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A modified Krasnoselskii-Mann algorithm for equilibrium and fixed point problems for nonexpansive mappings in Hilbert spaces

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Abstract. In this paper, we introduce two iterative shemes (one implicit and one explicit) by a modified Krasnoselskii-Mann algorithm for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive mappings in Hilbert spaces. We prove that both approaches converge strongly to a common element of the set of the equilibrium points and the set of fixed points of nonexpansive mappings. Such common element is the unique solution of a variational inequality, which is the minimum-norm element of the above two sets. Applications to the split feasibility problem and the optimization problem are given. Finally, numerical example is given to demonstrate the implementability of our algorithm.

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1. Introduction

Let H be a real Hilbert space, the associated product is denoted by $\langle ., . \rangle$, the corresponding norm is $\|.\|$ and let K be a nonempty subset of H. A map $T : K \to H$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K, \tag{1.1}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.
We denote by $Fix(T)$ the set of fixed points of the mapping T, that is $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. If T is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex. Historically, one of the most investigated methods of approximating fixed points of nonexpansive mappings dates back to 1953 and is known as Mann's method, in light of Mann [7]. Let C be a nonempty, closed and convex subset of a Banach space X. Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. But Mann's iteration process has only weak convergence, even in Hilbert spaces setting. Therefore, many authors try to modify Mann's iteration to have strong convergence for nonlinear operators (see, for example, [16, 12, 9, 8] and the references contained in them).

In 2017, Qinwei Fana and Zhangsong Yao [6], motivated by the fact that Krasnoselskii-Mann algorithm method is remarkably useful for finding fixed points of nonexpansive mapping, proved the following theorem.

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Theorem 1.1 (Qinwei Fana and Zhangsong Yao [6]). *Let C be a nonempty closed and convex subset of a real Hilbert space H_1 and $\theta \in C$, let $T : C \to C$ be such that $Fix(T) \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in $(0,1)$, the following conditions are satisfied:*
(i) $\lim_{n\to\infty} \alpha_n = 1$; $\lim_{n\to\infty} \beta_n = 1$ $\lim_{n\to\infty} \lambda_n = 1$,
$(ii)|\lambda_n - \beta_{n-1}\lambda_{n-1}| + \beta_n \leq 1$, $\sum_{n=0}^{\infty}(1-\lambda_n)(1-\beta_n) = \infty$, $(iii) \sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, $\sum_{n=0}^{\infty} |\beta_n - \beta_{n+1}| < \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. *Let $\{x_n\}$ be generated by $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = (1-\beta_n)(\lambda_n x_n) + \beta_n y_n, \end{cases} \tag{1.3}$$

Then, the sequence $\{x_n\}$ generated by (1.3) converges strongly to $x^* \in Fix(T)$.

However, we observe that in Theorem 1.1 recursion formula studied is not simpler.
Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the real numbers. The equilibrium problem for f is to find $x \in C$ such that

$$f(x,y) \geq 0, \ \forall y \in C. \tag{1.4}$$

The set of solutions is denoted by $EP(f)$. Equilibrium problems which were introduced by Fan [5] and Blum and Oettli [1] have had a great impact and influence on the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. It has been shown [15, 11] that equilibrium, problems include variational inequalities, fixed points, the Nash equilibrium, and game theory as special cases. A number of iterative algorithms have recently been studying for fixed points and equilibrium problems, see [10, 11, 1] and the references therein.

In 2007, Takahashi-Takahashi [13] investigated Moudafi's viscosity method [9] for finding a common element of the set of solutions of an equilibruim problem and the fixed points set of a nonexpansive mapping in a Hilbert space. They proved the following strong convergence theorem.

Theorem 1.2. *[13] Let C be a nonempty, closed and convex subset a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying the following assumptions:*
$(A1)$ *$F(x,x) = 0$ for all $x \in C$;*
$(A2)$ *F is monotone, i.e., $(Fx,y) + F(y,x) \leq 0$ for all $x, y \in C$;*
$(A3)$ *for each $x, y, z \in C$,*

$$\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$$

$(A4)$ *for each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous.*
Let $f : C \to C$ be a contraction and $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \cap EP(F) \neq \emptyset$.
Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in C$ by:

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset]0, \infty[$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.
(iii) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.
Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.5) converge strongly to $x^* \in Fix(T) \cap EP(F)$.

The above results naturally bring us to the following question.
Question 1: Can we construct an iterative method based on a modified Krasnoselskii-Mann algorithm for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive mappings in Hilbert spaces without imposing any compactness condition on the mapping or the space ?

Our aim in this paper is to give affirmative answer to the question raised. Thus, we introduce and study an implicit and explicit algorithm and prove strong convergence theorems for approximating a common element of the set of solution of equilibrium problems and the set of fixed points of nonexpansive mappings in Hilbert spaces. Applications are also considered. Finally, our method of proof is of independent interest.

