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On split equality for finite family of generalized mixed equilibrum problem and fixed point problem in real Banach spaces

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Abstract. The purpose of this paper is to introduce a simultaneous iterative algorithm for solving split equality for systems of generalized mixed equilibrium problem and split equality fixed point problem in p-uniformly convex and uniformly smooth Banach spaces using the Bregmann distance technique. Furthermore, we state and prove a strong convergence theorem for the approximation of a solution of split equality for systems of generalized mixed equilibrium problem and split equality fixed point problem in the framework of p-uniformly convex and uniformly smooth Banach spaces. Our result extends results on split equality generalized mixed equilibrium problems from Hilbert spaces to p-uniformly convex Banach spaces which are also uniformly smooth.

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1. Introduction

Let E be a p-uniformly convex and uniformly smooth Banach space, and C a nonempty, closed and convex subset of E. Throughout this paper, we shall denote the dual space of E by E^*. The norm and the duality pairing between E and E^* are denoted by $\|.\|$ and $\langle ., . \rangle$, respectively, and $\mathbb{R}$ stands for the set of real numbers. Let $f : E \to (-\infty, \infty]$ be a proper convex and lower semicontinuous functional. The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(\xi) = sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

Let $T : C \to C$ be a mapping, a point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$.

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Let $g : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function and $\Phi : C \to E^*$ be a nonlinear mapping. The *Generalized Mixed Equilibrium Problem* (GMEP) is to find $u \in C$ such that

$$g(u, y) + \langle \Phi u, y - u\rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote the set of solutions of Problem (1.1) by $GMEP(g, \Phi, \varphi)$. That is

$$GMEP(g, \Phi, \varphi) = \{u \in C : g(u, y) + \langle \Phi u, y - u\rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C\}.$$

If $\Phi = 0$, then the GMEP (1.1) reduces to the following *mixed equilibrium problem*: Find $u \in C$ such that

$$g(u, y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C.$$

If $\varphi = 0$, then the GMEP (1.1) becomes the *generalized equilibrium problem*, to find $u \in C$ such that

$$g(u, y) + \langle \Phi u, y - u\rangle \geq 0, \quad \forall y \in C.$$

Again if $\Phi = \varphi = 0$, then the GMEP (1.1) becomes the *equilibrium problem*, to find $u \in C$ such that

$$g(u, y) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was first introduced by Blum and Oettli [4], who denoted the solution set of (1.2) as $EP(g)$.

For solving equilibrium problem (1.2), the bifunction g is assumed to satisfy the following conditions:
(A1) $g(x, x) = 0$ for all $x \in C$;
(A2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C$, $\lim_{t\to 0} g(tz + (1 - t)x, y) \leq g(x, y)$;
(A4) for each $x \in C$; $y \mapsto g(x, y)$ is convex and lower semicontinuous.
Many mathematicians have found the study of equilibrium problems very interesting as it has been observed that the equilibrium problems and their generalizations have been widely applied to solve problems in various fields such as: linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed point problems and have also been widely applied to physics, structural analysis, management sciences, economics, etc (see, for example [4, 6, 27, 26]).
Many authors have proposed some efficient and implementable algorithms and obtained some convergence theorems for solving equilibrium problems, some of their generalizations and related optimization problems, (see for example, [1, 3, 6, 8, 9, 10, 11, 12, 14, 15, 18, 19, 20, 21, 22, 30, 31, 32, 33, 34, 36, 37] and the references therein).

Authors have started to study the Split Equilibrium Problem (SEP) defined as follows: Let H_1, H_2 be two real Hilbert spaces, let C, Q be closed convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_2$ a bounded linear operator. Let $g_1 : C \times C \to \mathbb{R}$, $g_2 : Q \times Q \to \mathbb{R}$ be bifunctions, $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$,

$\varphi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be functions and $\Phi_1 : C \to H_1$, $\Phi_2 : Q \to H_2$ be nonlinear mappings. Then the *split generalized mixed equilibrium problem* is to find $x^* \in C$ such that

$$g_1(x^*, x) + \langle \Phi_1 x^*, x - x^* \rangle + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall x \in C, \tag{1.3}$$

and $y^* = Ax^* \in Q$ solves

$$g_2(y^*, y) + \langle \Phi_2 y^*, y - y^* \rangle + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall y \in Q, \tag{1.4}$$

We shall denote the solution set of (1.3)-(1.4) by

$$\Omega_1 = \{x^* \in GMEP(g_1, \Phi_1, \varphi_1) : Ax^* \in GMEP(g_2, \Phi_2, \varphi_2)\}.$$

If $\Phi_1 = 0$ and $\Phi_2 = 0$, then (1.3)-(1.4) reduces to the following split mixed equilibrium problem: Find $x^* \in C$ such that

$$g_1(x^*, x) + \varphi_1(x) - \varphi_1(x^*) \geq 0, \quad \forall x \in C, \tag{1.5}$$

and $y^* = Ax^* \in Q$ solves

$$g_2(y^*, y) + \varphi_2(y) - \varphi_2(y^*) \geq 0, \quad \forall y \in Q, \tag{1.6}$$

with solution set $\Omega_\varphi = \{x^* \in MEP(g_1, \varphi_1) : Ax^* \in MEP(g_2, \varphi_2)\}$. Again in (1.3)-(1.4) if $\varphi_1 = \varphi_2 = 0$, we obtain the following split generalized equilibrium problem: Find $x^* \in C$ such that

$$g_1(x^*, x) + \langle \Phi_1 x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.7}$$

and $y^* = Ax^* \in Q$ solves

$$g_2(y^*, y) + \langle \Phi_2 y^*, y - y^* \rangle \geq 0, \quad \forall y \in Q, \tag{1.8}$$

with solution set $\Omega_\Phi = \{x^* \in GEP(g_1, \Phi_1) : Ax^* \in GEP(g_2, \Phi_2)\}$. Moreover, if $\Phi_1 = \Phi_2$ and $\varphi_1 = \varphi_2 = 0$, we have the split equilibrium problem, to find $x^* \in C$ such that

$$g_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.9}$$

and $y^* = Ax^* \in Q$ solves

$$g_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{1.10}$$

with solution set $\Omega_0 = \{x^* \in EP(g_1) : Ax^* \in EP(g_2)\}$.
Kazmi and Rizvi [13] studied the pair of equilibrium problems (1.9) and (1.10) called split equilibrium problem.

