in\Lambda}$ converges to x with respect to $\tau(\overline{\mathcal{U}})$. Given $U \in \mathcal{U}$, choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then there is $\alpha_0 \in \Lambda$ such that $(x_\alpha, x_\beta) \in \overline{V}$ whenever $\alpha_0 \le \beta \le \alpha$. We want to show that $(x, x_\alpha) \in \overline{U}$ for all $\alpha \in \Lambda$ with $\alpha_0 \le \alpha$. In fact, for such an α and any $n \in \mathbb{N}$, there is $\alpha(n) \in \Lambda$ with $\alpha \le \alpha(n)$ and $(x(n), x_{\alpha(n))}(n)) \in V$. Since $(x_{\alpha(n))}(n), x_\alpha(n)) \in V$, we conclude that $(x(n), x_\alpha(n)) \in V^2 \subseteq U$ and so $(x, x_\alpha) \in \overline{U}$. Therefore, $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is right K-complete.

(iv) We prove (iv) and (v) in parallel. Suppose $(X,\mathcal{U})$ is (U-complete) S-complete. Let $(\mathcal{F}, \mathcal{G})$ be a stable Cauchy pair of (ultra)filters on $\prod_{n\in\mathbb{N}} X$. Fix $n \in \mathbb{N}$. Then $(\mathcal{F}_n, \mathcal{G}_n)$ is a stable Cauchy pair of (ultra)filters on $(X,\mathcal{U})$. Hence $(\mathcal{F}_n, \mathcal{G}_n)$ converges to some point $x_0(n) \in X$. Thus we have defined $x_0 \in \prod_{n\in\mathbb{N}} X$ and we must show that the (ultra)filter $\mathcal{G}$ converges to x_0 with respect to $\tau(\overline{\mathcal{U}})$. Suppose $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Since $\mathcal{G}$ is $\overline{\mathcal{U}}$-stable, there is $H \in \mathcal{G}$ such that

$H \subseteq \overline{V}(G)$ for all $G \in \mathcal{G}$. Now let $n \in \mathbb{N}$. Then there is $G' \in \mathcal{G}$ such that $G'_n \subseteq V(x_0(n))$. Since for all $y \in H$ there is $x \in G'$ such that $(x, y) \in \overline{V}$ and $(x_0(n), x(n)) \in V$, it follows that $(x_0(n), y(n)) \in V^2 \subseteq U$. We conclude that $H \subseteq \overline{U}(x_0)$, so that $\mathcal{G}$ converges to x_0 with respect to $\tau(\overline{\mathcal{U}})$. Similarly, we can show that the (ultra)filter $\mathcal{F}$ converges to x_0 with respect to $\tau(\overline{\mathcal{U}}^{-1})$. Conversely, suppose that $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is (U-complete) S-complete and let $(\mathcal{F}, \mathcal{G})$ be a stable Cauchy pair of (ultra)filters on $(X, \mathcal{U})$. Then for each $U \in \mathcal{U}$, there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. Thus $B(F) \times B(G) \subseteq \overline{U}$. Denote by $\mathcal{F}_1$ and $\mathcal{G}_1$ the two (ultra)filters generated on $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ by $B(\mathcal{F})$ and $B(\mathcal{G})$ respectively. Then $(\mathcal{F}_1, \mathcal{G}_1)$ is a Cauchy pair of (ultra)filters on $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$. Furthermore, $\mathcal{G}_1$ is $\overline{U}$-stable and $\mathcal{F}_1$ is $\overline{U}^{-1}$-stable. Hence $(\mathcal{F}_1, \mathcal{G}_1)$ converges to x_0. Fix $n_0 \in \mathbb{N}$. Then it follows that the pair of (ultra)filters $(\mathcal{F}, \mathcal{G})$ converges to $x_0(n_0)$.

□

The following example shows that quietness cannot be omitted from Theorem 4.7.

Example 4.8. Suppose (X, d) is a quasi-metric space, where X is a set of positive integers and $d(n, m) = 1$ if $n < m$, $d(n, m) = (1/2)((1/n) + (1/m))$ if $n > m$ and $m \neq 1$, $d(n, 1) = 1/n$ if $n \neq 1$, and $d(n, n) = 0$ for all $n \in X$. Then $\tau(d)$ is the discrete topology on X. We now show that the quasi-uniformity generated by d is D-complete. Suppose $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair on X. For the case when $\mathcal{F}$ is generated by a singleton, $\mathcal{G}$ is $\tau(d)$-convergent. Therefore, we consider the other case. Since $F_1 \times G_1 \subseteq \{(n, m) \in X \times X : d(n, m) < 1\}$ for some $F_1 \in \mathcal{F}$ and $G_1 \in \mathcal{G}$, the filter $\mathcal{G}$ contains necessarily a finite set. Let E be such a set of minimal cardinality. If its cardinality is greater than 1, then it contains a point n different from 1. This contradicts that $(\mathcal{F}, \mathcal{G}) \to 0$ because the filter $\mathcal{F}$ does not contain a singleton $\{n\}$. So E is a singleton and the filter $\mathcal{G}$ is $\tau(d)$-convergent.