2. Preliminaries

We start with the following demiclosedness principle for nonexpansive mappings.

Lemma 2.1 (demiclosedness principle, Browder [2]). *Let H be a real Hilbert space, K be a closed convex subset of H, and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed; that is,*

$$\{x_n\} \subset K,\ x_n \rightharpoonup x \in K \text{ and } (I-T)x_n \to y \text{ implies that } (I-T)x = y.$$

Lemma 2.2 ([4]). *Let H be a real Hilbert space. Then, for any $x, y \in H$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.3 (Xu, [14]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*
$a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n$ *for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:
$(A1)$ $f(x,x) = 0$ for all $x \in C$;
$(A2)$ f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$,

$$\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y);$$

$(A4)$ for each $x \in C$, $y \to f(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.4. *[1] Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \forall y \in C.$$

The following lemma was also given in [2].

Lemma 2.5. *[2] Assume that $f : C \times C \to \mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C,\ f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \ \forall y \in C\},$$

for all $x \in H$. Then, the following hold:

1. T_r *is single-valued;*
2. T_r *is firmly nonexpansive, i.e.,* $\|T_r(x)-T_r(y)\|^2 \leq \langle T_r x - T_r y, x-y\rangle$ *for any* $x, y \in H$*;*
3. $Fix(T_r) = EP(f)$*;*
4. $EP(f)$ *is closed and convex.*

Lemma 2.6. *Let H be a real Hilbert space, K a nonempty, closed and convex subset of H. Let $S : K \to K$ be a mapping such that $F := EP(f) \cap Fix(S) \neq \emptyset$. Then,*

$$\langle x - ST_r x,\ x - p\rangle \geq 0, \quad \forall x \in K,\ p \in F.$$

Proof. Using the Schwartz inequality and properties of S and T_r , we obtain

$$\begin{aligned}\langle x - ST_r x,\ x - p\rangle &= \langle x - ST_r x + p - p,\ x - p\rangle \\ &= \|x-p\|^2 - \langle ST_r x - p,\ x - p\rangle \\ &\geq \|x-p\|^2 - \|ST_r x - p\|\|x-p\| \\ &\geq \|x-p\|^2 - \|T_r x - T_r p\|\|x-p\| \\ &\geq \|x-p\|^2 - \|x-p\|^2 \geq 0.\end{aligned}$$

Hence, $\langle x - ST_r x,\ x - p\rangle \geq 0$. □

3. Implicit Method

We start with the following result.

Lemma 3.1. *Let K be a nonempty, closed convex cone of a real Hilbert space H and $S : K \to K$ be a nonexpansive mapping. Let λ be a constant in $(0,1)$ and $\{r_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $\lim_{t\to 0}\inf r_t > 0$. Let $\{T_{r_t}\}$ be a mapping defined as in Lemma 2.5. Then, for each $t \in (0,1)$, there exists $z_t \in K$ such that*

$$z_t = t(\lambda z_t) + (1-t)ST_{r_t} z_t.$$

Proof. For each $t \in (0,1)$, define the mapping $S_t : K \to CB(K)$ by

$$S_t x = t(\lambda x) + (1-t)ST_{r_t} x,\ \forall x \in K.$$

We show that S_t is a contraction. For this, let $x, y \in K$. We have

$$\begin{aligned}\|S_t x - S_t y\| &= \|[t(\lambda x) + (1-t)ST_{r_t} x] - [t(\lambda y) + (1-t)ST_{r_t} y]\| \\ &\leq t\lambda\|x-y\| + (1-t)\|T_{r_t} x - T_{r_t} y\| \\ &\leq [1-(1-\lambda)t]\|x-y\|.\end{aligned}$$

Therefore, S_t is a contraction. Using Banach's contraction principle, there exists z_t in K, such that

$$z_t = t(\lambda z_t) + (1-t)ST_{r_t} z_t. \tag{3.1}$$

□

We now prove the following theorem.

Theorem 3.2. *Let K be a nonempty, closed convex cone of a real Hilbert space H. Let f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4), let $S : K \to K$ be a nonexpansive mapping such that $F := EP(f) \cap Fix(S) \neq \emptyset$ and λ be a constant in $(0,1)$. Let $\{z_t\}$ and $\{u_t\}$ be defined implicitly by:*

$$\begin{cases} f(u_t, y) + \dfrac{1}{r_t}\langle y - u_t, u_t - z_t\rangle \geq 0,\ \forall y \in K, \\ z_t = t(\lambda z_t) + (1-t)Su_t. \end{cases} \tag{3.2}$$

Then as $t \to 0$, the net $\{z_t\}$ defined by (3.2) converges strongly to $x^* \in F$, where x^* is the minimum-norm element of F.

