Novi Sad J. Math. Vol. 51, No. 1, 2021, 133-161
https://doi.org/10.30755/NSJOM.10867

Solving a hyperbolic equation in the first canonical form

Victor Dévoué[1]

Abstract. Using regularization techniques, we give a meaning to a nonlinear second order partial differential Cauchy problem by replacing it by a two parameter family of Lipschitz regular problems in an appropriate algebra of generalized functions. We prove existence of a solution and we explain how it depends on the choices made. We study the relationship with the classical solution.

AMS Mathematics Subject Classification (2010): 35D05; 35L05; 35L70; 46F30

Key words and phrases: algebras of generalized functions; Cauchy problems; hyperbolic equations; nonlinear partial differential equations; regularization of problems

1. Introduction

The main purpose of this paper is to establish the existence of solutions to the non-linear non-Lipschitz Cauchy problem formally written as

$$(P_{form})\left\{ \ \frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u,u_x,u_y);\ u|_\gamma = \varphi,\ u_x|_\gamma = \phi,\ u_y|_\gamma = \psi. \right.$$

where $u_x = \dfrac{\partial u}{\partial x}$, $u_y = \dfrac{\partial u}{\partial y}$, with a smooth nonlinear function F, in the case of irregular data.
The notation $F(\cdot,\cdot,u,u_x,u_y)$ extends, with a meaning to be defined later, the expression $(x,y) \mapsto F(x,y,u(x,y),u_x(x,y),u_y(x,y))$ in the case where u is a generalized function of two variables x and y. Here φ, ϕ and ψ are one-variable generalized functions. The data are given along a smooth monotonic curve γ with equation $y = f(x)$. Further, suppose that no tangent of γ is parallel to either the $x-$ or $y-$axis. The "consistency condition" assumes form $\varphi'(x) = \phi(x) + \psi(x)f'(x)$.
To give a meaning to this problem we use the $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras of J.-A.Marti (see [13]- [14]). These algebras give an efficient algebraic framework which permits a precise study of solutions as in [4], [8], [9]. We investigate solutions with distributions or other generalized functions as initial data, thus we must search for solutions in algebras which are invariant under nonlinear functions and contain the space of distributions.

[1]Laboratoire MEMIAD, Université des Antilles, Campus de Shoelcher, BP 7209, 97275 Schoelcher Cedex, Martinique,
e-mail: devoue-vi@orange.fr

This ill-posed problem remains unsolvable in classical function spaces. To overcome this difficulty, by means of regularizations, we associate to problem (P_{form}) a generalized one (P_{gen}) well formulated in a convenient algebra $\mathcal{A}(\Omega)$. We extend the studies made in [7], [8] for $F(\cdot,\cdot,u)$ and in [10] for $F(\cdot,\cdot,u_y)$, but here we must restrict our study to a neighborhood $\Omega \neq \mathbb{R}^2$ of γ, (for a linear equation no restriction of this kind is needed, $\Omega = \mathbb{R}^2$) and we must assume a stronger hypotheses. We search for a generalized solution u in $\mathcal{A}(\Omega)$ in which we have some conditions on F. The general idea goes as follows. The problem (P_{form}) is approached by a two-parameter family of classical smooth problems (P_λ) where $\lambda = (\varepsilon, \rho) \in (0,1]^2$. We then get a two-parameter family of classical solutions. A generalized solution is defined as the class of this family of smooth functions satisfying some asymptotical growth restrictions.

The article is organized as follows. This section is followed by Section 2 which introduces the algebras of generalized functions.

In Section 3 we define a well formulated generalized differential problem (P_{gen}) associated with the ill posed classical one. It is constructed by means of a family (P_λ) of regularized problems. We replace F with a family of Lipschitz functions (F_ε) given by suitable cutoff techniques which gives rise to a family of regularized Lipschitz problems. We use a family mollifiers $(\theta_\rho)_\rho$ to regularize the data in the singular case. Then parameter ε is used to render the problem Lipschitz, ρ making it regular. Then we can built a $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra, $\mathcal{A}(\Omega)$, stable under the family (F_ε), adapted to the generalized Cauchy problem in which the irregular problem can be solved.

Then we proceed in Section 4 with the proof of the existence of a generalized solution in the case where the irregular data are given along the monotonic curve γ. To prove the existence of a solution, a two parametric representative $(u_\lambda)_\lambda$, with $\lambda = (\varepsilon, \rho)$, is constructed from the existence of smooth solutions u_λ for each regularized Lipschitz problem (P_λ). The class of $(u_\lambda)_\lambda$ is the expected generalized solution. With regard to the regularization, we show that this solution depends solely on the class of cutoff functions as a generalized function, not on the particular representative. In the case of irregular data, the solution of the problem depends on the family of mollifiers but not on a class of that family. Moreover, we show that if the initial problem admits a smooth solution v satisfying appropriate growth estimates on some open subset O of Ω, then this solution and the generalized one are equal in a meaning given in Theorem 4.6.

In the Appendix we specify the results and estimates obtained in the classical problem.

2. Algebras of generalized functions

2.1. The presheaves of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras

2.1.1. Definitions

We refer the reader to [13], [14]. Let

- Λ be a set of indices;

- A be a solid subring of the ring $\mathbb{K}^\Lambda$, ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), that is, A has the following stability property: if $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leq r_\lambda$) for any pair $((s_\lambda)_\lambda,(r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times |A|$, then $(s_\lambda)_\lambda \in A$, with $|A| = \{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$;
- I_A be a solid ideal of A;
- $\mathcal{E}$ be a sheaf of $\mathbb{K}$-topological algebras on a topological space X, such that for any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (p_i)_{i\in I(\Omega)}$ of seminorms satisfying

$$\forall i \in I(\Omega),\ \exists (j,k,C) \in I(\Omega)\times I(\Omega)\times \mathbb{R}_+^*,\qquad \forall f,g \in \mathcal{E}(\Omega) : p_i(fg) \leq C p_j(f) p_k(g).$$

Assume that for any two open subsets Ω_1, Ω_2 of X such that $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$, then, for each $p_i \in \mathcal{P}(\Omega_1)$, the seminorm $\widetilde{p_i} = p_i \circ \rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$;

Assume that for any family $\mathcal{F} = (\Omega_h)_{h\in H}$ of open subsets of X, if $\Omega = \cup_{h\in H}\Omega_h$, then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily $\Omega_1,...,\Omega_{n(i)}$ of $\mathcal{F}$ and corresponding seminorms $p_1 \in \mathcal{P}(\Omega_1),...,p_{n(i)} \in \mathcal{P}(\Omega_{n(i)})$, such that, for each $u \in \mathcal{E}(\Omega)$, $p_i(u) \leq p_1\left(u_{|\Omega_1}\right) + ... + p_{n(i)}(u_{|\Omega_{n(i)}})$.
Set

$$\begin{aligned}
\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ ((p_i(u_\lambda))_\lambda \in |A|\},\\
\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in |I_A|\},\\
\mathcal{C} &= A/I_A.
\end{aligned}$$

Then, $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ is a sheaf of subalgebras of the sheaf $\mathcal{E}^\Lambda$ and $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ is a sheaf of ideals of $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ [13]. The constant sheaf $\mathcal{X}_{(A,\mathbb{K},|.|)}/\mathcal{N}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathcal{C} = A/I_A$.

Definition 2.1. We call presheaf of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra the factor presheaf of algebras over the ring $\mathcal{C} = A/I_A$: $\mathcal{A} = \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$.
We denote by $[u_\lambda]$ the class in $\mathcal{A}(\Omega)$ defined by the representative $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.

Definition 2.2. Let $B_p = \left\{(r_{n,\lambda})_\lambda \in (\mathbb{R}_+^*)^\Lambda : n = 1,...,p\right\}$ and B be the subset of $(\mathbb{R}_+^*)^\Lambda$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$ of elements in B_p as variables. Define
$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\}.$

We say that A is overgenerated by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^\Lambda$). If I_A is some solid ideal of A, we also say that $\mathcal{C} = A/I_A$ is overgenerated by B_p. See [5].

Remark 2.3. With this definition B is stable by inverse.

2.1.2. Relationship with distribution theory

Set Ω an open subset of $\mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ can be embedded into $\mathcal{A}(\Omega)$. If $(\theta_\rho)_{\rho\in(0,1]}$ is a family of mollifiers $\theta_\rho(x) = \frac{1}{\rho^n}\theta\left(\frac{x}{\rho}\right)$, $x \in \Omega$,

$\int \theta(x)\, dx = 1$ and if $T \in \mathcal{D}'(\Omega)$, the convolution product family $(T * \theta_\rho)_\rho$ is a family of smooth functions slowly increasing in $\frac{1}{\rho}$. Taking ρ as a component of the multi-index $\lambda \in \Lambda$, we shall choose the subring A overgenerated by some B_p of $(\mathbb{R}_+^*)^\Lambda$ containing the family $(\rho)_\lambda$ [3].

2.1.3. The association process

Assume that Λ is left-filtering for a given partial order relation $\prec$. Denote by Ω an open subset of X, E a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathcal{E}$ as a subsheaf, a a given map from Λ to $\mathbb{K}$ such that $(a(\lambda))_\lambda = (a_\lambda)_\lambda$ is an element of A. Assume that

$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{E(\Omega),\Lambda} u_\lambda = 0 \right\}.$$

Definition 2.4. We say that $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathcal{E}(\Omega)$ are a-E associated if $\lim_{E(\Omega),\Lambda} a_\lambda(u_\lambda - v_\lambda) = 0$. *That is to say, for each neighborhood V of 0 for the E-topology, there exists $\lambda_0 \in \Lambda$ such that $\lambda \prec \lambda_0 \Longrightarrow a_\lambda(u_\lambda - v_\lambda) \in V$.* We write : $u \underset{E(\Omega)}{\overset{a}{\sim}} v$.

Remark 2.5. *We define an association process between $u = [u_\lambda]$ and $T \in \mathcal{E}(\Omega)$ by $u \sim T \iff \lim_{E(\Omega),\Lambda} u_\lambda = T$. Taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda = (0,1]$,* we recover the association process defined in the literature.

2.2. *$\mathcal{D}'$-singular support*

Assume that $\mathcal{N}_{\mathcal{D}'}^{\mathcal{A}}(\Omega) = \left\{ (u_\lambda)_\lambda \in \mathcal{X}(\Omega) : \lim_{\lambda\to 0} u_\lambda = 0 \text{ in } \mathcal{D}'(\Omega) \right\} \supset \mathcal{N}(\Omega)$. Set

$$\mathcal{D}'_{\mathcal{A}}(\Omega) = \left\{ [u_\lambda] \in \mathcal{A}(\Omega) : \exists T \in \mathcal{D}'(\Omega),\ \lim_{\lambda\to 0} (u_\lambda) = T \text{ in } , \mathcal{D}'(\Omega) \right\}.$$

$\mathcal{D}'{}_{\mathcal{A}}(\Omega)$ is well defined because the limit is independent of the chosen representative; indeed, if $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$ we have $\lim_{\substack{\lambda\to 0\\ \mathcal{D}'(\mathbb{R})}} i_\lambda = 0$.

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is an $\mathbb{R}$-vector subspace of $\mathcal{A}(\Omega)$. Therefore we can consider the set $\mathcal{O}_{D'_{\mathcal{A}}}$ of all x having a neighborhood V on which u is associated with a distribution:

$$\mathcal{O}_{D'_{\mathcal{A}}}(u) = \{x \in \Omega : \exists V \in \mathcal{V}(x),\ \ u|_V \in \mathcal{D}'_{\mathcal{A}}(V)\},$$

$\mathcal{V}(x)$ being the set of all neighborhoods of x.

Definition 2.6. *The $\mathcal{D}'$-singular support of $u \in \mathcal{A}(\Omega)$, denoted* $\mathrm{singsupp}_{\mathcal{D}'}(u)$, is the set $S_{\mathcal{D}'_{\mathcal{A}}}^{\mathcal{A}}(u) = \Omega \backslash \mathcal{O}_{D'_{\mathcal{A}}}(u)$.