Recently Bnouhachem [5] stated and proved the following strong convergence result.

Theorem 1.1. *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subset H_1$ and $Q \subset H_2$ be nonempty closed and convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying $(A1) - (A4)$ and f_2 is upper semicontinuous in the first argument. Let $S, T : C \to C$ be nonexpansive mappings such that $\Omega_0 \cap F(T) \neq \emptyset$. Let $f : C \to C$ be a k-Lipschitzian mapping and η-strongly monotone and let $U : C \to C$ be τ-Lipschitzian mapping. For a given arbitrary $x_0 \in C$, let the iterative sequence $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by*

$$(1.11) \begin{cases} u_n = T_{r_n}^{f_1}(x_n + \gamma A^*(T_{r_n}^{f_2} - I)Ax_n); \\ y_n = \beta_n S x_n + (1 - \beta_n)u_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu f)(T(y_n))], \quad \forall n \geq 0; \end{cases}$$

where $\{r_n\} \subset (0, 2\zeta)$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A. Suppose the parameters satisfy $0 < \mu < \left(\frac{2\eta}{k^2}\right)$, $0 \leq \rho\eta < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty}(\frac{\beta_n}{\alpha_n}) = 0$,
(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$
(d) $\liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\zeta$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.
Then $\{x_n\}$ converges strongly to $z \in \Omega_0 \cap F(T)$.

Let E_1, E_2 and E_3 be three real Banach spaces and C, Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $g_1 : C \times C \to \mathbb{R}$ and $g_2 : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1 : C \to E_1^*$ and $\Phi_2 : Q \to E_2^*$ be two continuous and monotone mappings,$\varphi_1 : C \to \mathbb{R} \cup +\{\infty\}$ and $\varphi_2 : Q \to \mathbb{R} \cup +\{\infty\}$ be two proper lower semicontinuous and convex functions. Then the split equality generalized mixed equilibrium problem is: find $\bar{x} \in C$ and $\bar{y} \in Q$ such that

$$(1.12) \qquad g_1(\bar{x}, x) + \langle \Phi_1 \bar{x}, x - \bar{x} \rangle + \varphi_1(x) - \varphi_1(\bar{x}) \geq 0, \quad \forall x \in C,$$

$$(1.13) \qquad g_2(\bar{y}, y) + \langle \Phi_2 \bar{y}, y - \bar{y} \rangle + \varphi_2(y) - \varphi_2(\bar{y}) \geq 0, \quad \forall y \in Q,$$

and $A\bar{x} = B\bar{y}$. We can see that if $E_2 = E_3$ and B is the identity operator on E_2, then the split equality generalized mixed equilibrium problem (1.12)-(1.13) reduces to the split generalized mixed equilibrium problem (1.3)-(1.4).

Let E_1, E_2 and E_3 be three real Banach spaces and C, Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators. Let $g_1^i : C \times C \to \mathbb{R}$ $(i = 1, 2, ..., N)$ and $g_2^j : Q \times Q \to \mathbb{R}$ $(j = 1, 2, ..., M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1^i : C \to E_1^*$ $(i = 1, 2, ..., N)$ and $\Phi_2^j : Q \to E_2^*$ $(j = 1, 2, ..., M)$ be two finite families of continuous and monotone mappings, $\varphi_1^i : C \to \mathbb{R} \cup +\{\infty\}$ $(i = 1, 2, ..., N)$ and $\varphi_2^j : Q \to \mathbb{R} \cup +\{\infty\}$ $(j = 1, 2, ..., M)$ be two finite families of proper lower semicontinuous and convex functions. Let $T : C \to C$ and $S : Q \to Q$ be nonlinear mappings. Then, we consider the following problem: find $\bar{x} \in F(T)$ and $\bar{y} \in F(S)$ such that

$$g_1^i(\bar{x}, x) + \langle \Phi_1^i \bar{x}, x - \bar{x} \rangle + \varphi_1^i(x) - \varphi_1^i(\bar{x}) \geq 0, \tag{1.14}$$

$\forall x \in C, i = 1, 2, \cdots, N$;

$$g_2^j(\bar{y}, y) + \langle \Phi_2^j \bar{y}, y - \bar{y} \rangle + \varphi_2^j(y) - \varphi_2^j(\bar{y}) \geq 0, \tag{1.15}$$

$\forall y \in Q, j = 1, 2, \cdots, M$; and $A\bar{x} = B\bar{y}$. We shall denote the solution set of (1.14)-(1.15) by $\Omega = \{(\bar{x}, \bar{y}) : \bar{x} \in F(T) \cap (\cap_{i=1}^N GMEP(g_1^i, \Phi_1^i, \varphi_1^i)), \bar{y} \in F(S) \cap (\cap_{j=1}^M GMEP(g_2^j, \Phi_2^j, \varphi_2^j)), A\bar{x} = B\bar{y}\}$.
This problem (1.14)-(1.15) that we are considering has as special cases the split equality equilibrium problem, the split equality variational inequality problem, the split equality convex minimisation problem and the split generalized mixed equilibrium problem. Furthermore, results on split equilibrium problem and split equality equilibriums problems, to the best our knowledge, only exists in the framework of Hilbert spaces, but in this paper we give a strong convergence result for split equality for system of generalized mixed equilibrium problem and fixed point problems in $p-$uniformly convex and uniformly smooth Banach spaces. Thus, the result of this paper extends results on split equality equilibrium problems in the literature from Hilbert spaces to $p-$uniformly convex and uniformly smooth Banach spaces.