Proof.
We split the proof into four steps.
Step 1. We prove that $\{z_t\}$ is bounded. Let $p \in F$. Then from $u_t = T_{r_t} z_t$, we have

$$\|u_t - p\| = \|T_{r_t} z_t - T_{r_t} p\| \leq \|z_t - p\|.$$

Using (3.2) and the fact that S is nonexpansive, we have

$$\begin{aligned}
\|z_t - p\| &= \|t(\lambda z_t) + (1-t)Su_t - p\| \\
&\leq \lambda t\|z_t - p\| + (1-t)\|Su_t - p\| + t(1-\lambda)\|p\| \\
&\leq \lambda t\|z_t - p\| + (1-t)\|Su_t - Sp\| + t(1-\lambda)\|p\| \\
&\leq [1-(1-\lambda)t]\|z_t - p\| + t(1-\lambda)\|p\|,
\end{aligned}$$

which implies that

$$\|z_t - p\| \leq \|p\|.$$

Hence, $\{z_t\}$ is bounded and so is $\{Su_t\}$.
Step 2. We show that $\{z_t\}$ is relatively norm compact as $t \to 0$. Using (3.2) and the boundeness of $\{z_t\}$, we have

$$\|z_t - Su_t\| = t\|\lambda z_t - Su_t\| \to 0, \quad \text{as } t \to 0. \tag{3.3}$$

For $p \in F$, we have

$$\begin{aligned}
\|u_t - p\|^2 &= \|T_{r_t} z_t - T_{r_t} p\|^2 \\
&\leq \langle T_{r_t} z_t - T_{r_t} p, z_t - p\rangle \\
&\leq \langle u_t - p, z_t - p\rangle \\
&= \frac{1}{2}(\|u_t - p\|^2 + \|z_t - p\|^2 - \|z_t - u_t\|^2)
\end{aligned}$$

and hence

$$\|u_t - p\|^2 \leq \|z_t - p\|^2 - \|z_t - u_t\|^2. \tag{3.4}$$

Therefore, from (3.2) and (3.4), we get that

$$\begin{aligned}
\|z_t - p\|^2 &= \|t(\lambda z_t) + (1-t)Su_t - p\|^2 \\
&\leq \|t((\lambda z_t) - p) + (1-t)(Su_t - p)\|^2 \\
&\leq (1-t)^2\|Su_t - p\|^2 + 2t\langle(\lambda z_t) - p, z_t - p\rangle \\
&\leq (1-t)^2\|u_t - p\|^2 + 2t\lambda\langle z_t - p, z_t - p\rangle + 2(1-\lambda)t\langle p, p - z_t\rangle \\
&\leq (1-t)^2(\|z_t - p\|^2 - \|z_t - u_t\|^2) + 2t\lambda\|z_t - p\|^2 \\
&\quad +2\alpha_t(1-\lambda)\|p\|\|z_t - p\| \\
&= (1-2t+t^2)\|z_t - p\|^2 - (1-t)^2\|z_t - u_t\|^2 + 2t\lambda\|z_t - p\|^2 \\
&\quad +2(1-\lambda)t\|p\|\|z_t - p\| \\
&\leq \|z_t - p\|^2 + t\|z_t - p\|^2 - (1-t)^2\|z_t - u_t\|^2 + 2t\lambda\|z_t - p\|^2 \\
&\quad +2(1-\lambda)t\|p\|\|z_t - p\|,
\end{aligned}$$

and hence

$$(1-t)^2\|z_t - u_t\|^2 \leq t\|z_t - p\|^2 + 2t\lambda\|z_t - p\|^2 + 2(1-\lambda)t\|p\|\|z_t - p\|.$$

So, we have $\|z_t - u_t\| \to 0$, as $t \to 0$. Since $\|Su_t - u_t\| \leq \|z_t - Su_t\| + \|z_t - u_t\|$, it follows that

$$\lim_{t\to 0}\|Su_t - u_t\| = 0. \tag{3.5}$$

Let $p \in F$. From (3.2), we have

$$\begin{aligned}
\|z_t - p\|^2 &= \langle t(\lambda z_t) + (1-t)Su_t - p, z_t - p\rangle \\
&= t\lambda\langle z_t - p, z_t - p\rangle + (1-t)\langle Su_t - p, z_t - p\rangle \\
&\quad -(1-\lambda)t\langle p, z_t - p\rangle \\
&\leq [1-(1-\lambda)t]\|z_t - p\|^2 - (1-\lambda)t\langle p, z_t - p\rangle.
\end{aligned}$$