2.3. Algebraic framework for our problem

Set $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and Λ a set of indices, $\lambda \in \Lambda$. For any open set Ω, in $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of seminorms

$$P_{K,l}(u_\lambda) = \sup_{|\alpha| \le l} P_{K,\alpha}(u_\lambda) \ \text{ with } P_{K,\alpha}(u_\lambda) = \sup_{x \in K} |D^\alpha u_\lambda(x)|\,,\ \ K \Subset \Omega$$

and $D^\alpha = \dfrac{\partial^{\alpha_1 + \ldots + \alpha_d}}{\partial z_1^{\alpha_1} \ldots \partial z_d^{\alpha_d}}$ for $z = (z_1, \ldots, z_d) \in \Omega$, $l \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. Consider a solid ideal I_A of A. Then we have

$$\begin{aligned}
&\mathcal{X}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |A|\},\\
&\mathcal{N}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |I_A|\},\\
&\mathcal{A}(\Omega) = \mathcal{X}(\Omega)/\mathcal{N}(\Omega).
\end{aligned}$$

The generalized derivation $D^\alpha : u(= [u_\lambda]) \mapsto D^\alpha u = [D^\alpha u_\lambda]$ provides $\mathcal{A}(\Omega)$ with a differential algebraic structure.

We have the analogue of Theorem 1.2.3. of [12] for $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. We suppose that Λ is left filtering.

Proposition 2.7. *Assume that there exists $(a_\lambda)_\lambda \in B$ with $\lim_\Lambda a_\lambda = 0$. Consider $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$ such that : $\forall K \Subset \Omega$, $\left(P_{K,0}\left(u_\lambda\right)\right)_\lambda \in |I_A|$. Then $(u_\lambda)_\lambda \in \mathcal{N}\left(\Omega\right)$.*

We refer the reader to [2], [5] for a detailed proof.

Definition 2.8. Tempered generalized functions, [6], [12]. For any open set Ω in $\mathbb{R}^n$ and $f \in \mathrm{C}^\infty\left(\Omega\right)$, $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, we put

$$\mu_{r,m}(f) = \sup_{x \in \Omega, |\alpha| \le m} \left(1 + |x|\right)^r \left|\mathcal{D}^\alpha f(x)\right|.$$

The space of functions with slow growth is

$$\mathcal{O}_M\left(\Omega\right) = \left\{f \in \mathrm{C}^\infty\left(\Omega\right) : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \mu_{-q,m}(f) < +\infty\right\}.$$

2.4. Some regularizing conditions

2.4.1. Generalized operator associated with a stability property

Set $\Lambda = \Lambda_1 \times \Lambda_2$ where $\Lambda_1 = \Lambda_2 = (0, 1]$, denote by $\lambda = (\varepsilon, \rho)$ an element of Λ.

Definition 2.9. *Let Ω be an open subset of $\mathbb{R}^2$, $\Omega' = \Omega \times \mathbb{R}^3 \subset \mathbb{R}^5$. Let $F_\varepsilon \in \mathrm{C}^\infty(\Omega', \mathbb{R})$. We say that the algebra $\mathcal{A}(\Omega)$ is stable under the family $(F_\varepsilon)_\lambda$ if for all $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$ and $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$, we have*
$\Big(F_\varepsilon(\cdot, \cdot, u_\lambda, (u_\lambda)_x, (u_\lambda)_y)\Big)_\lambda \in \mathcal{X}(\Omega)$ and

$$\Big(F_\varepsilon(\cdot, \cdot, u_\lambda + i_\lambda, (u_\lambda + i_\lambda)_x, (u_\lambda + i_\lambda)_y) - F_\varepsilon(\cdot, \cdot, u_\lambda, (u_\lambda)_x, (u_\lambda)_y)\Big)_\lambda \in \mathcal{N}(\Omega).$$

If $\mathcal{A}(\Omega)$ if stable under $(F_\varepsilon)_\lambda$, for $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $\left[F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right]$ is a well defined element of $\mathcal{A}(\Omega)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u).

Definition 2.10. *Let Ω be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega\times\mathbb{R}^3,\mathbb{R})$. We say that F is smoothly tempered if the following two conditions are satisfied*
(i) *For each $K \Subset \Omega$, $l \in \mathbb{N}$ and $u \in \mathrm{C}^\infty(\Omega,\mathbb{R})$, there is a positive finite sequence $C_0,\ldots, C_l$ such that* $P_{K,l}(F(\cdot,\cdot,u,(u)_x,(u)_y)) \leq \sum_{i=0}^{l} C_i P^i_{K,l+1}(u)$.
(ii) *For each $K \Subset \Omega$, $l \in \mathbb{N}$, u and $v \in \mathrm{C}^\infty(\Omega,\mathbb{R})$, there is a positive finite sequence $D_1,\ldots, D_l$ such that*

$$P_{K,l}(F(\cdot,\cdot,v,(v)_x,(v)_y) - F(\cdot,\cdot,u,(u)_x,(u)_y)) \leq \sum_{j=1}^{l} D_j P^j_{K,l+1}(v-u).$$

Proposition 2.11. *Let Ω be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega\times\mathbb{R},\mathbb{R})$. For any ε assume that F_ε is smoothly tempered then $\mathcal{A}(\Omega)$ is stable under $(F_\varepsilon)_\lambda$.*

We define
$\mathrm{C}^\infty(\Omega) \to \mathrm{C}^\infty(\Omega), f \mapsto H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f,f_x,f_y)$.

$$H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f,f_x,f_y) : (x,y) \mapsto F_\varepsilon(x,y,f(x,y),f_x(x,y),f_y(x,y))$$

Clearly, the family $(H_\lambda)_\lambda$ maps $(\mathrm{C}^\infty(\Omega))^\Lambda$ into $(\mathrm{C}^\infty(\Omega))^\Lambda$ and allows to define a map from $\mathcal{A}(\Omega)$ into $\mathcal{A}(\Omega)$. For $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $\left[F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y\right]$ is a well defined element of $\mathcal{A}(\Omega)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u). This leads to the following definition [5]:

Definition 2.12. If $\mathcal{A}(\Omega)$ if stable under $(F_\varepsilon)_\lambda$, the operator

$$\mathcal{F} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega), \quad u = [u_\lambda] \mapsto \left[F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right] = [H_\lambda(u_\lambda)]$$

is called the *generalized operator associated with the family* $(F_\varepsilon)_\lambda$.

Definition 2.13. Let $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$ and $(g_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$, we define

$$F_\varepsilon(x,y,z,p,q) = F(x,y,zg_\varepsilon(z),pg_\varepsilon(p),qg_\varepsilon(q)).$$

The family $(F_\varepsilon)_\lambda$ is called the *family associated with F via the family* $(g_\varepsilon)_\varepsilon$. If $\mathcal{A}(\Omega)$ is stable under $(F_\varepsilon)_\lambda$, the operator

$$\mathcal{F} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega), \quad u = [u_\lambda] \mapsto \left[F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right] = [H_\lambda(u_\lambda)]$$

is called the *generalized operator associated with F via the family* $(g_\varepsilon)_\varepsilon$.

2.4.2. Generalized restriction mappings

Set $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^2$, $f \in \mathrm{C}^\infty(\mathbb{R})$. For each $g \in \mathrm{C}^\infty(\Omega)$ we define $R(g)$ by : $\mathrm{C}^\infty(\Omega) \to \mathrm{C}^\infty(\Omega_1)$, $g \mapsto (x \mapsto g(x, f(x)))$.

Definition 2.14. The smooth function f is *compatible with second side restriction* if

$$\forall (u_\lambda)_\lambda \in \mathcal{X}((\Omega), \ (u_\lambda(\cdot, f(\cdot)))_\lambda \in \mathcal{X}(\Omega_1);$$
$$\forall (i_\lambda)_\lambda \in \mathcal{N}(\Omega), \ (i_\lambda(\cdot, f(\cdot)))_\lambda \in \mathcal{N}(\Omega_1).$$

Clearly, if $u = [u_\lambda] \in \mathcal{A}(\Omega)$ then $[u_\lambda(\cdot, f(\cdot))]$ is a well defined element of $\mathcal{A}(\Omega_1)$. The mapping

$$\mathcal{R}_f : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega_1), \quad u = [u_\lambda] \mapsto [u_\lambda(\cdot, f(\cdot))] = [R(u_\lambda)]$$

is called the *generalized second side restriction mapping* associated with f.

Definition 2.15. Let $f \in \mathrm{C}^\infty(\mathbb{R})$. The function f is c-bounded if for all compact set $K \subset \mathbb{R}$ there exists another compact set $K' \subset \mathbb{R}$ such that $f(K) \subset K'$. Thus f is compatible with second side restriction.

We refer the reader to [12], [1].

3. A non Lipschitz Cauchy problem

We study the differential Cauchy problem formally written as

$$(P_{form}) \left\{ \ \frac{\partial^2 u}{\partial x \partial y} = F(\cdot, \cdot, u, u_x, u_y); \, u|_\gamma = \varphi, \quad u_y|_\gamma = \psi, \right.$$

where F, a nonlinear function of its arguments, may be non Lipschitz, γ is the monotonic curve of the equation $y = f(x)$, the data φ, ψ may be as irregular as distributions. We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem.

3.1. Cut off procedure

Let ε be a parameter belonging to the interval $(0, 1]$. Let $(r_\varepsilon)_\varepsilon$ be in $\mathbb{R}_*^{(0,1]}$ such that $r_\varepsilon > 0$ and $\lim\limits_{\varepsilon \to 0} r_\varepsilon = +\infty$. Set $L_\varepsilon = [-r_\varepsilon, r_\varepsilon]$. Consider a family of smooth one-variable functions $(g_\varepsilon)_\varepsilon$ such that

$$\sup_{z \in L_\varepsilon} |g_\varepsilon(z)| = 1, \, g_\varepsilon(z) = \left\{ \begin{array}{c} 0, \text{ if } \ |z| \geq r_\varepsilon \\ 1, \text{ if } \ -r_\varepsilon + 1 \leq z \leq r_\varepsilon - 1 \end{array} \right. \tag{11}$$

and $\dfrac{\partial^n g_\varepsilon}{\partial z^n}$ is bounded on L_ε for any integer n, $n > 0$. Set $\sup\limits_{z \in L_\varepsilon} \left| \dfrac{\partial^n g_\varepsilon}{\partial z^n}(z) \right| = M_n$. Let $\phi_\varepsilon(z) = z g_\varepsilon(z)$. We approximate the function F by the family of functions $(F_\varepsilon)_\varepsilon$ defined by $F_\varepsilon(x, y, z, p, q) = F(x, y, \phi_\varepsilon(z), \phi_\varepsilon(p), \phi_\varepsilon(q))$.

3.2. Estimates for a parametrized regular problem

Set γ a smooth non characteristic curve, whose equation is $y = f(x)$ such that

$$(12)\qquad \begin{cases} f \in \mathrm{C}^{\infty}(\mathbb{R}),\ f \text{ strictly increasing},\ f(\mathbb{R}) = \mathbb{R}, \\ \forall x \in \mathbb{R},\ f'(x) \neq 0; f, f^{-1} \in \mathcal{O}_M(\mathbb{R}),\ f \text{ is c-bounded.} \end{cases}$$

Assume that there exists $T > 0$ and an open set Ω such that $\Omega = \{(x, y) : |y - f(x)| < T\}$ verifies the following property: for any ε, there exists some positive number M_ε such that, for any $K \Subset \Omega$

$$(H)\qquad \begin{cases} \sup\limits_{(x,y)\in K;(z,p,q)\in\mathbb{R}^3} \{|\partial_z F_\varepsilon(x, y, z, p, q)|, |\partial_p F_\varepsilon(x, y, z, p, q)|\} < M_\varepsilon, \\ \sup\limits_{(x,y)\in K;(z,p,q)\in\mathbb{R}^3} |\partial_q F_\varepsilon(x, y, z, p, q)| < M_\varepsilon, \end{cases}$$

where the notation $K \Subset \mathbb{R}^2$ means that K is a compact subset of Ω. Let s be chosen so small that

$$\begin{aligned} O_s &= \{(x, y, z, p, q) \in \Omega \times \mathbb{R}^3 : |z - z_0| < s, |p - p_0| < s, |q - q_0| < s\} \\ &= \{(x, y, z, p, q) \in \mathbb{R}^5, |y - f(x)| < T, |z - z_0| < s, |p - p_0| < s, |q - q_0| < s\} \end{aligned}$$

with $z_0 = u_0(x, y), p_0 = (u_0)_x(x, y), q_0 = (u_0)_y(x, y)$.