2. Preliminaries

Let E be a Banach space and let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup\{\frac{1}{2}(||x + y|| + ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq t\}.$$

E is uniformly smooth if and only if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0,$$

q-uniformly smooth if there exists a $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$.

Definition 2.1. The duality mapping $J_P^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\}.$$

Lemma 2.2. *Let $x, y \in E$. If E is q-uniformly smooth, then there exists a $C_q > 0$ such that*

$$||x - y||^q \leq ||x||^q - q\langle J_p^E(x), y \rangle + C_q||y||^q. \tag{2.1}$$

Let $\dim E \geq 2$ ($\dim E$ denotes the dimension of E). The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \Big|\Big|\frac{x+y}{2}\Big|\Big| : ||x|| = ||y|| = 1; \epsilon = ||x-y||\Big\}.$$

E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$ and *p-uniformly convex* if there is a $C_p > 0$ so that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for any $\epsilon \in (0,2]$.
It is known that E is p-uniformly convex and uniformly smooth if and only if its dual E^* is q-uniformly smooth and uniformly convex. It is also a common knowledge that the duality mapping J_p^E is one-to-one, single valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$ where $J_q^{E^*}$ is the duality mapping of E^* (see [2]).

The duality mapping J_p^E is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_p^E x_n, y\rangle \to \langle J_p^E x, y\rangle$$

holds for any $y \in E$. We note here that l_p-spaces for $p > 1$ have such a property, but the L_p-spaces for $p > 2$ do not share this property. The domain of a convex function $f : E \to \mathbb{R}$ is defined as

$$domf := \{x \in E : f(x) < +\infty\}.$$

When $domf \neq \emptyset$, we say that f is proper.

Definition 2.3. Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the *Bregman distance* with respect to f is defined as:

$$\Delta_f(x,y) := f(y) - f(x) - \langle f'(x), y - x\rangle, \quad x, y \in E.$$

The duality mapping J_p^E is actually the derivative of the function $f_p(x) = (\frac{1}{p})||x||^p$. Given that $f = f_p$, then the Bregman distance with respect to f_p now becomes

$$\begin{aligned}\Delta_p(x,y) &= \frac{1}{q}||x||^p - \langle J_p^E x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_p^E x, x - y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_p^E x - J_p^E y, y\rangle.\end{aligned}$$

The Bregman distance is not symmetric and so is not a metric but it possesses the following important properties

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) + \langle z - y, J_p^E x - J_p^E y\rangle, \tag{2.2}$$

$\forall x, y, z \in E$.

$$\Delta_p(x,y) + \Delta_p(y,x) = \langle x - y, J_p^E x - J_p^E y\rangle, \quad \forall x, y \in E. \tag{2.3}$$

For the p-uniformly convex space, the metric and Bregman distance has the following relation:

$$\tau||x-y||^p \leq \Delta_p(x,y) \leq \langle x-y, J_p^E x - J_p^E y\rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number. Similar to the metric projection, the Bregman projection is defined as

$$\Pi_C x = arg\min_{y\in C} \Delta_p(x,y), \quad x \in E,$$

the unique minimizer of the Bregman distance. The Bregman projection is also characterized by the variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \leq 0, \ \forall z \in C, \tag{2.5}$$

from which it follows that

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \ \forall z \in C. \tag{2.6}$$

The resolvent of a bifunction $g : C \times C \to \mathbb{R}$ (see [29]) is the operator $Res_g^f : E \to C$ defined by

$$Res_g^f(x) = \{z \in C : g(z,y) + \langle J_p^E(z) - J_p^E(x), y - z\rangle \geq 0, y \in C\}, \tag{2.7}$$

$\forall x \in E$.

Recall from [29] that, for any $x \in E$, there exists $z \in C$ such that $z = Res_g^f(x)$.

Let C be a convex subset of $\text{int}(\text{dom} f_p)$, where $f_p(x) = (\frac{1}{p})||x||^p, \ \ 2 \leq p < \infty$ and let T be a self-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Recalling that the Bregman distance is not symmetric, we define the following operators.

Definition 2.4. A mapping T with a nonempty asymptotic fixed point set is said to be:
(i) *left Bregman strongly nonexpansive* (see [17]) with respect to a nonempty $\widehat{F}(T)$ if

$$\Delta_p(Tx,p) \leq \Delta_p(x,p), \quad \forall x \in C, \quad p \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n,p) - \Delta_p(Tx_n,p)) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

Martin-Marquez *et al.* [17], noted that a left Bregman strongly nonexpansive mapping T with respect to a nonempty $\widehat{F}(T)$ is called *strictly left Bregman strongly nonexpansive mapping.*
(ii) An operator $T : C \to \text{int } (\text{dom})f$ is said to be *left Bregman firmly nonexpansive* (L-BFNE) if

$$\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \le \langle J_p^E(Tx) - J_p^E(Ty), x - y\rangle$$

for any $x, y \in C$, or equivalently,

$$\Delta_p(Tx, Ty) + \Delta_p(Ty, Tx) + \Delta_p(x, Tx) + \Delta_p(y, Ty) \le \Delta_p(x, Ty) + \Delta_p(y, Tx).$$

It is known that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Following [2], we make use of the function $V_p : E^* \times E \to [0, +\infty)$ which is defined by

$$V_p(x^*, x) := \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p,\ \forall x^* \in E^*, x \in E.$$