Now consider the quasi-uniform box product $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$, where $\mathcal{U}$ is the natural quasi-uniformity inherited from d. Consider the sequences $\{x_m\}_{m\in\mathbb{N}}$ and $\{y_m\}_{m\in\mathbb{N}}$ on $\prod_{n\in\mathbb{N}} X$ defined as follows: $x_0(n) = x_1(n) = y_0(n) = y_1(n) = 1$ for all $n \in \mathbb{N}$, $x_m(n) = m$ if $n < m$, $x_m(n) = m + 1$ if $n = m$, $x_m(n) = n + 1$ if $n > m$, $y_m(n) = 1$ if $n < m - 1$ and $y_m(n) = m$ if $n \geq m - 1$.

A computation of different cases shows that $d(x_m(n), y_i(n)) \leq 1/k$ for all $i, m > k$ and for all $n \in \mathbb{N}$. Thus the filter $\mathcal{G}$ generated on $\prod_{n\in\mathbb{N}} X$ by the sequence $\{y_n\}_{n\in\mathbb{N}}$ is D-Cauchy. Suppose $\mathcal{G}$ converges to $y_0 \in \prod_{n\in\mathbb{N}} X$. Then $y_0(n) = 1$ for all $n \in \mathbb{N}$. Now for any $m \in \mathbb{N}$, choose $\epsilon > 0$ such that $\epsilon < \frac{1}{m}$. Thus for all $n \in \mathbb{N}$, $d(y_m(n), y_0(n)) = \frac{1}{y_m(n)} = \frac{1}{m} > \epsilon$ and this implies that $\mathcal{G}$

is not convergent to y_0 with respect to $\tau(\overline{\mathcal{U}})$. Therefore, we conclude that the quasi-uniform box product $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is not D-complete.

If the quasi-uniformity $(X,\mathcal{U})$ is strongly D-complete, then quietness can be omitted as the next result shows.

Proposition 4.9. *Suppose $(X,\mathcal{U})$ is a strongly D-complete quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. Then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is D-complete.*

Proof. Let $(\mathcal{F},\mathcal{G})$ be a Cauchy pair of filters on $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$. Fix $n \in \mathbb{N}$. Then $(\mathcal{F}_n,\mathcal{G}_n)$ is a Cauchy pair of filters on $(X,\mathcal{U})$ so that $\mathcal{F}_n$ has a $\tau(\mathcal{U})$-cluster point $x_0(n) \in X$. Thus we have defined $x_0 \in \prod_{n\in\mathbb{N}} X$ and we need to show that the filter $\mathcal{G}$ is $\tau(\overline{\mathcal{U}})$-convergent to x_0. Suppose $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then there are $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$ such that $F' \times G' \subseteq \overline{V}$. We shall show that $G' \subseteq \overline{U}(x_0)$. Let $y \in G'$ and $n \in \mathbb{N}$. Since $x_0(n)$ is a $\tau(\mathcal{U})$-cluster point of $\mathcal{F}_n$, there exists $x \in F'$ such that $x(n) \in V(x_0(n))$. Since $(x(n), y(n)) \in V$, we obtain that $(x_0(n), y(n)) \in V^2 \subseteq U$. We conclude that $y \in \overline{U}(x_0)$. Hence $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is D-complete. $\square$

5. Topological properties of quasi-uniform box products

In this section, we present some topological properties of quasi-uniform box products. We begin the section by presenting some separation properties of a quasi-uniform space that are preserved by its quasi-uniform box product. Thereafter, we present connectedness in quasi-uniform box products.

The following result was observed by [12] for the quasi-uniformity of uniform convergence. We adapt this result to quasi-uniform box products.

Lemma 5.1. *Suppose $(X,\mathcal{U})$ is a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product. If $(X,\mathcal{U})$ is a T_i space (for $i = 1, 2$), then $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is a T_i space (for $i = 1, 2$).*

We now extend this result to quasi-uniform box products in the framework of bitopological spaces.