Assume that F_ε satisfies the following condition

$$|F_\varepsilon(x, y, z, p, q) - F_\varepsilon(x', y', z', p', q')| \leq M_\varepsilon(|z - z'| + |p - p'| + |q - q'|)$$

for all (x, y, z, p, q), $(x', y', z', p', q') \in O_s$.

Recall that $\lambda = (\varepsilon, \rho) \in \Lambda_1 \times \Lambda_2 = \Lambda$, $\Lambda_1 = \Lambda_2 = (0, 1]$ where the parameter ρ is used to regularize the data. Denote by (P_λ) the problem which consists of searching for a function $u_\lambda \in \mathrm{C}^2(\Omega)$ satisfying

$$(13)\qquad \frac{\partial^2 u_\lambda}{\partial x \partial y}(x, y) = F_\varepsilon(x, y, u_\lambda(x, y), (u_\lambda)_x(x, y), (u_\lambda)_y(x, y)),$$

$$(14)\qquad u_\lambda(x, f(x)) = \varphi_\rho(x), \quad (u_\lambda)_y(x, f(x)) = \psi_\rho(x),$$

where $f, \varphi_\rho, \psi_\rho : \mathbb{R} \to \mathbb{R}$ are some smooth one-variable functions, γ is the curve of the equation $y = f(x)$ and F is a smooth function of all its arguments. According the Appendix 5, we can say that (P_λ) is equivalent to the integral formulation

(Int)

$$u_\lambda(x, t) = u_{0,\lambda}(x, t) - \iint_{D(x,t,f)} F_\varepsilon(\xi, \zeta, u_\lambda(\xi, \zeta), (u_\lambda)_x(\xi, \zeta), (u_\lambda)_y(\xi, \zeta))\, \mathrm{d}\xi\, \mathrm{d}\zeta,$$

where $u_{0,\lambda}(x, t) = \Upsilon_\rho(t) - \Upsilon_\rho(f(x)) + \varphi_\rho(x)$ and Υ_ρ denotes a primitive of $\psi_\rho \circ f^{-1}$, with

$$D(x, y, f) = \begin{cases} \{(\xi, \zeta) : f^{-1}(y) \leq \xi \leq x, y \leq \zeta \leq f(\xi)\} & \text{if } y \leq f(x) \\ \{(\xi, \zeta) : x \leq \xi \leq f^{-1}(y), f(\xi) \leq \zeta \leq y\} & \text{if } y \geq f(x). \end{cases}$$

First, we are going to prove that (P_λ) has a unique smooth solution under the following assumption

$$(H_\lambda)\left\{F_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^5,\mathbb{R});\varphi_\rho \text{ and } \psi_\rho \in \mathrm{C}^\infty(\mathbb{R}); f \in \mathrm{C}^\infty(\mathbb{R}),\ f'>0,\ f(\mathbb{R})=\mathbb{R}.\right.$$

Each compact $K \Subset \Omega$ is contained in some compact $K_a = \left[f^{-1}(-a), f^{-1}(a)\right] \times [-a,a]\cap\Omega$. Set

(15)
$$\begin{cases} a_K = 2\max(f^{-1}(a), \left|f^{-1}(-a)\right|), \\ K_a = K_{1,a}\times K_{2,a}\cap\Omega \text{ with } K_{1,a} = [-a_K/2, a_K/2] \text{ and } K_{2,a} = [-a,a]\,. \end{cases}$$

By construction we have $K \subset K_a$, $\forall (x,y)\in K_a, D(x,y,f)\subset K_a$.

Theorem 3.1. *Under Assumption (H_λ), Problem (P_λ) has a unique solution in $C^\infty(\Omega)$.*

Corollary 3.2. *With the previous notations, for every compact subset $K \Subset \Omega$, there exists a compact subset $K_a \Subset \Omega$, containing K, such that*

$$\sup_{(x,y)\in K_a;(z,p,q)\in\mathbb{R}^3} \{|\partial_z F_\varepsilon(x,y,z,p,q)|\,,\ |\partial_p F_\varepsilon(x,y,z,p,q)|\} < M_\varepsilon,$$
$$\sup_{(x,y)\in K_a;(z,p,q)\in\mathbb{R}^3} |\partial_q F_\varepsilon(x,y,z,p,q)| < M_\varepsilon.$$

$$\Phi_{a,\lambda} = \|F(\cdot,\cdot,0,0,0)\|_{\infty,K_a} + \left(\|u_{0,\lambda}\|_{\infty,K_a} + \left\|(u_{0,\lambda})_x\right\|_{\infty,K_a} + \left\|(u_{0,\lambda})_y\right\|_{\infty,K_a}\right) M_\varepsilon$$

$$\beta_a = \sup_{x\in[f^{-1}(-a),f^{-1}(a)]} \left|(f'(x))^{-1}\right|;\quad b_{a,\lambda} = (Ta_K/2 + T + a_K)\,\Phi_{a,\lambda}.$$

Then we have

$$\|u_\lambda\|_{\infty,K} \le \|u_\lambda\|_{\infty,K_a} \le \|u_{0,\lambda}\|_{\infty,K_a} + b_{a,\lambda} e^{M_\varepsilon(a_K+1+\beta_a)T}. \tag{16}$$

These results are proved in Appendix 5.

3.3. Construction of $\mathcal{A}(\Omega)$

Consider the previous family $(r_\varepsilon)_\varepsilon$. We make the following assumptions to generate a convenient $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra adapted to our problem

$$\exists p>0, \forall n\in\mathbb{N}, \exists c_n>0, \forall\varepsilon\in(0,1]\,,$$
$$\forall K\Subset\Omega, \forall\alpha\in\mathbb{N}^5, \sup_{(x,y)\in K;(z,p,q)\in\mathbb{R}^3,|\alpha|=n} |D^\alpha F_\varepsilon(x,y,z,p,q)| \le c_n r_\varepsilon^p,$$

in particular $M_\varepsilon \leq c_1 r_\varepsilon^p$. We take $\Lambda = \Lambda_1 \times \Lambda_2 = (0,1] \times (0,1]$, and $\lambda = (\varepsilon, \rho)$.

$$\varphi, \phi, \psi \in \mathcal{O}_M(\mathbb{R}). \tag{17}$$

$$\begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{(0,1]\times(0,1]} \\ (\varepsilon)_\lambda, (\rho)_\lambda, (r_\varepsilon)_\lambda, (e^{r_\varepsilon})_\lambda . \end{cases} \tag{18}$$

Then $\mathcal{A}(\Omega) = \mathcal{X}(\Omega)/\mathcal{N}(\Omega)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\Omega), (P_{K,l})_{K \Subset \Omega, l \in \mathbb{N}}\right)$ and $\mathcal{A}(\mathbb{R}) = \mathcal{X}(\mathbb{R})/\mathcal{N}(\mathbb{R})$ are built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E}, \mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}), (P_{K,l})_{K \Subset \mathbb{R}, l \in \mathbb{N}}\right)$.

As the data φ and ψ are irregular, we set $\varphi_\rho = r{*}\theta_\rho$ and $\varphi = [\varphi_\rho]$, $\psi_\rho = s{*}\theta_\rho$ and $\psi = [\psi_\rho]$ where $(\theta_\rho)_\rho$ is a chosen family of mollifiers. Then the data φ, ψ belong to $\mathcal{A}(\mathbb{R})$ and u is searched in the algebra $\mathcal{A}(\Omega)$.

3.4. Stability of $\mathcal{A}(\Omega)$

Proposition 3.3. *Set $S_n = \left\{\alpha \in \mathbb{N}^5 : |\alpha| = n\right\}$ when $n \in \mathbb{N}^*$. Let $F \in \mathrm{C}^\infty(\mathbb{R}^5, \mathbb{R})$, F_ε defined as above in Section 3.1. Assume that*

$$\forall \varepsilon \in (0,1], \forall (x,y) \in \Omega, F_\varepsilon(x,y,0,0,0) = 0, \tag{19}$$

$$\begin{aligned} &\exists p > 0, \forall n \in \mathbb{N}, \exists c_n > 0, \forall \varepsilon \in (0,1], \\ &\forall K \Subset \Omega, \sup_{(x,y)\in K; (z,p,q)\in\mathbb{R}^3, \alpha \in Sn} |D^\alpha F_\varepsilon(x,y,z,p,q)| \leq c_n r_\varepsilon^p, \end{aligned} \tag{20}$$

then $\mathcal{A}(\Omega)$ is stable under the family $(F_\varepsilon)_\varepsilon$.

We refer the reader to [8] for a similar proof.

3.5. A generalized differential problem associated with the formal one

Our goal is to give a meaning to the differential Cauchy problem formally written as (P_{form}).

Let $(g_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ and $\mathcal{F}$ the generalized operator associated with F *via* the family $(g_\varepsilon)_\varepsilon$ in Definition 2.13. Let $f \in \mathrm{C}^\infty(\mathbb{R})$ and $\mathcal{R}_f$ given by Definition 2.14.

The problem associated problem to (P_{form}) can be written as the well formulated one

$$(P_{gen}) \left\{ \ \frac{\partial^2 u}{\partial x \partial t} = \mathcal{F}(u); \mathcal{R}_f(u) = \varphi, \ \mathcal{R}_f(u_y) = \psi \right.$$

where u is in the algebra $\mathcal{A}(\Omega)$ and $\mathcal{F}$, $\mathcal{R}_f$ are defined as previously by taking into account the family $(g_\varepsilon)_\varepsilon$and f.

In terms of representatives, and thanks to the stability and restriction hypothesis, solving (P_{gen}) amounts to find a family $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$ such that

$$\begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) - F_\varepsilon(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y),(u_\lambda)_y(x,y)) = i_\lambda(x,y), \\ u_\lambda(x,f(x)) - \varphi_\rho(x) = j_\rho(x), (u_\lambda)_y(x,f(x)) - \psi_\rho(x) = l_\rho(x), \end{cases}$$

where $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$, $(j_\rho)_\lambda$, $(l_\rho)_\lambda \in \mathcal{N}(\mathbb{R})$.

Suppose we can find $u_\lambda \in \mathrm{C}^\infty(\Omega)$ verifying

$$(P_\lambda) \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,t) = F_\varepsilon(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y),(u_\lambda)_y(x,y)), \\ u_\lambda(x,f(x)) = \varphi_\rho(x), (u_\lambda)_y(x,f(x)) = \psi_\rho(x), \end{cases}$$

then, if we can prove that $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$, $u = [u_\lambda]$ is a solution of (P_{gen}).

Remark 3.4. Uniqueness in the algebra $\mathcal{A}(\Omega)$. Let $v = [v_\lambda]$ another solution to (P_{gen}). There are $(k_\lambda)_\lambda \in \mathcal{N}(\Omega)$, $(\alpha_\rho)_\lambda$, $(\beta_\rho)_\lambda \in \mathcal{N}(\mathbb{R})$, such that

$$\begin{cases} \dfrac{\partial^2 v_\lambda}{\partial x \partial y}(x,y) - F_\varepsilon(x,y,v_\lambda(x,y),(v_\lambda)_x(x,y),(v_\lambda)_y(x,y)) = k_\lambda(x,y), \\ v_\lambda(x,f(x)) = \varphi_\rho(x) + \alpha_\rho(x), \; (v_\lambda)_y(x,f(x)) = \psi_\rho(x) + \beta_\rho(x). \end{cases}$$

The uniqueness of the solution to (P_{gen}) will be the consequence of $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Omega)$.