Clearly, V_p is nonnegative and $V_p(x^*, x) = \Delta_p(J_q^{E^*}(x^*), x)$ for all $x^* \in E^*$ and $x \in E$. Moreover, by the subdifferential inequality,

$$\langle f'(x), y - x\rangle \le f(y) - f(x),$$

with $f(x) = \frac{1}{q}||x||^q, x \in E^*,$ then $f'(x) = J_q^{E^*}$.
So, we have

$$\langle J_q^{E^*}(x), y\rangle \le \frac{1}{q}||x + y||^q - \frac{1}{q}||x||^q. \tag{2.8}$$

From (2.8), we obtain

$$\begin{aligned} V_p(x^* + y^*, x) &= \frac{1}{q}||x^* + y^*||^q - \langle x^* + y^*, x\rangle + \frac{1}{p}||x||^p \\ &\ge \frac{1}{q}||x^*||^q + \langle y^*, J_q^{E^*}(x^*)\rangle - \langle x^* + y^*, x\rangle + \frac{1}{p}||x||^p \\ &= \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p + \langle y^*, J_q^{E^*}(x^*)\rangle - \langle y^*, x\rangle \\ &= \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p + \langle y^*, J_q^{E^*}(x^*) - x\rangle \\ &= V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle. \qquad (2.9) \end{aligned}$$

for all $x \in E$ and $x^*, y^* \in E^*$. In addition, since $f = f_p$ is a proper lower semi-continuous and convex function, we have that $f^* = f_p^*$ is a proper *weak** lower semi-continuous and convex function (see, for example, [25]). Hence V_p

is convex in the second variable. Thus for all $z \in E$,

$$\begin{aligned}
&\Delta_p\big(J_q^{E^*}\big(\sum_{i=1}^N t_i J_p^E(x_i)\big), z\big) \\
&= V_p\Big(\sum_{i=1}^N t_i J_p^E(x_i), z\Big) \\
&= \frac{1}{q}||\sum_{i=1}^N t_i J_p^E(x_i)||^q - \langle\sum_{i=1}^N t_i J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p \\
&\le \frac{1}{q}\sum_{i=1}^N t_i||J_p^E(x_i)||^q - \sum_{i=1}^N t_i\langle J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p \\
&= \frac{1}{q}\sum_{i=1}^N t_i||(x_i)||^{(p-1)q} - \sum_{i=1}^N t_i\langle J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p \\
&= \frac{1}{q}\sum_{i=1}^N t_i||(x_i)||^p - \sum_{i=1}^N t_i\langle J_p^E(x_i), z\rangle + \frac{1}{p}||z||^p \\
&= \sum_{i=1}^N t_i\Delta_p(x_i, z), \qquad (2.10)
\end{aligned}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.5. *([24, 35]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \ge 0,$$

where
(i) $\{\alpha_n\} \subset [0,1],\ \sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \le 0$;
(iii) $\gamma_n \ge 0,\ \sum \gamma_n < \infty$.
Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.6. *([16, 23]) Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.7. *(Reich and Sabach [28]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}_{n=1}^\infty$ is also bounded.*

Lemma 2.8. *(Reich and Sabach [29]) Let $f : E \to (-\infty, +\infty)$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then,*
1. Res_g^f is single-valued;
2. Res_g^f is a Bregman firmly nonexpansive mapping;
3. $F(Res_g^f) = EP(g)$;
4. $EP(g)$ is a closed and convex subset of C;
5. for all $x \in E$ and $q \in F(Res_g^f)$,

$$\Delta_p(q, Res_g^f(x)) + \Delta_p(Res_g^f(x), x) \leq \Delta_p(q, x).$$

3. Main Results

Theorem 3.1. *Let E_1, E_2 and E_3 be three real Banach spaces which are $p-$uniformly convex and uniformly smooth and C, Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$ the adjoints of A and B, respectively. Let $g_1^i : C \times C \to \mathbb{R}$ $(i = 1, 2, ..., N)$ and $g_2^j : Q \times Q \to \mathbb{R}$ $(j = 1, 2, ..., M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1^i : C \to E_1^*$ $(i = 1, 2, ..., N)$ and $\Phi_2^j : Q \to E_2^*$ $(j = 1, 2, ..., M)$ be two finite families of continuous and monotone mappings, $\varphi_1^i : C \to \mathbb{R} \cup \{+\infty\}$ $(i = 1, 2, ..., N)$ and $\varphi_2^j : Q \to \mathbb{R} \cup \{+\infty\}$ $(j = 1, 2, ..., M)$ be two finite families of proper lower semicontinuous and convex functions. Let $T : C \to C$ and $S : Q \to Q$ be left Bregman strongly nonexpansive mappings such that $\Omega \neq \emptyset$ and let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For a fixed $u \in E_1$ and a fixed $v \in E_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by $x_0 \in E_1$ and $y_0 \in E_2$:*

$$(3.1)\quad \begin{cases} u_n = Res^f_{G_1^N} \circ Res^f_{G_1^{N-1}} \circ ... \circ Res^f_{G_1^2} \\ \circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], \\ v_n = Res^f_{G_2^M} \circ Res^f_{G_2^{M-1}} \circ ... \circ Res^f_{G_2^2} \\ \circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], \\ y_{n+1} = J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_1}(S(v_n))\Big], \end{cases}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n, \gamma_n \leq d < 1$;
(iv) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q + D_q\|B\|^q}\right)^{\frac{1}{q-1}}$;
$G_\iota(x, y) := g_\iota(x, y) + \langle \Phi_\iota x, y - x\rangle + \varphi_\iota(y) - \varphi_\iota(x)$, $(\iota = 1, 2)$.
Then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$.