Definition 5.2. A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is pairwise T_1 iff for each pair x, y of distinct points of X, there is a $\mathscr{P}$-open set U and a $\mathscr{Q}$-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Lemma 5.3. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ is pairwise T_1, then $\left(\prod_{n\in\mathbb{N}} X, \tau(\overline{\mathcal{U}}), \tau(\overline{\mathcal{U}}^{-1})\right)$ is pairwise T_1 as well.*

Proof. Suppose $(x(n))_{n\in\mathbb{N}}$ and $(y(n))_{n\in\mathbb{N}}$ are distinct elements in $\prod_{n\in\mathbb{N}} X$. Then for a fixed $n_0 \in \mathbb{N}$, $x_{n_0} \neq y_{n_0}$. Since $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ is pairwise T_1, there exist $G \in \tau(\mathcal{U})$ and $M \in \tau(\mathcal{U}^{-1})$ such that $x_{n_0} \in G$, $y_{n_0} \notin G$ and $y_{n_0} \in M$, $x_{n_0} \notin M$. Suppose $x_{n_0} \in G$, $y_{n_0} \notin G$. Then there exists $U \in \mathcal{U}$ such that $U(x_{n_0}) \subset G$, $U(y_{n_0})$ is not a subset of G since G is $\tau(\mathcal{U})$-open. It follows that $\overline{U}((x(n))_{n\in\mathbb{N}})$ contains a $\tau(\overline{\mathcal{U}})$-open set say $\overline{G}$ such that $(x(n))_{n\in\mathbb{N}} \in \overline{G}$, $(y(n))_{n\in\mathbb{N}} \in \overline{G}$. One can use the same argument to show the existence of a $\tau(\overline{\mathcal{U}}^{-1})$-open set $\overline{M}$ such that $(y(n))_{n\in\mathbb{N}} \in \overline{M}$, $(x(n))_{n\in\mathbb{N}} \in \overline{M}$. □

Definition 5.4. A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is pairwise T_2 (or pairwise Hausdorff) iff for each pair of distinct points x and y in X there is a $\mathscr{P}$-open set U and a $\mathscr{Q}$-open set V disjoint from U such that $x \in U$, $y \in V$.

Proposition 5.5. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ is pairwise T_2, then $\left(\prod_{n\in\mathbb{N}} X, \tau(\overline{\mathcal{U}}), \tau(\overline{\mathcal{U}}^{-1})\right)$ is pairwise T_2 as well.*

Proof. Suppose $(x(n))_{n\in\mathbb{N}}$ and $(y(n))_{n\in\mathbb{N}}$ are two distinct points in $\prod_{n\in\mathbb{N}} X$ then there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \neq y_{n_0}$. Since $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ is pairwise T_2, there exist $G \in \tau(\mathcal{U})$ and $M \in \tau(\mathcal{U}^{-1})$ such that $x_{n_0} \in G$ and $y_{n_0} \in M$ and $G \cap M = \emptyset$. Therefore, there exist $U \in \mathcal{U}$ and $V \in \mathcal{U}^{-1}$ such that $U(x_0) \subset G$ and $V(y_{n_0}) \subset M$. It follows that

$$U(x_{n_0}) \cap V(y_{n_0}) = \emptyset.$$

This implies that

$$\prod_{n\in\mathbb{N}} U(x(n)) \cap \prod_{n\in\mathbb{N}} V(y(n)) = \emptyset,$$

and so

$$\overline{U}((x(n))_{n\in\mathbb{N}}) \cap \overline{V}((y(n))_{n\in\mathbb{N}}) = \emptyset.$$

Thus $\overline{U}((x(n))_{n\in\mathbb{N}})$ contains a $\tau(\overline{\mathcal{U}})$ open set say $\overline{G}$ such that $(x(n))_{n\in\mathbb{N}} \in \overline{G}$ and $\overline{V}((y(n))_{n\in\mathbb{N}})$ contains a $\tau(\overline{\mathcal{U}}^{-1})$ open set say $\overline{M}$ such that $(y(n))_{n\in\mathbb{N}} \in \overline{M}$. Clearly $\overline{G} \cap \overline{M} = \emptyset$ □

We now present connectedness in quasi-uniform box products. We present two types of connectedness; uniform connectedness and topological connectedness. Let us start with uniform connectedness.

Definition 5.6. ([10, p.243]) Let $(X,\mathcal{U})$ be a quasi-uniform space. Then

(i) $(X,\mathcal{U})$ is called *bounded* provided that for each $U \in \mathcal{U}$, there exists a positive integer m such that $U^m = X \times X$.

(ii) $(X,\mathcal{U})$ is called *uniformly strongly connected* if for all $U \in \mathcal{U}$ and $x, y \in X$, there exists $m \in \mathbb{N}$ such that $(x, y) \in U^m$.