4. Non characteristic non Lipschitz problem with irregular data

4.1. Solution to (P_{gen})

Theorem 4.1. *With the previous assumptions, if u_λ is the solution to problem (P_λ), then problem (P_{gen}) admits $[u_\lambda]_{\mathcal{A}(\Omega)}$ as solution.*

Proof. We have

$$u_\lambda(x,y) = u_{0,\lambda}(x,y) - \iint_{D(x,y,f)} F_\varepsilon(\xi,\zeta,u_\lambda(\xi,\zeta),(u_\lambda)_x(\xi,\zeta),(u_\lambda)_y(\xi,\zeta))\,\mathrm{d}\xi\,\mathrm{d}\zeta,$$

where $u_{0,\lambda}(x,y) = \Upsilon_\rho(y) - \Upsilon_\rho(f(x)) + \varphi_\rho(x)$ and $\Upsilon'_\rho = \psi_\rho \circ f^{-1}$. Then

$$(u_{0,\lambda})_x(x,y) = \psi_\rho \circ f^{-1}(y) - f'(x)\,\psi_\rho(x) + \varphi'_\rho(x).$$

We will actually prove that $\forall K \Subset \Omega$, $(P_{K,n}(u_\lambda))_\lambda \in |A|$ for all n in $\mathbb{N}$.
We have $f^{-1}(K_{2,a}) = K_{1,a}$ and $\psi \in \mathcal{A}(\mathbb{R})$, as $f^{-1} \in \mathcal{O}_M(\mathbb{R})$, we have $(\Upsilon_\rho)_\lambda \in \mathcal{A}(\mathbb{R})$, so

$$\forall l \in \mathbb{N}, \left(P_{K_{2,a},l}(\Upsilon_\rho)\right)_\lambda \in |A|, \left(P_{K_{2,a},l}(\psi_\rho \circ f^{-1}\right)_\lambda \in |A|.$$

Moreover, as $\varphi \in \mathcal{A}(\mathbb{R})$, we also have : $\forall l \in \mathbb{N}, \left(P_{K_{1,a}l}(\varphi_\rho)\right)_\lambda \in |A|$ and as $(\Upsilon_\rho \circ f)' = f'\psi_\rho$ and $f' \in \mathcal{O}_M(\mathbb{R})$ we can conclude that

$$\forall l \in \mathbb{N}, \left(P_{K,l}\left(u_{0,\lambda}\right)\right)_\lambda \in |A|\,, \left(P_{K,l}\left(\left(u_{0,\lambda}\right)_x\right)\right)_\lambda \in |A|\,.$$

We have $\forall K \Subset \Omega$, $\exists K_a = K_{1,a} \times K_{2,a} \cap \Omega$, $K \subset K_a$,

$$\|u_\lambda\|_{\infty,K} \leq \|u_\lambda\|_{\infty,K_a} \leq \|u_{0,\lambda}\|_{\infty,K_a} + b_{a,\lambda} e^{M_\varepsilon(a_K+1+\beta_a)T}. \tag{21}$$

With the notations of Corollary 3.2 we have $(\|u_{0,\lambda}\|_{\infty,K_a})_\lambda \in A$, $(\left\|(u_{0,\lambda})_x\right\|_{\infty,K_a})_\lambda \in A$ and $(\left\|(u_{0,\lambda})_y\right\|_{\infty,K_a})_\lambda \in A$, thus $\Phi_{a,\lambda} \in A$ then

$$\|u_{0,\lambda}\|_{\infty,K_a} + b_{a,\lambda} e^{M_\varepsilon(2a+1+\beta_a)T} \in A.$$

A being stable, we have $(\|u_\lambda\|_{\infty,K_a})_\lambda \in |A|$ and then $(\|u_\lambda\|_{\infty,K})_\lambda \in |A|$, that is $(P_{K,0}\left(u_\lambda\right))_\lambda \in |A|$.

Let us show that $(P_{K,1}(u_\lambda))_\lambda \in |A|$. We have

$$\frac{\partial u_\lambda}{\partial x}(x,y) = \frac{\partial u_{0,\lambda}}{\partial x}(x,y) + \int_{f(x)}^{y} F_\varepsilon(x,\zeta,u_\lambda(x,\zeta),\left(\left(u_\lambda\right)_x\right)(x,\zeta),\left(\left(u_\lambda\right)_y\right)(x,\zeta))\,\mathrm{d}\zeta,$$

thus

$$P_{K,(1,0)}(u_\lambda) \leq \sup\nolimits_K \left|\frac{\partial u_{0,\lambda}}{\partial x}(x,y)\right| + 2a \sup\nolimits_{K_a} \left|F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x,\left(u_\lambda\right)_y))\right|.$$

We have

$$P_{K_a,(0,0)}(F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x,\left(u_\lambda\right)_y))) \leq P_{K_a,0}(F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x,\left(u_\lambda\right)_y))) \leq c_0 r_\varepsilon^p.$$

Then

$$P_{K,(1,0)}(u_\lambda) \leq \|\partial/\partial x\ u_{0,\lambda}\|_{\infty,K} + c_0 r_\varepsilon^p 2a.$$

Moreover, $(\|\partial/\partial x\ u_{0,\lambda}\|_{\infty,K})_\lambda \in |A|$, and then we get $(P_{K,(1,0)}(u_\lambda))_\lambda \in |A|$. We have

$$\frac{\partial u_\lambda}{\partial y}(x,y) = \frac{\partial u_{0,\lambda}}{\partial y}(x,y) - \int_{x}^{f^{-1}(y)} F_\varepsilon(\cdot,\cdot,,u_\lambda,\left(u_\lambda\right)_x,\left(u_\lambda\right)_y)(\xi,y))\,\mathrm{d}\xi,$$

thus

$$P_{K,(0,1)}(u_\lambda) \leq \sup\nolimits_K \left|\frac{\partial u_{0,\lambda}}{\partial y}(x,y)\right| + a_K \sup\nolimits_{K_a} \left|F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x,\left(u_\lambda\right)_y))\right|.$$

We obtain $P_{K,(0,1)}(u_\lambda) \leq \|\partial/\partial y\ u_{0,\lambda}\|_{\infty,K} + \nu_K a_\eta c_0 r_\varepsilon^p$ and then $(\|\partial/\partial y\ u_\lambda\|_{\infty,K_a})_\lambda \in |A|$.

Now we proceed by induction. Suppose that $(P_{K,l}(u_\lambda))_\lambda \in |A|$ for every $l \le n$, and let us show that implies $(P_{K,n+1}(u_\lambda))_\lambda \in |A|$. We have $P_{K,n+1} = \max(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n})$ with

$$P_{1,n} = P_{K,(n+1,0)},\ P_{2,n} = P_{K,(0,n+1)},$$
$$P_{3,n} = \sup\nolimits_{\alpha+\beta=n;\ \beta\ge 1} P_{K,(\alpha+1,\beta)},\ P_{4,n} = \sup\nolimits_{\alpha+\beta=n;\ \alpha\ge 1} P_{K,(\alpha,\beta+1)}.$$

First let us show that $(P_{1,n}(u_\lambda))_\lambda, (P_{2,n}(u_\lambda))_\lambda \in |A|$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \ge 1$,

$$\begin{aligned}\frac{\partial^{n+1} u_\lambda}{\partial x^{n+1}}(x,y) &= \frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}(x,y)\\ &\quad - \sum_{j=0}^{n-1} C_n^j f^{(n-j)}(x)\ \left(\frac{\partial^j}{\partial x^j} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,f(x))\\ &\quad + \int_{f(x)}^{y} \left(\frac{\partial^n}{\partial x^n} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

As $K \subset K_a$, we can write

$$\begin{aligned}\sup_{(x,y)\in K}\left|\frac{\partial^{n+1} u_\lambda}{\partial x^{n+1}}(x,y)\right| &\le \left\|\frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}\right\|_{\infty,K} +\\ &\sup_{x\in K_{1,a}} \sum_{j=0}^{n-1} C_n^j \left|f^{(n-j)}(x)\right| \left|\frac{\partial^j}{\partial x^j} F_\varepsilon(x,f(x),\varphi_\rho(x),(u_\lambda)_x(x,f(x)),\psi_\rho(x))\right|\\ &\quad + a_K \sup_{(x,y)\in K}\left|\left(\frac{\partial^n}{\partial x^n} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,y)\right|.\end{aligned}$$

We have $f \in \mathcal{O}_M(\mathbb{R})$ then for all j, we can find $k \in \mathbb{R}$ such that

$$\sup_{K_{1,a}} (1+|x|)^{-p}\left|f^{(j)}(x)\right| \le k,$$

but then we have

$$\left\|f^{(j)}\right\|_{K_{1,a}} \le \max\left\{(1+\left|f^{-1}(a)\right|)^p, (1+\left|f^{-1}(-a)\right|)^p\right\} k \in |A|.$$

Moreover,

$$\begin{aligned}\sup\nolimits_{(x,y)\in K}&\left|\left(\frac{\partial^n}{\partial x^n} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,y)\right|\\ &\le P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \le c_n r_\varepsilon^p\end{aligned}$$

and

$$\begin{aligned}\sup\nolimits_{x\in K_{1,a}}&\left|\left(\frac{\partial^j}{\partial x^j} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,f(x))\right|\\ &\le P_{K_a,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \le c_n r_\varepsilon^p,\end{aligned}$$

and $(\left\|\partial^{n+1}/\partial x^{n+1}\ u_{0,\lambda}\right\|_{\infty,K})_\lambda \in |A|$. According to the stability hypothesis, a simple calculation shows that, for every $K \Subset \Omega$, $(P_{K,(n+1,0)}\,(u_\lambda))_\lambda \in |A|$.

Let us show that $(P_{2,n}(u_\lambda))_\varepsilon \in |A|$, for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$

$$\frac{\partial^{n+1}u_\lambda}{\partial y^{n+1}}(x,y) = \frac{\partial^{n+1}u_{0,\lambda}}{\partial y^{n+1}}(x,y) - \int\limits_x^{f^{-1}(t)} \frac{\partial^n}{\partial y^n}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)(\xi,y)\,\mathrm{d}\xi$$
$$-\sum_{j=0}^{n-1} C_n^j \left(f^{-1}\right)^{(n-j)}(y)\left(\frac{\partial^j}{\partial x^j}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(f^{-1}(y),y)\,.$$

As $K \subset K_a$, we can write

$$\sup_{(x,y)\in K}\left|\frac{\partial^{n+1}u_\lambda}{\partial y^{n+1}}(x,y)\right|$$
$$\leq \left\|\frac{\partial^{n+1}u_{0,\lambda}}{\partial y^{n+1}}\right\|_{\infty,K} + a_{K,\eta}\sup_{(x,y)\in K}\left|\left(\frac{\partial^n}{\partial y^n}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,y)\right| +$$
$$\sup_{y\in[-a,a]}\sum_{j=0}^{n-1} C_n^j\left|\left(f^{-1}\right)^{(n-j)}(y)\right|\left|\left(\frac{\partial^j}{\partial x^j}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(f^{-1}(y),y)\right|.$$

For $0 \leq j \leq n$, we have

$$\sup\nolimits_{(x,y)\in K}\left|\left(\frac{\partial^j}{\partial y^j}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,y)\right|$$
$$\leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \leq c_n r_\varepsilon^p.$$

We have $f \in \mathcal{O}_M(\mathbb{R})$ then for all j, we can find $k \in \mathbb{R}$ such that

$$\left\|f^{(j)}\right\|_{K_{1,a}} \leq \max\left\{(1+\left|f^{-1}(a)\right|)^p,(1+\left|f^{-1}(-a)\right|)^p\right\}k \in |A|\,.$$

According to the stability hypothesis, a simple calculation shows that, for every $K \Subset \Omega$ and $n \in \mathbb{N}$, $(P_{K,(0,n+1)}\,(u_\lambda))_\lambda \in |A|$. For $\alpha+\beta = n$ and $\beta \geq 1$, we now have

$$\begin{aligned} P_{K,(\alpha+1,\beta)}(u_\lambda) &= \sup\nolimits_{(x,y)\in K}\left|\left(D^{(\alpha,\beta-1)}D^{(1,1)}u_\lambda\right)(x,y)\right| \\ &= \sup\nolimits_{(x,y)\in K}\left|\left(D^{(\alpha,\beta-1)}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\right)(x,y)\right| \\ &= P_{K,(\alpha,\beta-1)}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \\ &\leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \leq c_n r_\varepsilon^p. \end{aligned}$$