So,

$$\|z_t - p\|^2 \leq \langle p, p - z_t\rangle. \tag{3.6}$$

Now, let $\{t_n\} \subseteq (0,1)$ be a sequence such that $t_n \to \infty$ as $n \to \infty$. Set $z_n := z_{t_n}$ and $u_n := u_{t_n}$. Since H is reflexive and $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which converges weakly to $x^* \in K$. From (3.5) and Lemma 2.2, we obtain $x^* \in Fix(S)$. Let us show $x^* \in EP(f)$. It follows by (3.2) and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq f(y, u_n)$$

and hence

$$\langle y - u_{n_k}, \frac{u_{n_k} - z_{n_k}}{r_{n_k}}\rangle \geq f(y, u_{n_k}).$$

Since $\dfrac{u_{n_k} - z_{n_k}}{r_{n_k}} \to 0$ and $u_{n_k} \rightharpoonup x^*$, it follows $(A4)$ that $f(y, x^*) \leq 0$ for all $y \in K$. For t with $0 < t < 1$ and $y \in K$, let $y_t = ty + (1-t)x^*$. Since $y \in K$ and $x^* \in K$, we have $y_t \in K$ and hence $f(y_t, x^*) \leq 0$. So, from $(A1)$ and $(A4)$ we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x^*) \leq tf(y_t, y)$$

and hence $0 \leq f(y_t, y)$. From $(A3)$, we have $f(x^*, y) \geq 0$ for all $y \in K$ and hence $x^* \in EP(f)$. Therefore, $x^* \in Fix(S) \cap EP(f) = F$.
Since $z_{n_k} \rightharpoonup x^*$ as $k \to \infty$, it follows from (3.6) that $z_{n_k} \to x^*$ as $k \to \infty$. This proves the relative compactness of the net $\{z_t\}$.
Step 3. We show that the entire net $\{z_t\}$ converges strongly to $x^* \in F$. We claim that the net $\{z_t\}$ has a unique cluster point. From **Step 2**, the net $\{z_t\}$ has a cluster point. Now suppose that $x^* \in K$ and $x^{**} \in K$ are two cluster points of $\{z_t\}$. Let $\{z_{n_k}\}$ and $\{z_{n_p}\}$ be two subsequences of $\{z_n\}$ such that $z_{n_k} \to x^{**}$, as $k \to \infty$ and $z_{n_p} \to x^{**}$, as $p \to \infty$.

Following the same arguments as in **Step 2**, it follows that $x^*, x^{**} \in F$, and the following estimates hold:

$$\|z_{n_k} - x^{**}\|^2 \leq \langle x^{**}, x^{**} - z_{nk}\rangle, \tag{3.7}$$

and

$$\|z_{n_p} - x^*\|^2 \leq \langle x^*, x^* - z_{np}\rangle. \tag{3.8}$$

Letting $k \to \infty$ and $p \to \infty$ in (3.7) and (3.8) gives

$$\|x^* - x^{**}\|^2 \leq \langle x^{**}, x^{**} - x^*\rangle \tag{3.9}$$

and

$$\|x^{**} - x^*\|^2 \leq \langle x^*, x^* - x^{**}\rangle. \tag{3.10}$$

Adding up (3.9) and (3.10) yields

$$2\|x^* - x^{**}\|^2 \leq \|x^* - x^{**}\|^2,$$

which implies that $x^* = x^{**}$.
Step 4. Finally, we show that x^* is the minimum-norm element of F.

Following the same arguments as in **Step 3**, it follows that

$$\|x^* - p\|^2 \leq \langle -p, x^* - p\rangle, \ \ \forall p \in F.$$

Equivalently,

$$\|x^*\|^2 \leq \langle p, x^*\rangle, \ \ \forall p \in F.$$

This clearly implies that

$$\|x^*\| \leq \|p\|, \ \forall p \in F.$$

Therefore, x^* is the minimum-norm element of F. This completes the proof. □

We now apply Theorem 3.2 for solving variational inequality problems.

Theorem 3.3. *The net $\{z_t\}$ defined by (3.2) converges strongly to a unique solution of the following variational inequality*

$$\langle x^*, x^* - p\rangle \le 0, \quad \forall p \in F. \tag{3.11}$$

Proof. It follows from (3.2) that,

$$z_t = -\frac{1-t}{(1-\lambda)t}(z_t - Su_t).$$

Using Lemma 2.6, for any $p \in F$, we have

$$\langle z_t, z_t - p\rangle = -\frac{1-t}{(1-\lambda)t}\langle z_t - Su_t,\, z_t - p\rangle \le 0.$$

Letting $t \to 0$, noting the fact that $z_t \to x^*$, we obtain

$$\langle x^*, x^* - p\rangle \le 0. \tag{3.12}$$

Finally, we show that the uniqueness of the solution of the variational inequality (3.11). Suppose both $x^* \in F$ and $x^{**} \in F$ are solutions to (3.11), then

$$\langle x^*, x^* - x^{**}\rangle \le 0 \tag{3.13}$$

and

$$\langle x^{**}, x^{**} - x^*\rangle \le 0 \tag{3.14}$$

Adding up (3.13) and (3.14) yields

$$\langle x^{**} - x^*, x^{**} - x^*\rangle \le 0, \tag{3.15}$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. □

4. Explicit Method

We now apply Theorems 3.2 and 3.3 to find a common element of the set of fixed points of nonexpansive mappings and the set of solutions of equilibrium problems.