Proof. It is known (see [38]), that the function
$G(x,y) := g(x,y) + \langle \Phi x, y-x\rangle + \varphi(y) - \varphi(x)$ satisfies $(A1)-(A4)$ and $GMEP(g,\Phi,\varphi)$ is closed and convex.
For any $(x,y)\in\Omega$, it follows from (3.1) that

$$\begin{aligned}
&\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y)\\
&= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(u_n)+\gamma_n J_p^{E_1}(T(u_n))\Big],x)\\
&\quad +\Delta_p(J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v)+\beta_n J_p^{E_2}(v_n)+\gamma_n J_p^{E_1}(S(v_n))\Big],y)\\
&\leq \alpha_n\Delta_p(u,x)+\beta_n\Delta_p(u_n,x)+\gamma_n\Delta_p(T(u_n),x)\\
&\quad +\alpha_n\Delta_p(v,y)+\beta_n\Delta_p(v_n,y)+\gamma_n\Delta_p(S(v_n),y)\\
&\leq \alpha_n\Delta_p(u,x)+\beta_n\Delta_p(u_n,x)+\gamma_n\Delta_p(u_n,x)\\
&\quad +\alpha_n\Delta_p(v,y)+\beta_n\Delta_p(v_n,y)+\gamma_n\Delta_p(v_n,y)\\
&= \alpha_n(\Delta_p(u,x)+\Delta_p(v,y))+(1-\alpha_n)(\Delta_p(u_n,x)+\Delta_p(v_n,y)).
\end{aligned}\tag{3.2}$$

Noting that $Ax=By$, we obtain from (3.1)

$$\begin{aligned}
&\Delta_p(u_n,x)+\Delta_p(v_n,x)\\
&= \Delta_p\Big(Res^f_{G_1^N}\circ Res^f_{G_1^{N-1}}\circ\ldots\circ Res^f_{G_1^2}\\
&\qquad \circ Res^f_{G_1^1}J_q^{E_1^*}[J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)],x\Big)\\
&\quad +\Delta_p\Big(Res^f_{G_2^M}\circ Res^f_{G_2^{M-1}}\circ\ldots\circ Res^f_{G_2^2}\\
&\qquad \circ Res^f_{G_2^1}J_q^{E_2^*}[J_p^{E_2}(y_n)+t_nB^*J_p^{E_3}(Ax_n-By_n)],y\Big)\\
&\leq \Delta_p\Big(J_q^{E_1^*}[J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)],x\Big)\\
&\quad +\Delta_p\Big(J_q^{E_2^*}[J_p^{E_2}(y_n)+t_nB^*J_p^{E_3}(Ax_n-By_n)],y\Big)\\
&= \frac{1}{q}||J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)||^q-\langle J_p^{E_1}(x_n),x\rangle\\
&\quad +t_n\langle J_p^{E_3}(Ax_n-By_n),Ax\rangle+\frac{1}{p}||x||^p\\
&\quad +\frac{1}{q}||J_p^{E_2}(y_n)+t_nB^*J_p^{E_3}(Ax_n-By_n)||^q-\langle J_p^{E_2}(y_n),x\rangle\\
&\quad -t_n\langle J_p^{E_3}(Ax_n-By_n),By\rangle+\frac{1}{p}||y||^p\\
&\leq \frac{1}{q}||J_p^{E_1}(x_n)||^q-t_n\langle J_p^{E_3}(Ax_n-By_n),Ax_n\rangle\\
&\quad +\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n-By_n)||^q\\
&\quad -\langle J_p^{E_1}(x_n),x\rangle+t_n\langle J_p^{E_3}(Ax_n-By_n),Ax\rangle+\frac{1}{p}||x||^p
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{q}||J_p^{E_2}(y_n)||^q + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n\rangle \\
&+\frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&-\langle J_p^{E_2}(y_n), y\rangle - t_n\langle J_p^{E_3}(Ax_n - By_n), By\rangle + \frac{1}{p}||y||^p \\
= \;& \frac{1}{q}||x_n||^p - \langle J_p^{E_1}(x_n), x\rangle + \frac{1}{p}||x||^p \\
&+t_n\langle J_p^{E_3}(Ax_n - By_n), Ax - Ax_n\rangle \\
&+\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&+\frac{1}{q}||y_n||^p - \langle J_p^{E_2}(y_n), y\rangle + \frac{1}{p}||y||^p \\
&+t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - By\rangle \\
&+\frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
= \;& \Delta_p(x_n, x) + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax - Ax_n\rangle \\
&+\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&+\Delta_p(y_n, y) + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - By\rangle \\
&+\frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
= \;& \Delta_p(x_n, x) + \Delta_p(y_n, y) + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - Ax_n\rangle \\
&+\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&+\frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q. \qquad (3.3)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Delta_p(u_n, x) + \Delta_p(v_n, x) \\
&\le \Delta_p(x_n, x) + \Delta_p(y_n, y) + t_n\langle J_p^{E_3}(Ax_n - By_n), By_n - Ax_n\rangle \\
&\quad +\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&\quad +\frac{D_q(t_n||B||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&= \Delta_p(x_n, x) + \Delta_p(y_n, y) \\
&\quad -[t_n - (\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q})]||(Ax_n - By_n)||^p, \qquad (3.4)
\end{aligned}
$$

which implies

$$
\Delta_p(u_n, x) + \Delta_p(v_n, x) \le \Delta_p(x_n, x) + \Delta_p(y_n, y). \qquad (3.5)
$$

Substituting (3.5) into (3.2), we have

$$\begin{aligned}
&\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y)\\
&\quad\le \alpha_n(\Delta_p(u,x)+\Delta_p(v,y))\\
&\qquad +(1-\alpha_n)(\Delta_p(x_n,x)+\Delta_p(y_n,y))\\
&\quad\le \max\{(\Delta_p(u,x)+\Delta_p(v,y)),\\
&\qquad (\Delta_p(x_n,x)+\Delta_p(y_n,y))\}\\
&\qquad \vdots\\
&\quad\le \max\{(\Delta_p(u,x)+\Delta_p(v,y)),\\
&\qquad (\Delta_p(x_0,x)+\Delta_p(y_0,y))\}. \qquad (3.6)
\end{aligned}$$

Therefore, $(\{\Delta_p(x_n,x)\},\{\Delta_p(x_n,x)\})$ are bounded and consequently we have that $\{x_n\}$, $\{y_n\}$,$\{u_n\}$, $\{v_n\}$, $\{T(u_n)\}$ and $\{S(v_n)\}$ are all bounded.