Then we finally have $P_{3,n}(u_\lambda) = \sup_{\alpha+\beta=n;\beta\geq1} P_{K,(\alpha+1,\beta)}(u_\lambda) \leq c_n r_\varepsilon^p$ and the stability hypothesis ensures that $(P_{3,n}(u_\lambda))_\lambda \in |A|$. In the same way, for

$\alpha + \beta = n$ and $\alpha \geq 1$, we have

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_\lambda) &= \sup_{(x,y)\in K} \left| \left(D^{(\alpha-1,\beta)} D^{(1,1)} u_\lambda \right)(x,y) \right| \\ &= \sup_{(x,y)\in K} \left| \left(D^{(\alpha-1,\beta)} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y) \right)(x,y) \right| \\ &= P_{K,(\alpha-1,\beta)}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \\ &\leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)) \leq c_n r_\varepsilon^p. \end{aligned}$$

Thus we have $P_{4,n}(u_\lambda) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\lambda) \leq c_n r_\varepsilon^p$ and the stability hypothesis ensures that $(P_{4,n}(u_\lambda))_\lambda \in |A|$. Finally, we clearly have $(P_{K,n+1}(u_\lambda))_\lambda \in |A|$, consequently $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$. □

4.2. Dependence of the generalized solution on the class of cut off functions

See [9]. Recall that $\Lambda_1 = (0,1]$, set

$$\begin{aligned} &\mathcal{X}_1(\mathbb{R}) = \{(g_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N},\ (P_{K,l}(g_\varepsilon))_\varepsilon \in |A|\}, \\ &\mathcal{N}_1(\mathbb{R}) = \{(g_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N},\ (P_{K,l}(g_\varepsilon))_\varepsilon \in |I_A|\}, \\ &\mathcal{A}_1(\mathbb{R}) = \mathcal{X}_1(\mathbb{R})/\mathcal{N}_1(\mathbb{R}). \end{aligned}$$

Consider $\mathcal{T}(\mathbb{R})$ the set of families of smooth one-variable functions $(h_\varepsilon)_{\varepsilon\in\Lambda_1} \in \mathcal{X}_1(\mathbb{R})$, verifying the following assumptions

$$\exists\, (s_\varepsilon)_\varepsilon \in \mathbb{R}_*^{(0,1]} : \sup_{z\in[-s_\varepsilon,s_\varepsilon]} |h_\varepsilon(z)| = 1,\ h_\varepsilon(z) = \left\{ \begin{array}{c} 0, \text{ if } \ |z| \geq s_\varepsilon \\ 1, \text{ if } \ -s_\varepsilon + 1 \leq z \leq s_\varepsilon - 1 \end{array} \right. .$$

$$\exists q \in \mathbb{N}^*, \forall\, (h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R}), \forall \varepsilon, s_\varepsilon \leq r_\varepsilon^q. \tag{22}$$

Moreover, assume that $\dfrac{\partial^n h_\varepsilon}{\partial z^n}$ is bounded on $J_\varepsilon = [-s_\varepsilon, s_\varepsilon]$ for any integer n, $n > 0$.

We have $(g_\varepsilon)_{\varepsilon\in\Lambda_1} \in \mathcal{T}(\mathbb{R})$. Recall that $\phi_\varepsilon(z) = z g_\varepsilon(z)$ for $z \in \mathbb{R}$, $F_\varepsilon(x,y,z,p,q) = F(x,y,\phi_\varepsilon(z),\phi_\varepsilon(p),\phi_\varepsilon(q))$ for $(x,y,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-r_\varepsilon,r_\varepsilon]} \left| \frac{\partial^n g_\varepsilon}{\partial z^n}(z) \right| = M_n.$$

Let $g \in \mathcal{T}(\mathbb{R})/\mathcal{N}_1(\mathbb{R})$ be the class of $(g_\varepsilon)_\varepsilon$. Take $(h_\varepsilon)_\varepsilon$ another representative of g, that is to say $(h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R})$ and $(g_\varepsilon - h_\varepsilon)_\varepsilon \in \mathcal{N}_1(\mathbb{R})$.

Set $\sigma_\varepsilon(z) = z h_\varepsilon(z)$ for $z \in \mathbb{R}$, $H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z))$ for $(x,y,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-s_\varepsilon,s_\varepsilon]} \left| \frac{\partial^n h_\varepsilon}{\partial z}(z) \right| = M_n'.$$

Our choice is made such that $(\mathrm{supp}\,(h_\varepsilon))_\varepsilon$ have the same growth as $(\mathrm{supp}\,(f_\varepsilon))_\varepsilon$ with respect to the scale $(r_\varepsilon^q)_\varepsilon$, in this way the corresponding solutions are lying in the same algebra $\mathcal{A}\,(\Omega)$.

Proposition 4.2. *Set* $S_n = \left\{\alpha \in \mathbb{N}^5 : |\alpha| = n\right\}$ *when* $n \in \mathbb{N}^*$. *Let* $F \in \mathrm{C}^\infty(\mathbb{R}^5, \mathbb{R})$, H_ε *defined by* $H_\varepsilon(x, y, z, p, q) = F(x, y, \sigma_\varepsilon(z), \sigma_\varepsilon(p), \sigma_\varepsilon(q))$, *where* σ_ε *is as before. Assume that*

$$\forall\,(x, y) \in \Omega, F(x, y, 0, 0, 0) = 0\ ,$$

$$\exists p_0 > 0, \forall \alpha \in \mathbb{N}^5, |\alpha| = n > p_0, D^\alpha F(x, y, z, p, q) = 0,$$

$$\begin{gathered}\forall n \in \mathbb{N}, n \leq p_0, \exists d_n > 0, \forall \varepsilon \in (0,1]\,, \forall K \Subset \Omega, \\ \sup_{(x,y)\in K;\ z\in J_\varepsilon;\alpha\in Sn} |D^\alpha F(x, y, z, p, q)| \leq d_n r_\varepsilon^{p_0},\end{gathered} \tag{23}$$

then

$$\begin{gathered}\forall n \in \mathbb{N}, n \leq p_0, \exists c_n > 0, \forall \varepsilon \in (0,1]\,, \forall K \Subset \Omega, \\ \sup_{(x,y)\in K;\ z\in\mathbb{R};\alpha\in Sn} |D^\alpha H_\varepsilon(x, y, z, p, q)| \leq c_n r_\varepsilon^{p_0(1+q)}\end{gathered}$$

and $\mathcal{A}\,(\Omega)$ *is stable under the family* $(H_\varepsilon)_{(\varepsilon,\rho)}$.

We refer the reader to [9] for a similar detailed proof.

Theorem 4.3. *Assume that* $p = p_0(1 + q)$ *and the hypotheses of Proposition 4.2 are verified. Let* $\mathcal{F}$ *be the generalized operator associated with* F *via the family* $(g_\varepsilon)_\varepsilon$. *Let* $(h_\varepsilon)_\varepsilon \in \left(\mathrm{C}^\infty(\mathbb{R})\right)^{\Lambda_1}$ *be another family representative of the class* $[g_\varepsilon] = g$ *and leading to another generalized operator* $\mathcal{H}$ *associated with* F. *Then we have* $\mathcal{H} = \mathcal{F}$, *that is to say* $\mathcal{H}\,(u) = \mathcal{F}\,(u)$ *for any* $u \in \mathcal{A}(\Omega)$. *In terms of representatives, that is to say, if* $(u_\lambda)_\lambda$, $(v_\lambda)_\lambda \in \mathcal{X}(\Omega)$ *and* $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Omega)$, *if*

$$\begin{aligned}&F\left(\cdot, \cdot, \sigma_\varepsilon\left(v_\lambda\right), \sigma_\varepsilon\left(\left(v_\lambda\right)_x\right), \sigma_\varepsilon\left(\left(v_\lambda\right)_y\right)\right)\\ &- F\left(\cdot, \cdot, \phi_\varepsilon\left(v_\lambda\right), \phi_\varepsilon\left(\left(v_\lambda\right)_x\right), \phi_\varepsilon\left(\left(v_\lambda\right)_y\right)\right) = L\left(\sigma_\varepsilon\left(v_\lambda\right), \phi_\varepsilon\left(v_\lambda\right)\right)\end{aligned}$$

then $\left(L\left(\sigma_\varepsilon\left(v_\lambda\right), \phi_\varepsilon\left(v_\lambda\right)\right)\right)_\lambda \in \mathcal{N}(\Omega)$.

We refer the reader to [9] for a similar detailed proof.

Corollary 4.4. *Problem* (P_{gen}), *a fortiori its solution, does not depend of the choice of the representative* $(f_\varepsilon)_\varepsilon$ *of the class* $f \in \mathcal{T}(\mathbb{R})/\mathcal{N}_1(\mathbb{R})$.

Proof. $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Omega)$ then $((w_\lambda)_x)_\lambda \in \mathcal{N}(\Omega)$, $\left((w_\lambda)_y\right)_\lambda \in \mathcal{N}(\Omega)$. We deduce that $\left(L\left(\sigma_\varepsilon\left(v_\lambda\right), \phi_\varepsilon\left(u_\lambda\right)\right)\right)_\lambda \in \mathcal{N}(\Omega)$, that is to say $\mathcal{H}\,(u) = \mathcal{F}\,(u)$ for any $u \in \mathcal{A}(\Omega)$. □

4.3. Comparison with classical solutions

Even if the data are as irregular as distributions, it may happen that the initial formal ill-posed problem (P_{form}) has nonetheless a local smooth solution. We are going to prove that this solution is exactly the restriction (in the sense of sheaf theory) of the generalized one.

The generalized solution to Problem (P_{gen}) is defined from the integral representation (Int). Thus, we are going to study the relationship between this generalized function and the classical solutions to (P_{form}) (when they exist) on a domain O such that $\forall\,(x,y) \in O$, $D(x,y,f) \subset O$. This justified to choose $O = \left]f^{-1}(\mu), f^{-1}(\nu)\right[\times \left]\mu,\nu\right[\cap \Omega$ when $(\mu,\nu) \in O$ with $\mu < 0 < \nu$.

Remark 4.5. If the non regularized problem (P_{form}) has a smooth solution v on O then, necessarily, we have $O \subset \mathbb{R}^2 \backslash \operatorname{singsupp}(u)$.

Recall that there exists a canonical sheaf embedding of $\mathrm{C}^\infty(\cdot)$ into $\mathcal{A}(\cdot)$, through the morphism of algebra $\sigma_O : \mathrm{C}^\infty(O) \to \mathcal{A}(O)\,, \quad f \mapsto [f_{\varepsilon,\rho}]\,,$ where O is any open subset of Ω and $f_{\varepsilon,\rho} = f$. The presheaf $\mathcal{A}$ allows to restriction and as usually we denote by $u|_O$ the restriction on O of $u \in \mathcal{A}(\Omega)$.