In what follows, we use the following explicit scheme: let K be a nonempty, closed convex cone of a real Hilbert space H and $S : K \to K$ be a nonexpansive mapping.
Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:

$$\begin{cases} f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in K \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Su_n, \ \ n \ge 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ and $\{r_n\} \subset]0,\infty[$ satisfying:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$; (ii) $\sum\limits_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim\limits_{n\to\infty} \lambda_n = 1$;

(iii) $\lim\limits_{n\to\infty} \inf r_n > 0$ and $\sum\limits_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$;

(iv) $\sum\limits_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ and $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

Theorem 4.1. *Let K be a nonempty, closed convex cone of a real Hilbert space H. Let f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4), let $S : K \to K$ be a nonexpansive mapping such that $F := EP(f) \cap Fix(S) \ne \emptyset$. Then, $\{x_n\}$ and $\{u_n\}$ defined by (4.1) converge strongly to $x^* \in F$, where x^* is the minimum-norm element of F.*

Proof. We prove that the sequence $\{x_n\}$ is bounded. Let $p \in F$. Then from $u_n = T_{r_n}x_n$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|, \ \ \forall n \geq 0.$$

From (4.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)Su_n - p\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| + (1-\alpha_n)\|Su_n - p\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| + (1-\alpha_n)\|Su_n - Sp\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| + (1-\alpha_n)\|x_n - p\| \\
&= [1-(1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\|.
\end{aligned}$$

$$\|x_{n+1} - p\| \leq \max\{\|x_n - p\|, \|p\|\}. \tag{4.2}$$

Hence, $\{x_n\}$ and $\{Su_n\}$ are bounded.

From (4.1), it follows that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)Su_n - \alpha_{n-1}(\lambda_{n-1}x_{n-1}) - (1-\alpha_{n-1})Su_{n-1}\| \\
&= \|\alpha_n\lambda_n(x_n - x_{n-1}) + \alpha_n(\lambda_n - \lambda_{n-1})x_{n-1} + (\alpha_n - \alpha_{n-1})(\lambda_{n-1}x_{n-1}) \\
&\quad + (1-\alpha_n)(Su_n - Su_{n-1}) + (\alpha_{n-1} - \alpha_n)Su_{n-1}\| \\
&\leq \alpha_n\lambda_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|Su_n - Su_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\| \\
&\quad + \|Su_{n-1}\|) + \alpha_n|\lambda_n - \lambda_{n-1}|\|x_{n-1}\| \\
&\leq \alpha_n\lambda_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\| + \|Su_{n-1}\|) + \alpha_n|\lambda_n - \lambda_{n-1}|\|x_{n-1}\|.
\end{aligned}$$

Hence,

$$\|x_{n+1} - x_n\| \leq \alpha_n\lambda_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \alpha_n|\lambda_n - \lambda_{n-1}|)M_1, \tag{4.3}$$

where $M_1 > 0$ is such that $\sup_n\{\|x_{n-1}\| + \|Su_{n-1}\|\} \leq M_1$.

On other hand, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \tag{4.4}$$

and

$$f(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{4.5}$$

Putting $y = u_{n+1}$ in (4.4) and $y = u_n$ in (4.5), we have

$$f(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$f(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from $(A2)$, we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_n}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + \Big(1 - \frac{r_n}{r_{n+1}}\Big)(u_{n+1} - x_{n+1})\rangle \\
&\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\},
\end{aligned}$$

and hence

$$\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|.$$

This implies that

$$\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L, \tag{4.6}$$

where $L > 0$ is such that $\sup_n\{\|u_{n+1} - x_{n+1}\|\} \le L$.