Moreover,

$$\begin{aligned}
&\Delta_p(x_{n+1},x)\\
&\quad= \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}(u_n)+\gamma_nJ_p^{E_1}(T(u_n))\Big],x\Big)\\
&\quad= V_p\Big(\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}(u_n)+\gamma_nJ_p^{E_1}(T(u_n)),x\Big)\\
&\quad\le V_p\Big(\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}(u_n)+\gamma_nJ_p^{E_1}(T(u_n))\\
&\qquad -\alpha_n\Big(J_p^{E_1}(u)-J_p^{E_1}(x)\Big),x\Big)\\
&\qquad -\langle-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x)),\\
&\qquad J_q^{E_1^*}\Big[\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}(y_n)+\gamma_nJ_p^{E_1}(T(u_n))\Big]-x\rangle\\
&\quad= V_p\Big(\alpha_nJ_p^{E_1}(x)+\beta_nJ_p^{E_1}(u_n)+\gamma_nJ_p^{E_1}(T(u_n)),x\Big)\\
&\qquad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle\\
&\quad= \Delta_p\Big(J_q^{E_1^*}\Big[\alpha_nJ_p^{E_1}(x)+\beta_nJ_p^{E_1}(u_n)+\gamma_nJ_p^{E_1}(T(u_n))\Big],x\Big)\\
&\qquad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle\\
&\quad\le \alpha_n\Delta_p(x,x)+\beta_n\Delta_p(u_n,z)+\gamma_n\Delta_p(T(u_n),x)\\
&\qquad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle\\
&\quad\le (1-\alpha_n)\Delta_p(u_n,x)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle\\
&\quad\le (1-\alpha_n)\Delta_p(x_n,x)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle. \qquad (3.7)
\end{aligned}$$

Similarly, we have

$$\Delta_p(y_{n+1},y)\le(1-\alpha_n)\Delta_p(y_n,y)+\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y),y_{n+1}-y\rangle. \qquad (3.8)$$

We divide into two cases to obtain the strong convergence.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n,x)+\Delta_p(y_n,y)\}$ is monotonically non-increasing. Then obviously $\{\Delta_p(x_n,x)+\Delta_p(y_n,y)\}$ converges and

$$(\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y))-(\Delta_p(x_n,x)+\Delta_p(y_n,y))\to 0. \tag{3.9}$$

Let

$$w_n := J_q^{E_1^*}\Big(\frac{\beta_n}{1-\alpha_n}J_p^{E_1}(u_n)+\frac{\gamma_n}{1-\alpha_n}T(u_n)\Big)$$

and

$$z_n := J_q^{E_2^*}\Big(\frac{\beta_n}{1-\alpha_n}J_p^{E_2}(v_n)+\frac{\gamma_n}{1-\alpha_n}S(v_n)\Big).$$

Then,

$$\begin{aligned}
&\Delta_p(w_n,x)+\Delta_p(z_n,y)\\
&= \Delta_p(J_q^{E_1^*}\Big(\frac{\beta_n}{1-\alpha_n}J_p^{E_1}(u_n)+\frac{\gamma_n}{1-\alpha_n}(T(u_n))\Big),x)\\
&\quad +\Delta_p(J_q^{E_1^*}\Big(\frac{\beta_n}{1-\alpha_n}J_p^{E_1}(v_n)+\frac{\gamma_n}{1-\alpha_n}(S(v_n))\Big),y)\\
&\le \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n,x)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(T(u_n),x)\\
&\quad +\frac{\beta_n}{1-\alpha_n}\Delta_p(v_n,y)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(S(v_n),y)\\
&\le \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n,x)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(u_n,x)\\
&\quad +\frac{\beta_n}{1-\alpha_n}\Delta_p(v_n,y)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(v_n,y)\\
&= \Delta_p(u_n,x)+\Delta_p(v_n,y).
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}
0 &\le (\Delta_p(u_n,x)+\Delta_p(v_n,y))-(\Delta_p(w_n,x)+\Delta_p(z_n,y))\\
&= \Delta_p(u_n,x)-\Delta_p(x_{n+1},x)+\Delta_p(x_{n+1},x)-\Delta_p(w_n,x)\\
&\quad +\Delta_p(v_n,y)-\Delta_p(y_{n+1},y)+\Delta_p(y_{n+1},y)-\Delta_p(z_n,y)\\
&\le \Delta_p(x_n,x)-\Delta_p(x_{n+1},x)+\Delta_p(x_{n+1},x)-\Delta_p(w_n,x)\\
&\quad +\Delta_p(y_n,y)-\Delta_p(y_{n+1},y)+\Delta_p(y_{n+1},y)-\Delta_p(z_n,y)\\
&\le \Delta_p(x_n,x)-\Delta_p(x_{n+1},x)+\alpha_n\Delta_p(u,x)\\
&\quad +(1-\alpha_n)\Delta_p(w_n,x)-\Delta_p(w_n,x)\\
&\quad +\Delta_p(y_n,y)-\Delta_p(y_{n+1},y)+\alpha_n\Delta_p(v,y)\\
&\quad +(1-\alpha_n)\Delta_p(z_n,y)-\Delta_p(z_n,y)\\
&= (\Delta_p(x_n,x)+\Delta_p(y_n,y))-(\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y))\\
&\quad +\alpha_n((\Delta_p(u,x)+\Delta_p(v,y))\\
&\quad -(\Delta_p(w_n,x)+\Delta_p(z_n,y)))\to 0,\quad n\to\infty.
\end{aligned} \tag{3.11}$$