Theorem 4.6. *Let $u = [u_{\varepsilon,\rho}]$ be the solution to Problem (P_{gen}). Let O be an open subset of Ω such that $O \subset \mathbb{R}^2 \backslash \operatorname{singsupp}(u)$. Assume that $O = \bigcup_{\varepsilon\in\Lambda_1} O_\varepsilon$ with $(O_\varepsilon)_\varepsilon$ is an increasing family of open subsets of Ω such that $O_\varepsilon = \left]f^{-1}(a_\varepsilon), f^{-1}(b_\varepsilon)\right[\times \left]a_\varepsilon, b_\varepsilon\right[\cap \Omega$ when $(a_\varepsilon, b_\varepsilon) \in \Omega$ with $a_\varepsilon < 0 < b_\varepsilon$. Assume that problem (P_{form}) has a smooth solution v on O such that $\sup_{(x,y)\in O_\varepsilon} |v(x,y)| < r_\varepsilon - 1$, $\sup_{(x,y)\in O_\varepsilon} |v_x(x,y)| < r_\varepsilon - 1$ and $\sup_{(x,y)\in O_\varepsilon} |v_y(x,y)| < r_\varepsilon - 1$ for any ε. Then v (an element of $\mathrm{C}^\infty(O)$ canonically embedded in $\mathcal{A}(O)$) is the restriction (in the sense of sheaf theory) of u to O, $v = u|_O$.*

Proof. We clearly have $\forall\,(x,y) \in O, \exists \varepsilon_0, \forall \varepsilon \le \varepsilon_0, (x,y) \in O_\varepsilon$.
Then $D(x,y,g) \subset O_\varepsilon \subset O$; we have

$$v(x,y) = v_0(x,y) - \iint_{D(x,y,g)} F(\cdot,\cdot,v_\lambda,(v_\lambda)_x,(v_\lambda)_y)(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta.$$

We take as an representative of u the family $(u_{\varepsilon,\rho})_{(\varepsilon,\rho)}$; we have: $\forall\,(x,y) \in O$,

$$u_\lambda(x,y) \quad = \quad u_{0,\lambda}(x,y) \ - \iint_{D(x,y,f_\eta)} F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta,$$

where $u_{0,\lambda}(x,y) = \Upsilon_\rho(y) - \Upsilon_\rho(f(x)) + \varphi_\rho(x)$ and $\Upsilon'_\rho = \psi_\rho \circ f^{-1}$. Moreover, we have $v_0(x,y) = u_{0,\lambda}(x,y)$ and

$$\frac{\partial u_\lambda}{\partial x}(x,y) = \frac{\partial u_{0,\lambda}}{\partial x}(x,y) + \int_{f(x)}^{y} F(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)(x,\zeta)\,\mathrm{d}\zeta,$$

$$\frac{\partial u_\lambda}{\partial y}(x,y) = \frac{\partial u_{0,\lambda}}{\partial y}(x,y) - \int_{x}^{f^{-1}(y)} F(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)(\xi,y)\,\mathrm{d}\xi.$$

Set $(w_\lambda)_\lambda = (u_\lambda|_O - v)_\lambda$ and take $K \Subset O$. There exists ε_1 such that, for all $\varepsilon < \varepsilon_1$, $K \Subset O_\varepsilon$. According to the definition of O_ε, there exists a, $0 < a < (b_\varepsilon - a_\varepsilon)/2$, such that $K \subset Q_a \subset O$ with $Q_a = \left[f^{-1}(a_\varepsilon + a), f^{-1}(b_\varepsilon - a)\right] \times [a_\varepsilon + a, b_\varepsilon - a] \cap \Omega$. Take $(x,y) \in K$, then $D(x,y,g) \subset Q_a$.

$$w_\lambda(x,y) = \iint_{D(x,y,g)} \left(F(\cdot,\cdot,v,v_x,v_y) - F_\varepsilon(\cdot,\cdot,v,v_x,v_y)\right)(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta + \iint_{D(x,y,g)} \Big(F_\varepsilon(\cdot,\cdot,v,v_x,v_y) - F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x,(u_\lambda)_y)\Big)(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta$$

Note that, for $(\xi,\varsigma,z,p,q) \in O_\varepsilon \times (]-r_\varepsilon + 1, r_\varepsilon - 1[)^3$, we have $F(\xi,\varsigma,z,p,q) = F_\varepsilon(\xi,\varsigma,z,p,q)$ by construction of F_ε.

As values of v, v_x, v_y are in $]-r_\varepsilon + 1, r_\varepsilon - 1[$, we have $F(\cdot,\cdot,v,v_x,v_x) - F_\varepsilon(\cdot,\cdot,z,v_x,v_x) = 0$. Moreover, we have

$$|w_\lambda(x,y)| \quad \leq \quad c_1 r_\varepsilon^p \int_{f^{-1}(b_\varepsilon - a)}^{f^{-1}(a_\varepsilon + a)} \int_{f(x)}^{y} \Big(|w_\lambda| + |(w_\lambda)_x| + \Big|(w_\lambda)_y\Big|\Big)(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta,$$

then $w_\lambda = 0$. Thus v and u_λ are solutions of the same integral equation, which admits a unique solution since F_ε is a smooth function of its arguments. Thus, for all $\varepsilon \leq \varepsilon_1$, v and u_λ, v_x and $(u_\lambda)_x$, v_y and $(u_\lambda)_y$ are equal on O_ε. We deduce that v and u_λ are solutions of the same integral equation, which admits a unique solution. Thus $(P_{K,n}(v))_\lambda \in |A|$ for any $K \Subset O$ and $n \in \mathbb{N}$. Then v (identified with $[(v)_\lambda]$) belongs to $\mathcal{A}(OO)$.
Moreover, for all $\varepsilon \leq \varepsilon_1$, $\sup_{(x,y)\in Q_a} |w_\lambda(x,y)| = 0$, hence $(P_{K,l}(w_\lambda))_\lambda \in |I_A|$ for any $l \in \mathbb{N}$ as w_λ vanishes on K. Thus $(w_\lambda)_\lambda \in \mathcal{N}(O)$ and $v = u|_O$, as claimed. □

Remark 4.7. Construction of $\mathcal{A}(\Omega)$ in the case of regular data. If the data s and t are smooth, we take $\varepsilon \in \Lambda = \Lambda_1 = (0,1]$. Let $(r_\varepsilon)_\varepsilon$ be in $(\mathbb{R}_*^+)^{(0,1]}$ such that $\lim_{\varepsilon\to 0} r_\varepsilon = +\infty$. We take $\mathcal{C} = A/I_A$ the ring overgenerated by $(\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon$, $(e^{r_\varepsilon})_{(\varepsilon,\eta)}$, elements of $(\mathbb{R}_*^+)^{(0,1]}$. Then $\mathcal{A}(\Omega) = \mathcal{X}(\Omega)/\mathcal{N}(\Omega)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P}) = \Big(\mathrm{C}^\infty(\Omega), (P_{K,l})_{K\Subset\Omega, l\in\mathbb{N}}\Big)$ and $\mathcal{A}(\mathbb{R}) = \mathcal{X}(\mathbb{R})/\mathcal{N}(\mathbb{R})$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P}) = \Big(\mathrm{C}^\infty(\mathbb{R}), (P_{K,l})_{K\Subset\mathbb{R}, l\in\mathbb{N}}\Big)$. Nonetheless, the algebra $\mathcal{A}(\Omega)$ is not the same in the two cases, regular data and irregular data. We can take $r_\varepsilon = \frac{1}{\varepsilon}$.
We set $\varphi = s$ and $\psi = h$, elements of $\mathrm{C}^\infty(\mathbb{R})$ canonically embedded in $\mathcal{A}(\mathbb{R})$. If $\alpha \in \mathcal{A}(\mathbb{R})$ we take $\alpha_\rho = \alpha$, if $\alpha \in \mathcal{N}(\mathbb{R})$ we take $\alpha_\rho = 0$. Then we can rewrite this section and we get similar results. We have the same definitions as previously and we obtain the same theorems, the same proofs replacing φ_ρ by φ and ψ_ρ by ψ. As previously, we can prove that Problem (P_{gen}) has a generalized solution $u = [u_\varepsilon]$ in the algebra $\mathcal{A}(\Omega)$.

5. Appendix

The Appendix is devoted to the construction of global smooth solutions to the Cauchy problem when the data are smooth. This is achieved by rewriting the differential equation as an integral equation and making a thorough investigation on the method of successive approximations [11]. Several improvements to classical methods and results are needed to obtain precise estimates used in the previous sections. Namely, the growth in the parameter ε of the families of solutions has to be known to choose the good algebraic structure to solve the regularized problems. So the results of the Appendix form an essential basis for the construction of generalized solutions.

5.1. Global smooth solutions to the Cauchy problem

For the following study of generalized situation, we will need precise estimates for the case of smooth data.

5.1.1. Formulation of the problem

We consider the Cauchy problem

$$(P)\left\{ \frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u,u_x,u_y);\ u|_\gamma = \varphi, \qquad u_y|_\gamma = \psi, \right.$$

where $f, \varphi, \psi : \mathbb{R} \to \mathbb{R}$ are some smooth one-variable functions, the function F must satisfy smoothness requirements in its dependence on the arguments x, y, z, p, q which will be specified later. Therefore we have to impose the following hypothesis on the initial curve: that it should nowhere be tangent to a characteristic. Since the equation appears in canonical form, its characteristics are simply the vertical lines and the horizontal lines. Hence the initial curve has non-parametric representation $y = f(x)$, where f is strictly monotonic.

We shall establish that the Cauchy problem is well posed for the hyperbolic partial differential equation. In all cases the following hypothesis will be satisfied

$$(H) \qquad \left\{ \begin{array}{l} F \in \mathrm{C}^\infty(\mathbb{R}^5, \mathbb{R}); \forall x \in \mathbb{R}, f'(x) \neq 0 \\ f \text{ is defined and strictly increasing on } \mathbb{R} \text{ with image } \mathbb{R}. \end{array} \right.$$

Assume that there exists $T > 0$ and an open set Ω such that
$\Omega = \{(x,y) : |y - f(x)| < T\}$ verifies the following property: we fix some positive number M such that, for any $K \Subset \Omega$,

$$\sup_{(x,y)\in K;(z,p,q)\in\mathbb{R}^3} |F(x,y,z,p,q)| < M,$$

and

$$(H2) \quad \left\{ \begin{array}{l} \displaystyle\sup_{(x,y)\in K;(z,p,q)\in\mathbb{R}^3} \{|\partial_z F(x,y,z,p,q)|\,,\ |\partial_p F(x,y,z,p,q)|\} < M, \\ \displaystyle\sup_{(x,y)\in K;(z,p,q)\in\mathbb{R}^3} |\partial_q F(x,y,z,p,q)| < M. \end{array} \right.$$

Let s be chosen to be small and
$O_s = \{(x,y,z,p,q) \in \Omega \times \mathbb{R}^3 : |z| < s, |p| < s, |q| < s\}$.

We denote by (P_∞) the problem which consists in searching for a function $u \in \mathrm{C}^2(\Omega)$ satisfying

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y),u_x(x,y),u_y(x,y)), \tag{1.2}$$

$$u\,(x,f(x)) = \varphi(x), u_y\,(x,f(x)) = \psi(x). \tag{1.3}$$

We denote by (P_i) the problem which consists in searching for a function $u \in \mathrm{C}^0(\Omega)$ satisfying

$$u(x,y) = u_0(x,y) - \iint\limits_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta),u_y(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{1.4}$$

where $u_0(x,y) = \Upsilon\,(y) - \Upsilon(f(x)) + \varphi(x)$ and Υ denotes a primitive of $\psi \circ f^{-1}$, with

$$D(x,y,f) = \begin{cases} \{(\xi,\eta) :\ f^{-1}(y) \le \xi \le x,\ y \le \eta \le f(\xi)\}, & \text{if } y \le f(x), \\ \{(\xi,\eta) : x \le \xi \le f^{-1}(y),\ f(\xi) \le \eta \le y\}, & \text{if } y \ge f(x). \end{cases}$$

Theorem 5.1. *Let $u \in \mathrm{C}^0(\Omega)$. The function u is a solution to (P_∞) if and only if u is a solution to (P_i).*

Proof. The existence of f^{-1} is ensured by (H). Hypothesis (H) also ensures that the domain $D(x,y,f)$ is bounded. We consider the points $M(x,y)$, $P(f^{-1}(y),y)$, $Q(x,f(x))$, the domain $D(x,y,f)$ is the "curvilinear triangle" MPQ. If u is a solution to (P_∞), suppose that $y \ge f(x)$. We have

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta &= \int\limits_{f(x)}^{y} \left(\int\limits_{x}^{f^{-1}(\eta)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\,\mathrm{d}\xi \right) \mathrm{d}\eta \\ &= \int\limits_{f(x)}^{y} \frac{\partial u}{\partial y}(f^{-1}(\eta),\eta)\,\mathrm{d}\eta - \int\limits_{f(x)}^{y} \frac{\partial u}{\partial y}(x,\eta)\,\mathrm{d}\eta \\ &= \Upsilon\,(y) - \Upsilon(f(x)) - u(x,y) + \varphi(x), \end{aligned}$$

where Υ denotes a primitive of $\psi \circ f^{-1}$. Then

$$u(x,y) = u_0(x,y) - F(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta),u_y(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,,$$

where $u_0(x,y) = \Upsilon\,(y) - \Upsilon(f(x)) + \varphi(x)$. We obtain the same result if we suppose $y \le f(x)$. Thus u satisfies (P_i). We remark moreover that, if u is of class C^n, then $(x,y) \mapsto F(x,y,u(x,y),u_x(x,y),u_y(x,y))$ is of class C^n. Therefore

$$W : (x,y) \mapsto u_0(x,y) - (F(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta),u_y(\xi,\eta))\,\mathrm{d}\eta)\mathrm{d}\xi$$

has a partial derivative with respect to x of class C^n, and

$$W : (x, y) \mapsto u_0(x, y) - (F(\xi, \eta, u(\xi, \eta), u_x(\xi, \eta), u_y(\xi, \eta))\, d\xi) d\eta$$

has a partial derivative with respect to y of class C^n. As

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x, y) = F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x, y)$$

is of class C^n we conclude that u is of class C^{n+1}. By induction, u is therefore of class C^∞. □

We have, of course, the following corollary.