So, from (4.3) we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \alpha_n\lambda_n\|x_n - x_{n-1}\| + (1-\alpha_n)(\|x_n - x_{n-1}\| \\
&+ \frac{1}{b}|r_n - r_{n-1}|L) + (|\alpha_n - \alpha_{n-1}| + \alpha_n|\lambda_n - \lambda_{n-1}|)M_1 \\
&= [1 - (1-\lambda_n)\alpha_n]\|x_n - x_{n-1}\| + (1-\alpha_n)\frac{1}{b}|r_n - r_{n-1}|L \\
&+ (|\alpha_n - \alpha_{n-1}| + \alpha_n|\lambda_n - \lambda_{n-1}|)M_1 \\
&= [1 - (1-\lambda_n)\alpha_n]\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|L + (|\alpha_n - \alpha_{n-1}| \\
&\quad +\alpha_n|\lambda_n - \lambda_{n-1}|)M_1.
\end{aligned}$$

Using Lemma 2.3, we deduce $\lim_{n\to+\infty}\|x_{n+1}-x_n\| \to 0$. From (4.6) and $\lim_{n\to+\infty}|r_n - r_{n-1}| \to 0$, we have

$$\lim_{n\to+\infty}\|u_{n+1} - u_n\| = 0.$$

Since $x_n = \alpha_{n-1}(\lambda_{n-1}x_{n-1}) + (1-\alpha_{n-1})Su_{n-1}$, we have

$$\begin{aligned}
\|x_n - Su_n\| &\le \|x_n - Su_{n-1}\| + \|Su_{n-1} - Su_n\| \\
&\le \alpha_{n-1}\|\lambda_{n-1}x_{n-1} - Su_{n-1}\| + \|u_{n-1} - u_n\|.
\end{aligned}$$

From $\alpha_n \to 0$, as $n \to \infty$, we obtain, $\lim_{n\to+\infty}\|x_n - Su_n\| = 0$. For $p \in F$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\
&\le \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2)
\end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Therefore, from (4.1) and Lemma 2.1, we get that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)Su_n - p\|^2 \\
&\le \|\alpha_n((\lambda_n x_n) - p) + (1-\alpha_n)(Su_n - p)\|^2 \\
&\le (1-\alpha_n)^2\|Su_n - p\|^2 + 2\alpha_n\langle(\lambda_n x_n) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)^2\|u_n - p\|^2 + 2\alpha_n\lambda_n\langle x_n - p, x_{n+1} - p\rangle + 2(1-\lambda_n)\alpha_n\langle p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)^2(\|x_n - p\|^2 - \|x_n - u_n\|^2) \\
&\quad +2\alpha_n\lambda_n\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1} - p\| \\
&\le (1 - 2\alpha_n + \alpha_n^2)\|x_n - p\|^2 - (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\lambda_n\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1} - p\| \\
&\le \|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 - (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\lambda_n\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1} - p\|,
\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n)^2\|x_n-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|x_n-p\|^2\\ &\quad +2\alpha_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n\|p\|\|x_{n+1}-p\|\\ &\leq \|x_{n+1}-x_n\|\{\|x_n-p\|+\|x_{n+1}-p\|\}\\ &\quad +\alpha_n\|x_n-p\|^2+2\alpha_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n\|p\|\|x_{n+1}-p\|.\end{aligned}$$

So, we have $\|x_n-u_n\|\to 0$, as $n\to\infty$. Since $\|Su_n-u_n\|\leq\|x_n-Su_n\|+\|x_n-u_n\|$, it follows that

$$\lim_{n\to\infty}\|Su_n-u_n\|=0. \tag{4.7}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*,x^*-x_n\rangle\leq 0$, where $x^*=\lim_{t\to 0}z_t$.

We choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that:

$$\limsup_{n\to+\infty}\langle x^*,x^*-x_n\rangle=\lim_{k\to+\infty}\langle x^*,x^*-x_{n_k}\rangle.$$

Since H is reflexive and $\{u_{n_k}\}$ is bounded, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ which converges weakly to $a\in K$. From (4.7) and Lemma 2.1, we obtain $a\in Fix(S)$. Without loss of generality, we can assume that $u_{n_k}\rightharpoonup a$. By the same argument as in the proof of Theorem 3.2, we have $a\in Fix(S)\cap EP(f)=F$. Using Theorem 3.3, we have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^*,x^*-x_n\rangle &= \lim_{k\to+\infty}\langle x^*,x^*-x_{n_k}\rangle\\ &= \langle x^*,x^*-a)\rangle\leq 0.\end{aligned}$$

Finally, we show that $x_n\to x^*$. From (4.1) and Lemma 2.2, we get that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*,x_{n+1}-x^*\rangle=\alpha_n\lambda_n\langle x_n-x^*,x_{n+1}-x^*\rangle\\ &+ (1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle+(1-\alpha_n)\langle Su_n-x^*,x_{n+1}-x^*\rangle\\ &\leq \alpha_n\lambda_n\langle x_n-x^*,x_{n+1}-x^*\rangle\\ &+ (1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle+(1-\alpha_n)\|Su_n-x^*\|\|x_{n+1}-x^*\|\\ &\leq \alpha_n\lambda_n\langle x_n-x^*,x_{n+1}-x^*\rangle\\ &+ (1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle+(1-\alpha_n)H(Su_n,Sx^*)\|x_{n+1}-x^*\|\\ &\leq \alpha_n\lambda_n\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &+ (1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle+(1-\alpha_n)\|u_n-x^*\|\|x_{n+1}-x^*\|\\ &\leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|\|x_{n+1}-x^*\|+(1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle\\ &\leq \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\ &\quad +(1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1}-x^*\|^2\leq[1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|+2(1-\lambda_n)\alpha_n\langle x^*,x^*-x_{n+1}\rangle.$$