Remark 5.7. Note that Definition 5.6 (i) was introduced by Bushaw in the context of uniform spaces while Definition 5.6 (ii) is motivated by a similar terminology in graph theory and in the setting of uniform spaces it is called uniform connectedness.

Proposition 5.8. *Suppose* $(X,\mathcal{U})$ *is a quasi-uniform space and* $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ *is its quasi-uniform box product. Then the following are equivalent:*

(i) $(X,\mathcal{U})$ *is bounded*

(ii) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ *is bounded*

(iii) $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ *is uniformly strongly connected.*

Proof. $(i) \Rightarrow (ii)$ Suppose $U \in \mathcal{U}$. Then there exists $m \in \mathbb{N}$ such that $U^m = X \times X$. Therefore,

$$\overline{U}^m = \overline{U^m} = \overline{X \times X} = \prod_{n\in\mathbb{N}} X \times \prod_{n\in\mathbb{N}} X.$$

$(ii) \Rightarrow (iii)$ Follows from the fact that any bounded quasi-uniform space is uniformly strongly connected.

$(iii) \Rightarrow (i)$ Suppose $U \in \mathcal{U}$ and $x, y \in \prod_{n\in\mathbb{N}} X$. Suppose to the contrary that $U^m \neq X \times X$ for all $m \in \mathbb{N}$. Then one can choose $x, y \in \prod_{n\in\mathbb{N}} X$ such that $(x(m), y(m)) \notin U^m$ for all $m \in \mathbb{N}$. This implies that $(x, y) \notin \overline{U}^m$ for all $m \in \mathbb{N}$ contradicting that $\left(\prod_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is uniformly strongly connected. Therefore, we must have $U^m = X \times X$. □

We now show that topological connectedness is preserved by quasi-uniform box products. Let us first recall the definition of a totally bounded quasi-uniform space.

Definition 5.9. [6] A quasi -uniformity $\mathcal{U}$ on a set X is called totally bounded provided that for each $U \in \mathcal{U}$, there exists a cover $\mathcal{A}$ of X such that $A \times A \subset U$ whenever $A \in \mathcal{A}$.

It has been observed by [6] that total boundedness is preserved by arbitrary products of quasi-uniform spaces. Therefore, we have the following result.

Corollary 5.10. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space and* $\left(\prod\limits_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ *be its quasi-uniform box product. If* $(X,\mathcal{U})$ *is totally bounded, then* $\left(\prod\limits_{n\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ *is totally bounded.*

The following was observed by Bell in uniform box products (see [2, Theorem 4.1]). Since there are variations in the proof, we prove the result in the quasi-uniform setting.

Theorem 5.11. *Let* $(X,\mathcal{U})$ *be a totally bounded quasi-uniform space and* $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ *be its quasi-uniform box product. If* $(X,\mathcal{U})$ *is connected, then* $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ *is connected.*

Proof. Let $P = \{A_1, A_2, \cdots, A_n\}$ be a finite partition of natural numbers and $\prod_P = \left\{x \in \prod_{n\in\mathbb{N}} X : \forall i \leq n \ \forall j_1, j_2 \in A_i \quad x(j_1) = x(j_2)\right\}$. Note that the set $\prod_P$ is the set of points whose restriction to an element of the partition is constant. Also, since P is finite, $\prod_P$ is homeomorphic to the finite product $X^{|P|}$ of connected spaces and, as a consequence, it is connected. Now consider the constant point $y \in \prod_{\{\mathbb{N}\}}$. Then for every partition P, $y \in \prod_P$. Therefore, $Z = \bigcup\{\prod_P : P$ is a finite partition of $\mathbb{N}\}$ is connected in $\prod_{n\in\mathbb{N}} X$. For every $U \in \mathcal{U}$, we choose the set $F_U \subseteq X$ such that $U(F_U) \cap U^{-1}(F_U) = X$. Then for any $x \in \prod_{n\in\mathbb{N}} X$ and for any $n \in \mathbb{N}$, we have $F_U \cap U(x(n)) \cap U^{-1}(x(n)) \neq \emptyset$. Hence $\bigcap_{n\in\mathbb{N}} F_U \cap U(x) \cap U^{-1}(x) \neq \emptyset$. Therefore, there is a finite partition P of $\mathbb{N}$ such that some element of $\bigcap_{n\in\mathbb{N}} F_U \cap U(x) \cap U^{-1}(x)$ is constant on each member of P. So $\prod_P \cap U(x) \cap U^{-1}(x) \neq \emptyset$. Therefore, Z is dense in the quasi-uniform box product $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$. Since Z is connected, $\left(\prod_{n\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is also connected. $\square$

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