Corollary 5.2. *If u is a solution to (P_i) (or to (P_∞)), then u belongs to $C^\infty(\Omega)$.*

5.1.2. Existence and uniqueness of solutions

Theorem 5.3. *From hypothesis (H) it follows that problem (P_∞) has a unique solution in $C^\infty(\Omega)$.*

Proof. According to Theorem 5.1, solving problem (P_∞) amounts to solving problem (P_i), that is searching for $u \in C^0(\Omega)$ satisfying (1.4). For every compact subset of Ω, we can find $\lambda > 0$ so that this compact subset is contained in $K_\lambda = [-\lambda, \lambda] \times [f(-\lambda), f(\lambda)] \cap \Omega$.
Let us assume always that $y \geq f(x)$ and let us make the change of variables $X = x + \lambda$, $Y = y - f(-\lambda)$. The relation (1.4) can be written as

$$u(X - \lambda, Y + f(-\lambda)) = u_0(X - \lambda, Y + f(-\lambda)) - \iint_{D(X-\lambda, Y+f(-\lambda), f)} F(\cdot, \cdot, u, u_x, u_y)(\xi - \lambda, \eta + f(-\lambda))\, d\xi\, d\eta,$$

whose form is (P_{int})

$$U(X, Y) = U_0(X, Y) - \mathfrak{F}(\cdot, \cdot, U, U_X, U_Y)(\xi, \eta)\, d\xi\, d\eta, \tag{1.5}$$

with $g(X) = f(X - \lambda) - f(-\lambda)$; K_λ turns into the compact subset $Q_\lambda = [0, 2\lambda] \times [0, g(2\lambda)] \cap \Omega$. The equation of (γ) can then be written as $Y = g(X)$ and $g(0) = 0$. So we now have $X \geq 0$ and $Y \geq g(X)$. Picard's procedure for solving (P_{int}) is to set up a sequence of successive approximations U_n defined by the formula for any $n \in \mathbb{N}^*$,

$$U_n(X, Y) = U_0(X, Y) - \mathfrak{F}(., ., U_{n-1}, (U_{n-1})_X, (U_{n-1})_Y)(\xi, \eta)\, d\xi\, d\eta. \tag{1.6}$$

Our purpose is to establish that the limit $U = \lim U_n = U_0 + \sum_{n=0}^{+\infty} (U_{n+1} - U_n)$ of the successive approximations U_n exists and satisfies the integro-differential equation.

For every compact subset $H \Subset \Omega$, let us put $\|U_0\|_{\infty,H} = \sup_{(x,y)\in H} |U_0(x,y)|$. According to the mean value theorem, we can write

(1.7)
$$\mathfrak{F}(\xi,\eta,z,p,q) - \mathfrak{F}(\xi,\eta,z',p',q') = (z-z')\frac{\partial \mathfrak{F}}{\partial z} + (p-p')\frac{\partial \mathfrak{F}}{\partial p} + (q-q')\frac{\partial \mathfrak{F}}{\partial q},$$

where the partial derivatives $\frac{\partial \mathfrak{F}}{\partial z}$, $\frac{\partial \mathfrak{F}}{\partial p}$, $\frac{\partial \mathfrak{F}}{\partial q}$ on the right are evaluated at a set of arguments intermediate between (ξ,η,z,p,q) and (ξ,η,z',p',q'). Thus for all $(\xi,\eta) \in \mathfrak{D}(X,Y,g)$, according to $(H2)$, we can conclude that

(1.8) $$|\mathfrak{F}(\xi,\eta,z,p,q) - \mathfrak{F}(\xi,\eta,z',p',q')| \le |z-z'|\, M + |p-p'|\, M + |q-q'|\, M.$$

By differentiating 1.6 with respect to X and with respect to Y we obtain the formulas

$$(U_{n+1})_X (X,Y) = (U_0)_X (X,Y) + \int_{g(X)}^{Y} \mathfrak{F}(\cdot,\cdot,U_n,(U_n)_X,(U_n)_Y)(X,\eta)\, \mathrm{d}\eta,$$

$$(U_{n+1})_Y (X,Y) = (U_0)_Y (X,Y) - \int_{X}^{g^{-1}(Y)} \mathfrak{F}(\cdot,\cdot,U_n,(U_n)_X,(U_n)_Y)(\xi,Y)\, \mathrm{d}\xi,$$

It is necessary to show that the definitions of the functions U_n, $(U_n)_x$, $(U_n)_y$ are meaningful. We assume that for some value of n, the functions U_n, $(U_n)_x$, $(U_n)_y$ are defined and continuous in Ω and satisfy the inequalities

(1.9)
$$|U_n - U_0|\,(X,Y) < s, |(U_n)_x - (U_0)_x|\,(X,Y) < s, \left|(U_n)_y - (U_0)_y\right|(X,Y) < s.$$

Then $\mathfrak{F}(X,Y,U_n(X,Y),(U_n)_X (X,Y),(U_n)_Y (X,Y)$ is well defined.

Moreover,

$$\begin{aligned}&\mathfrak{F}(\xi,\eta,U_0(\xi,\eta),(U_0)_X(\xi,\eta),(U_0)_Y(\xi,\eta)) - \mathfrak{F}(\xi,\eta,0,0,0)\\ &= U_0(\xi,\eta)\frac{\partial \mathfrak{F}}{\partial z} + (U_0)_X(\xi,\eta)\frac{\partial \mathfrak{F}}{\partial p} + (U_0)_Y(\xi,\eta)\frac{\partial \mathfrak{F}}{\partial q}\end{aligned}$$

where the partial derivatives $\frac{\partial \mathfrak{F}}{\partial z}$, $\frac{\partial \mathfrak{F}}{\partial p}$, $\frac{\partial \mathfrak{F}}{\partial q}$ on the right are evaluated at a set of arguments intermediate between (ξ,η,z,p,q) and $(\xi,\eta,0,0,0)$.

$$\begin{aligned}&|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta),(U_0)_X(\xi,\eta),(U_0)_Y(\xi,\eta))|\\ &\le |\mathfrak{F}(\xi,\eta,0,0,0)| + \left(\|U_0\|_{\infty,Q_\lambda} + \|(U_0)_X\|_{\infty,Q_\lambda} + \|(U_0)_Y\|_{\infty,Q_\lambda}\right) M.\end{aligned}$$

Set

$$\Phi_\lambda = \|\mathfrak{F}(\cdot,\cdot,0,0,0)\|_{\infty,Q_\lambda} + \left(\|U_0\|_{\infty,Q_\lambda} + \|(U_0)_X\|_{\infty,Q_\lambda} + \|(U_0)_Y\|_{\infty,Q_\lambda}\right) M$$

and $\forall n \in \mathbb{N}^*$, $V_n = U_n - U_{n-1}$. In particular,

$$|V_1(X,Y)| \leq \iint\limits_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\cdot,\cdot,U_0,(U_0)_X,(U_0)_Y)|(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \leq \Phi_\lambda A(X,Y),$$

where $A(X,Y) = \iint\limits_{\mathfrak{D}(X,Y,g)} \mathrm{d}\xi\,\mathrm{d}\eta$ indicates the area of the domain $\mathfrak{D}(X,Y,g)$. We can notice that $A(X,Y) \leq (Y - g(X))\,(g^{-1}(Y) - X)/2 \leq T\lambda$ and then $|V_1(X,Y)| \leq \Phi_\lambda T\lambda$.

If $(U_n)_X$, $(U_n)_Y$ and $(U_{n-1})_X$, $(U_{n-1})_Y$ satisfy 1.9, the definition of U_n and the Lipschitz condition 1.8 lead to the inequality

$$\begin{aligned}
(2.0)\quad & |V_{n+1}(X,Y)| \\
\leq & \iint\limits_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\cdot,\cdot,U_n,(U_n)_X,(U_n)_Y) - \mathfrak{F}(\cdot,\cdot,U_{n-1},(U_{n-1})_X,(U_{n-1})_Y)|\,\mathrm{d}\xi\,\mathrm{d}\eta \\
\leq M & \iint\limits_{\mathfrak{D}(X,Y,g)} (|U_n - U_{n-1}| + |(U_n)_X - (U_{n-1})_X| + |(U_n)_Y - (U_{n-1})_Y|)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}$$

Furthermore, the formulas

$$(U_{n+1})_X(X,Y) = (U_0)_X(X,Y) + \int_{g(X)}^{Y} \mathfrak{F}(\cdot,\cdot,U_n(,(U_n)_X,(U_n)_Y)(X,\eta)\,\mathrm{d}\eta,$$

$$(U_{n+1})_Y(X,Y) = (U_0)_Y(X,Y) - \int_{X}^{g^{-1}(Y)} \mathfrak{F}(\cdot,\cdot,U_n,(U_n)_X,(U_n)_Y)(\xi,Y)\,\mathrm{d}\xi,$$

obtained by differentiating V_n with respect to X and with respect to Y, furnish in a similar way the inequalities

$$|(V_{n+1})_X(X,Y)| \leq M \int_{g(X)}^{Y} (|V_n| + |(V_n)_X| + |(V_n)_Y|)\,(X,\eta)\,\mathrm{d}\eta,$$

$$|(V_{n+1})_Y(X,Y)| \leq M \int_{X}^{g^{-1}(Y)} (|V_n| + |(V_n)_X| + |(V_n)_Y|)\,(\xi,Y)\,\mathrm{d}\xi.$$

Moreover,

$$|(V_1)_X(X,Y)| = \left| \int_{g(X)}^{Y} \mathfrak{F}(\cdot,\cdot,U_0,(U_0)_X,(U_0)_Y)(X,\eta)\,\mathrm{d}\eta \right|$$
$$\leq |Y-g(X)|\,\Phi_\lambda \leq T\Phi_\lambda,$$
$$|(V_1)_Y(X,Y)| = \left| -\int_{X}^{g^{-1}(Y)} \mathfrak{F}(\cdot,\cdot,U_0,(U_0)_X,(U_0)_Y)(\xi,Y)\,\mathrm{d}\xi \right|$$
$$\leq \left|g^{-1}(Y)-X\right|\Phi_\lambda \leq 2\lambda\Phi_\lambda.$$

We have $g(2\lambda) = f(\lambda) - f(-\lambda) < T$, moreover,

$$\Omega = \{(x,y) : |y-f(x)| < T\} = \{(X,Y) : |Y-g(X)| < T\}.$$

To exploit the similarity of the integrands, it is convenient to set

$$E_n(t) = \max_{t=Y-g(X)} \left(|V_n(X,Y)| + |(V_n)_X(X,Y)| + |(V_n)_Y(X,Y)|\right),$$

with the points (X,Y) restricted so that each of them generates a domain of dependence lying inside the region $Q_\lambda \subset \Omega$, [11]. We need to make exceptionally careful estimates of E_n here. Introducing new coordinates $\tau = \eta - g(\xi)$.