We can check that all assumptions of Lemma 2.3 are satisfied. Therefore, we deduce $x_n\to x^*$. This completes the proof. □

Corollary 4.2. *Let K be a nonempty, closed convex cone of a real Hilbert space H, let $S:K\to K$ be a nonexpansive mapping such that $Fix(S)\neq\emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0\in K$ by:*

$$x_{n+1}=\alpha_n(\lambda_n x_n)+(1-\alpha_n)Sx_n,\ \ n\geq 0, \tag{4.8}$$

where $\{\alpha_n\}\subset(0,1)$, and $\{\lambda_n\}\subset(0,1)$ satisfying:

(i) $\lim_{n\to\infty}\alpha_n=0$; *(ii)* $\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$; $\lim_{n\to\infty}\lambda_n=1$;

(*iii*) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty, \quad \sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty.$

Then, $\{x_n\}$ converges strongly to $x^* \in Fix(S)$, where x^* is the minimum-norm element of $Fix(S)$.

Proof. Put $f(x,y) = 0$ for all $x, y \in K$ and $r_n = 1$, we get $u_n = x_n$ in Theorem 4.1. The proof follows from Theorem 4.1. □

Remark 4.3. Recursion formula (4.8) is simpler than those of Qinwei Fana and Zhangsong Yao [6].

Remark 4.4. In our theorems, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$. Therefore, our results can be used to approximate common element of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive mappings from the closed unit ball to itself.

Corollary 4.5. *Let H be a real Hilbert space. Let B be the closed unit ball of H. Let f be a bifunction from $B \times B \to \mathbb{R}$ satisfying (A1)-(A4), let $S : B \to B$ be a nonexpansive mapping such that $F := EP(f) \cap Fix(S) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in B$ by:*

$$(4.9) \qquad \begin{cases} f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in B \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Su_n, \ \ n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ and $\{r_n\} \subset]0,\infty[$ satisfy:

(*i*) $\lim_{n\to\infty} \alpha_n = 0$; (*ii*) $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim_{n\to\infty} \lambda_n = 1$;

(*iii*) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\sum_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$

(*iv*) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty.$

Then, $\{x_n\}$ and $\{u_n\}$ defined by (4.9) converge strongly to $x^* \in F$, where x^* is the minimum-norm element of F.

5. Applications

The split feasibility problem. In this section, we apply our main results to solving the split feasiblity problem. The split feasibility problem (SFP) was first introdued by Censor and Elfving [3] in 1994. The SFP is to find

$$(5.1) \qquad x \in K, \text{ such that } \ Ax \in Q,$$

where K is a nonempty, closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.
The problem (5.1) arises in signal processing and image reconstruction with particular progress in intensity modulated therapy, and many iterative algorithms has been established for it (see e.g [1, 2, 15]and the reference therein). Let Ω be the solution set of the split feasibility problem.
From an optimization point of view, $x^* \in \Omega$ if and only if x^* is a solution of the following minimization problem with zero optimal value:

$$\min_{x\in K} f(x), \text{ where } f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2.$$

The following lemma appears in [4].

Lemma 5.1. *Given $x^* \in H$, then x^* solves SFP (5.1) if and only if x^* is the solution of the fixed point equation $x = P_K(I - \gamma A^*(I - P_Q)A)x$, where $\gamma > 0$ is a suitable constant.*

Proposition 5.2. *[6] Let K be a nonempty, closed and convex subset of a Hilbert space H_1, Q be a a nonempty, closed and convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator. Let P_K, P_Q denote the orthogonal projection onto set K, Q respectively. Let $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral raduis of A^*A, and A^* is the adjoint of A. Then, the operator $S := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K.*

Theorem 5.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, and $A^* : H_2 \to H_1$ be the adjoint operator of A. Let K be a nonempty, closed convex cone of a Hilbert space H_1 and f be a bifunction from $K \times K \to \mathbb{R}$ satisfying $(A1)$-$(A4)$ such that $F := EP(f) \cap \Omega \neq \emptyset$. Let $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral raduis of A^*A. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in K \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)P_K(I - \gamma A^*(I - P_Q)A)u_n, \quad n \geq 0, \end{cases} \tag{5.2}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim_{n\to\infty} \lambda_n = 1$;

(iii) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\sum_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$

(iv) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.