Furthermore,

$$
\begin{aligned}
&\Delta_p(w_n, x) + \Delta_p(z_n, y) \\
&\le \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n, x) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(T(u_n), x) \\
&\quad + \frac{\beta_n}{1-\alpha_n}\Delta_p(v_n, y) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(S(v_n), y) \\
&= \Delta_p(u_n, x) - (1 - \frac{\beta_n}{1-\alpha_n})\Delta_p(u_n, x) \\
&\quad + \frac{\gamma_n}{1-\alpha_n}\Delta_p(T(u_n), x) \\
&\quad + \Delta_p(v_n, y) - (1 - \frac{\beta_n}{1-\alpha_n})\Delta_p(v_n, y) \\
&\quad + \frac{\gamma_n}{1-\alpha_n}\Delta_p(S(v_n), y) \\
&= \Delta_p(u_n, x) + \Delta_p(v_n, y) \\
&\quad + \frac{\gamma_n}{1-\alpha_n}\Big(\Delta_p(T(u_n), x) - \Delta_p(u_n, x)\Big) \\
&\quad + \frac{\gamma_n}{1-\alpha_n}\Big(\Delta_p(S(v_n), y) + \Delta_p(v_n, y)\Big). \qquad (3.12)
\end{aligned}
$$

Thus, from (3.12)

$$
\begin{aligned}
&\frac{\gamma_n}{1-\alpha_n}\Big[(\Delta_p(u_n, x) - \Delta_p(T(u_n), x)) + (\Delta_p(v_n, y) - \Delta_p(S(v_n), y))\Big] \\
&\le \Big((\Delta_p(u_n, x) + \Delta_p(v_n, y)) - \Delta_p(w_n, x) + \Delta_p(z_n, y)\Big) \to 0, \qquad (3.13)
\end{aligned}
$$

which by condition (iii) implies

$$\Delta_p(u_n, x) - \Delta_p(T(u_n), x) \to 0, n \to \infty,$$

and

$$\Delta_p(v_n, y) - \Delta_p(S(v_n), y) \to 0, n \to \infty.$$

Since T and S are left Bregman strongly nonexpansive, we have

$$\lim_{n\to\infty} \Delta_p(Ty_n, y_n) = 0,$$

and

$$\lim_{n\to\infty} \Delta_p(Ty_n, y_n) = 0,$$

which implies

$$\lim_{n\to\infty} ||Tu_n - u_n|| = 0, \qquad (3.14)$$

and

$$\lim_{n\to\infty} ||Sv_n - v_n|| = 0, \qquad (3.15)$$

respectively. Since $\{u_n\}$ is bounded and E_1 is reflexive, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to $x^* \in E_2$. From (3.14), it follows that $x^* \in F(T)$ since $F(T) = \hat{F}(T)$. Also since $\{u_n\}$ is bounded and E_2 is reflexive, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that converges weakly to $y^* \in E_2$. From (3.15), it follows that $y^* \in F(S)$ since $F(S) = \hat{F}(S)$.

Next, we show that $Ax^* = By^*$.

Now from (3.4), we obtain

$$
\begin{aligned}
&\Big[t_n - \Big(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\Big)\Big]||(Ax_n - By_n)||^p \\
&\leq \Delta_p(x_n, x) + \Delta_p(y_n, y) - (\Delta_p(u_n, x) + \Delta_p(v_n, y)) \\
&= \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + \Delta_p(x_{n+1}, x) - \Delta_p(u_n, x) \\
&\quad +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) + \Delta_p(y_{n+1}, y) - \Delta_p(v_n, y) \\
&\leq \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) + (1 - \alpha_n)\Delta_p(u_n, x) \\
&\quad +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle - \Delta_p(u_n, x) \\
&\quad +\Delta_p(y_n, x) - \Delta_p(y_{n+1}, y) + (1 - \alpha_n)\Delta_p(v_n, y) \\
&\quad +\alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle - \Delta_p(v_n, y) \\
&= \Delta_p(x_n, x) - \Delta_p(x_{n+1}, x) \\
&\quad +\alpha_n(-\Delta_p(u_n, x) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x), x_{n+1} - x\rangle) \\
&\quad +\Delta_p(y_n, y) - \Delta_p(y_{n+1}, y) \\
&\quad +\alpha_n(-\Delta_p(v_n, y) + \alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y), y_{n+1} - y\rangle) \to 0, \quad n \to \infty,
\end{aligned} \tag{3.16}
$$

and since

$$
\begin{aligned}
0 &< t\Big(1 - \Big(\frac{C_q k^{q-1}(||A||)^q}{q} + \frac{D_q k^{q-1}(||B||)^q}{q}\Big)\Big) \\
&\leq \Big(t_n - \Big(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\Big)\Big),
\end{aligned}
$$

we have that $||(Ax_n - By_n)||^p \to 0, n \to \infty$.
Let $\mu_n = J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)]$ and
$\nu_n = J_q^{E_2^*}[J_p^{E_2}(y_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)]$.
Denote $\Theta_i = Res^f_{G_1^i} \circ Res^f_{G_1^{i-1}} \circ, ..., \circ Res^f_{G_1^1}$ for $i = 1, 2, ...N$ and $\Theta_0 = I$. We note that $u_n = \Theta_N \mu_n$. Also denote $\Psi_j = Res^f_{G_2^j} \circ Res^f_{G_2^{j-1}} \circ, ..., \circ Res^f_{G_2^1}$ for $j = 1, 2, ...M$ and $\Psi_0 = I$. We note that $v_n = \Psi_N \nu_n$.
Since $(x, y) \in \cap_{i=1}^N EP(G_1^i) \times \cap_{j=1}^M EP(G_2^j)$, then from (3.1) and Lemma 2.8(5),