If T is taken sufficiently small, the quantities U_n, $(U_n)_x$, $(U_n)_y$ fulfill 1.9 for all values of n. It follows

$$\iint_{\mathfrak{D}(X,Y,g)} \left(|V_n| + |(V_n)_X| + |(V_n)_Y|\right)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$\leq \int_{g(X)}^{Y}\int_{X}^{g^{-1}(\eta)} \left(|V_n| + |(V_n)_X| + |(V_n)_Y|\right)\,\mathrm{d}\xi\mathrm{d}\eta \leq \int_{0}^{2\lambda}\int_{X}^{g^{-1}(Y)} E_n(\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

and

$$\left| \int_{0}^{Y-g(X)}\int_{X}^{g^{-1}(Y)} E_n(\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \right| \leq 2\lambda\int_0^t E_n(\tau)\,\mathrm{d}\tau \leq 2\lambda\int_0^t E_n(\tau)\,\mathrm{d}\tau.$$

$$U_n(X,Y) = U_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\cdot,\cdot,U_{n-1},(U_{n-1})_X,(U_{n-1})_Y)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Moreover,

$$|(V_{n+1})_X(X,Y)| \leq M \int\limits_{g(X)}^{Y} (|V_n| + |(V_n)_X| + |(V_n)_Y|)\,\mathrm{d}\eta \leq M \int\limits_0^t E_n(\tau)\,\mathrm{d}\tau,$$

$$|(V_{n+1})_Y(X,Y)| \leq M \int\limits_{X}^{g^{-1}(Y)} (|V_n| + |(V_n)_X| + |(V_n)_Y|)(\xi,Y)\,\mathrm{d}\xi$$

$$\leq M \int\limits_0^t \beta_\lambda E_n(\tau)\,\mathrm{d}\tau,$$

where $\beta_\lambda = \sup_{X\in[0,2\lambda]} |(1/g'(X))|$. Thus we have

(2.1)

$$(|V_{n+1}| + |(V_{n+1})_X| + |(V_{n+1})_Y|)(X,Y) \leq (2\lambda M + M + M\beta_\lambda) \int\limits_0^t E_n(\tau)\,\mathrm{d}\tau$$

If we replace the left-hand side of (2.1) by its maximum value, for $0 \leq t \leq 2\lambda$, we obtain

$$E_{n+1}(t) \leq M(2\lambda + 1 + \beta_\lambda) \int\limits_0^t E_n(\tau)\,\mathrm{d}\tau. \tag{2.2}$$

We have

$$\begin{aligned} E_1(t) &= \max_{t=Y-g(X)} (|V_1(X,Y)| + |(V_1)_X(X,Y)| + |(V_1)_Y(X,Y)|) \\ &\leq \lambda t \Phi_\lambda + |Y - g(X)|\,\Phi_\lambda + \left|g^{-1}(Y) - X\right| \Phi_\lambda \leq b_\lambda, \end{aligned}$$

with $b_\lambda = (\lambda T + T + 2\lambda)\,\Phi_\lambda$. From (2.2) it may be deduced that

$$E_2(t) \leq M(2\lambda + 1 + \beta_\lambda) \int\limits_0^t E_1(\tau)\,\mathrm{d}\tau \leq M(2\lambda + 1 + \beta_\lambda) b_\lambda t.$$

Mathematical induction serves to establish that

$$E_n(t) \leq [M(2\lambda + 1 + \beta_\lambda)]^{n-1} b_\lambda \frac{t^{n-1}}{(n-1)!} \leq [M(2\lambda + 1 + \beta_\lambda)]^{n-1} (b_\lambda) \frac{T^{n-1}}{(n-1)!}.$$

Then the exponential series $\sum\limits_{n=0}^{\infty} [M(2\lambda + 1 + \beta_\lambda)]^n (b_\lambda) \frac{T^n}{n!} = r_\lambda e^{M(2\lambda+1+\beta_\lambda)T}$ is a majorant for the infinite series $\sum\limits_{n=0}^{+\infty} (U_{n+1} - U_n)$, as well as for formal partial derivatives with respect to X and Y which ensures the uniform convergence of the series $\sum_{n\geq 1} V_n$, $\sum_{n\geq 1} (V_n)_X$ and $\sum_{n\geq 1} (V_n)_Y$ on Q_λ and consequently on every compact subset of Ω. From the equality $\sum\limits_{k=1}^{n} V_k = U_n - U_0$ we deduce that the sequence $(U_n)_{n\in\mathbb{N}}$ converges uniformly on Q_λ to a function U. As every U_n is derivable with respect to X, from the equality $\sum\limits_{k=1}^{n} (V_k)_X = (U_n)_X - (U_0)_X$, we deduce that the uniform limit U is derivable with respect to X on every compact subset Q_λ, so on Ω, and the sequence $((U_n)_X)_{n\in\mathbb{N}}$ converges uniformly on Q_λ to the function $(U)_X$.

As every U_n is derivable with respect to Y, from the equality $\sum\limits_{k=1}^{n} (V_k)_Y = (U_n)_Y - (U_0)_Y$, we deduce that the uniform limit U is derivable with respect to Y on every compact subset Q_λ of Ω, so on Ω, and the sequence $((U_n)_Y)_{n\in\mathbb{N}}$ converges uniformly on Q_λ to the function $(U)_Y$.

Let us put $d_n(X,Y) = U(X,Y) - U_n(X,Y)$. Then

$$U(X,Y) - U_0(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta),U_X(\xi,\eta),U_Y(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= (U-U_n)(X,Y) + (U_n - U_0)(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\cdot,\cdot,U,U_X,U_Y)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= d_n(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\cdot,\cdot,U,U_X,U_Y) - \mathfrak{F}(\cdot,\cdot,U_n,(U_n)_X,(U_n)_Y)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

As for all $(\xi,\eta)\in\mathfrak{D}(X,Y,g)$,

$$\begin{aligned}&|(\mathfrak{F}(\cdot,\cdot,U,U_X,U_Y) - \mathfrak{F}(\cdot,\cdot,U_n,(U_n)_X,(U_n)_Y))(\xi,\eta)|\\ &\leq M\,(|U-U_n| + |U_X - (U_n)_X| + |U_Y - (U_n)_Y|)(\xi,\eta),\end{aligned}$$

the limit of the second member is 0 when n tends to $+\infty$. It follows that

$$U(X,Y) = U_0(X,Y) - \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta),U_X(\xi,\eta),U_Y(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta$$

for $(X,Y)\in Q_\lambda \cap \{(X,Y) : y \geq g(X)\} = Q_\lambda^+$.

Let us show the uniqueness of the solution. Let W be another solution to (1.4). Putting $\Delta = W - U$, we obtain

$$\Delta(X,Y) = \iint\limits_{\mathfrak{D}(X,Y,g)} (\mathfrak{F}(\cdot,\cdot,U,U_X,U_Y) - \mathfrak{F}(\cdot,\cdot,W,W_X,W_Y))(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Let $(X,Y)\in Q_\lambda$. As $\mathfrak{D}(X,Y,g)\subset Q_\lambda$, we have

$$|\Delta(X,Y)| \leq \iint\limits_{\mathfrak{D}(X,Y,g)} M\,(|U-W| + |U_X - W_X| + |U_Y - W_Y|)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

It follows that $E(t) \le M(2\lambda+1+\beta_\lambda)\int_0^t E(\tau)\,\mathrm{d}\tau$, where

$$E(t) \quad = \quad \max_{t=Y-g(X)} \left(|U-W|+|U_X-W_X|+|U_Y-W_Y|\,(X,Y)\right),$$

According to the Grunwall Lemma, $E(t)=0$. The conclusion to be drawn is that U and W are identical. This completes our proof that the solution U of the Cauchy problem is unique on Q_λ^+.

Then putting $v_\lambda(x,y)=U(x+\lambda, y-f(-\lambda))$, it follows that v_λ is the unique solution to (1.4) on $K_\lambda \cap \{(x,y) : y \ge f(x)\} = K_\lambda^+$.

Now consider the case $y \le f(x)$. We make the change of variables $X = -x+\lambda$, $Y=-y+f(\lambda)$ with which we can deal as previously.
It follows that $w_\lambda(x,y) = W(-x+\lambda, -y+f(\lambda))$ is a solution to (1.4) on $K_\lambda \cap \{(x,y) : y \le f(x)\} = K_\lambda^-$.
From the continuity of U on Q_λ^+ and of W on Q_λ^- we have the continuity of v_λ on K_λ^+ and of w_λ on K_λ^-. Moreover, v_λ and w_λ agree on γ because $v_\lambda(x,f(x)) = w_\lambda(x,f(x)) = \varphi(x)$. Finally, if we put

$$u_\lambda(x,y) = \begin{cases} v_\lambda(x,y) \text{ for } (x,y)\in K_\lambda^+, \\ w_\lambda(x,y) \text{ for } (x,y)\in K_\lambda^-, \end{cases}$$

then u_λ is the unique continuous solution to (P_i) on K_λ.

It remains to prove that the method actually gives a continuous global solution u to (1.4) on Ω, that is, which satisfies (P_i). If $\lambda_2 > \lambda_1$ then $K_{\lambda_1} \subset K_{\lambda_2}$, so we must prove that $u_{\lambda_2}|_{K_{\lambda_1}} = u_{\lambda_1}$. But for all $(x,y)\in K_{\lambda_2}$,

$$u_{\lambda_2}(x,y) = u_0(x,y) - \iint\limits_{D(x,y,f)} F(.,.,u_{\lambda_2},(u_{\lambda_2})_x,(u_{\lambda_2})_y)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and we have this equality, all the more so, for $(x,y)\in K_{\lambda_1}$. So $u_{\lambda_2}|_{K_{\lambda_1}}$ satisfies (1.4) on K_{λ_1} and so coincides on it with its unique solution u_{λ_1}. For every $(x,y)\in\Omega$ we can thus put

$$u(x,y) = u_\lambda(x,y) = u_0(x,y) - \iint\limits_{D(x,y,f)} F(.,.,u,u_x,u_y)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{2.3}$$

where u_λ satisfies (1.4) on K_λ and $(x,y)\in K_\lambda \subset \Omega$. The definition of u in (2.3), being independent of the compact subset K_λ, finally gives the unique solution to (P_i) or (P_∞) on Ω. □

Proposition 5.4. *With the previous notations, for every compact subset $K \Subset \Omega$, there exists a compact subset $K_\lambda \Subset \Omega$ containing K, verifying $H2$. Set*

$$\Phi_\lambda = \|F(\cdot,\cdot,0,0,0)\|_{\infty,K_\lambda} + \left(\|u_0\|_{\infty,K_\lambda} + \|(u_0)_x\|_{\infty,K_\lambda} + \left\|(u_0)_y\right\|_{\infty,K_\lambda}\right) M$$

$$\beta_\lambda = \sup_{x\in[-\lambda,\lambda]} \left| \frac{1}{f'(x)} \right| \quad \textit{and} \quad b_\lambda = (\lambda T + T + \lambda)\,\Phi_\lambda$$

We have

$$\|u\|_{\infty,K} \le \|u\|_{\infty,K_\lambda} \le \|u_0\|_{\infty,K_\lambda} + b_\lambda e^{M(2\lambda+1+\beta_\lambda)T}. \tag{2.4}$$

Proof. Keeping the previous notations, we have the result. From the relations

$$\begin{cases} \|v_\lambda\|_{\infty,K_\lambda^+} = \|U\|_{\infty,Q_\lambda}, \\ \|w_\lambda\|_{\infty,K_\lambda^-} = \|W\|_{\infty,Q_\lambda}, \end{cases} \quad \begin{cases} \|u_0\|_{\infty,K_\lambda^+} = \|U_0\|_{\infty,Q_\lambda}, \\ \|u_0\|_{\infty,K_\lambda^-} = \|W_0\|_{\infty,Q_\lambda}, \end{cases} \quad u_\lambda = \begin{cases} v_\lambda \text{ on } K_\lambda^+, \\ w_\lambda \text{ on } K_\lambda^-, \end{cases}$$

it may be deduced that

$$\|u\|_{\infty,K_\lambda^+} \le \|u_0\|_{\infty,K_\lambda^+} + b_\lambda e^{M(2\lambda+1+\beta_\lambda)T};$$
$$\|u\|_{\infty,K_\lambda^-} \le \|u_0\|_{\infty,K_\lambda^-} + b_\lambda e^{M(2\lambda+1+\beta_\lambda)T}.$$

So $\|u\|_{\infty,K_\lambda} \le \|u_0\|_{\infty,K_\lambda} + b_\lambda e^{M(2\lambda+1+\beta_\lambda)T}$. As $\|u\|_{\infty,K} \le \|u\|_{\infty,K_\lambda}$, the previous inequality implies the conclusion (2.4). □

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Received by the editors March 29, 2020
First published online August 14, 2020