Then, $\{x_n\}$ and $\{u_n\}$ defined by (5.2) converge strongly to $x^* \in F$, where x^* is the minimum-norm element of a common element of the set of solutions of equilibrium problems and the set of solutions of split feasibility problems.

Proof. From Lemma 5.1, we know $x^* \in \Omega$ if and only if $x^* = P_K(I - \gamma A^*(I - P_Q)A)x^*$. From Proposition 5.2, we have that the operator $S := P_K(I-\gamma A^*(I-P_Q)A)$ is nonexpansive on K. Using, Theorem 4.1, we can obtain that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to a solution of (5.1). □

Optimization problem. We now study the following optimization problem:

$$\min_{x\in K} h(x). \tag{5.3}$$

where K is a nonempty closed convex cone of a real Hilbert space H and $h : K \to \mathbb{R}$ is a convex and a lower semi-continuous functional. Let us denote the set of solutions to (5.3) by Ω_1. Let $f : K \times K \to \mathbb{R}$ be defined by $f(x, y) := h(y) - h(x)$. Let us now find the following equilibrium problem: find $x \in K$ such that

$$f(x, y) \geq 0, \tag{5.4}$$

for all $y \in K$. It is obvious that f satisfies conditions $(A1)$-$(A4)$ and $EP(f) = \Omega_1$. By Theorem 4.1, we have the following theorem.

Theorem 5.4. *Let K be a nonempty, closed convex cone of a real Hilbert space H. Let $h : K \to \mathbb{R}$ is a convex and a lower semi-continuous functional. Let $S : K \to K$ be a nonexpansive mapping such that $F := \Omega_1 \cap Fix(S) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} h(y) - h(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in K, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)Su_n, \ n \geq 0, \end{cases} \tag{5.5}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$; $\lim_{n\to\infty} \lambda_n = 1$;

(iii) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} (1 - \lambda_n)\alpha_n = \infty$.

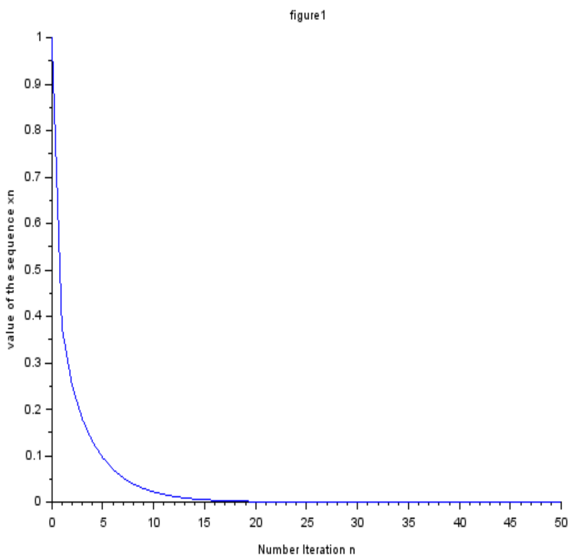
Then, the sequence $\{x_n\}$ *defined by* (5.5) *converges strongly to a solution of optimization problem* (5.3).

6. Numerical example

In this last section, we discuss the direct application of Theorem 4.1 on a real line. Consider the following:
$H = \mathbb{R}$, $K = [0, 1]$, $f(x, y) := y^2 + yx - 2x^2$, $Sx = \frac{1}{2}x$ and $T_r(x) = \{z \in K,\ f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K\}$. We can observe that $T_r(x) = \frac{1}{1+3r}x$ and $0 \in Fix(S) \cap EP(f)$. Choose $r = 1$, $\alpha_n = \frac{1}{\sqrt{n+1}}$ and $\lambda_n = 1 - \frac{1}{\sqrt{n+1}}$. Then, the scheme (4.1) can be simplified as

$$\begin{cases} u_n = \frac{1}{4}x_n, \\ x_{n+1} = \frac{\sqrt{n+1}-1}{n+1}x_n + \frac{\sqrt{n+1}-1}{2\sqrt{n+1}}u_n, \quad n \geq 0. \end{cases} \tag{6.1}$$

Take the initial point $x_0 = 1$, the numerical experiment result using MATLAB is given by Figure 1, which shows the iteration process of the sequence $\{x_n\}$ converge strongly to 0.



7. Conclusion

In this paper, we introduce and study an iterative method based on a modified Krasnoselskii-Mann algorithm for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive mappings in Hilbert spaces without imposing any compactness-type condition. This method can be applied in solving the relevant problem, such as optimization problem, the split feasiblity problem (SFF), and so on.

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