$$
\begin{aligned}
&\Delta_p(\Theta_{N-1}\mu_n, u_n) + \Delta_p(\Psi_{M-1}\nu_n, v_n) \\
&\quad = \Delta_p(\Theta_{N-1}\mu_n, Res^f_{G_1^N}\Theta_{N-1}\mu_n) + \Delta_p(\Psi_{M-1}\nu_n, Res^f_{G_2^M}\Psi_{M-1}\nu_n) \\
&\quad \leq \Delta_p(\Theta_{N-1}\mu_n, x) - \Delta_p(u_n, x) + \Delta_p(\Psi_{M-1}\nu_n, y) - \Delta_p(v_n, y) \\
&\quad \leq \Delta_p(\mu_n, x) - \Delta_p(u_n, x) + \Delta_p(\nu_n, y) - \Delta_p(v_n, y) \\
&\quad \leq \Delta_p(x_n, x) - \Delta_p(u_n, x) + \Delta_p(y_n, y) - \Delta_p(v_n, y)
\end{aligned}
$$

$$\begin{aligned}
&= \Delta_p(x_n,x)-\Delta_p(x_{n+1},x)+\Delta_p(x_{n+1},x)-\Delta_p(u_n,x)\\
&\quad +\Delta_p(y_n,y)-\Delta_p(y_{n+1},y)+\Delta_p(y_{n+1},y)-\Delta_p(v_n,y)\\
&\leq \Delta_p(x_n,x)-\Delta_p(x_{n+1},x)+(1-\alpha_n)\Delta_p(u_n,x)\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle-\Delta_p(u_n,x)+\Delta_p(y_n,y)\\
&\quad -\Delta_p(y_{n+1},y)+(1-\alpha_n)\Delta_p(v_n,y)\\
&\quad +\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y),y_{n+1}-y\rangle-\Delta_p(v_n,y)\\
&= \Delta_p(x_n,x)+\Delta_p(y_n,y)-(\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y))\\
&\quad +\alpha_n(-\Delta_p(u_n,x)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle)\\
&\quad +\alpha_n(-\Delta_p(v_n,y)+\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y),y_{n+1}-y\rangle)\to 0, \qquad (3.17)
\end{aligned}$$

as $n\to\infty$, which implies

$$||\Theta_{N-1}\mu_n-u_n||\to 0, n\to\infty, \qquad (3.18)$$

and

$$||\Psi_{M-1}\nu_n-v_n||\to 0, n\to\infty. \qquad (3.19)$$

Consequently, we have

$$||J_p^{E_1}(\Theta_{N-1}\mu_n)-J_p^{E_1}(\Theta_N\mu_n)||\to 0, n\to\infty, \qquad (3.20)$$

and

$$||J_p^{E_2}(\Psi_{M-1}\nu_n)-J_p^{E_2}(\Psi_M\nu_n)||\to 0, n\to\infty. \qquad (3.21)$$

Again

$$\begin{aligned}
&\Delta_p(\Theta_{N-2}\mu_n,\Theta_{N-1}\mu_n)+\Delta_p(\Psi_{M-2}\nu_n,\Psi_{M-1}\nu_n)\\
&\leq \Delta_p(\Theta_{N-2}\mu_n,x)-\Delta_p(\Theta_{N-1}\mu_n,x)\\
&\quad +\Delta_p(\Psi_{M-2}\nu_n,y)-\Delta_p(\Psi_{M-1}\nu_n,y)\\
&\leq \Delta_p(\mu_n,x)-\Delta_p(u_n,x)+\Delta_p(\nu_n,y)-\Delta_p(v_n,y)\\
&\leq \Delta_p(x_n,x)-\Delta_p(u_n,x)+\Delta_p(y_n,y)-\Delta_p(v_n,y)\\
&= \Delta_p(x_n,x)-\Delta_p(x_{n+1},x)+\Delta_p(x_{n+1},x)-\Delta_p(u_n,x)\\
&\quad +\Delta_p(y_n,y)-\Delta_p(y_{n+1},y)+\Delta_p(y_{n+1},y)-\Delta_p(v_n,y)\\
&\leq \Delta_p(x_n,x)-\Delta_p(x_{n+1},x)+(1-\alpha_n)\Delta_p(u_n,x)\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle-\Delta_p(u_n,x)\\
&\quad +\Delta_p(y_n,y)-\Delta_p(y_{n+1},y)+(1-\alpha_n)\Delta_p(v_n,y)\\
&\quad +\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y),y_{n+1}-y\rangle-\Delta_p(v_n,y)\\
&= \Delta_p(x_n,x)+\Delta_p(y_n,y)-(\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y))\\
&\quad +\alpha_n(-\Delta_p(u_n,x)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x),x_{n+1}-x\rangle)\\
&\quad +\alpha_n(-\Delta_p(v_n,y)\\
&\quad +\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y),y_{n+1}-y\rangle)\to 0, \quad n\to\infty, \qquad (3.22)
\end{aligned}$$

which implies

$$||\Theta_{N-2}\mu_n - \Theta_{N-1}\mu_n|| \to 0, n \to \infty, \tag{3.23}$$

and

$$||\Psi_{M-2}\nu_n - \Psi_{M-1}\nu_n|| \to 0, n \to \infty. \tag{3.24}$$

Consequently, we have

$$||J_p^{E_1}(\Theta_{N-2}\mu_n) - J_p^{E_1}(\Theta_{N-1}\mu_n)|| \to 0, n \to \infty, \tag{3.25}$$

and

$$||J_p^{E_2}(\Psi_{M-2}\nu_n) - J_p^{E_2}(\Psi_{M-1}\nu_n)|| \to 0, n \to \infty. \tag{3.26}$$

In a similar way, we can verify that

$$\lim_{n\to\infty} ||\Theta_{N-2}\mu_n - \Theta_{N-3}\mu_n|| = \cdots = \lim_{n\to\infty} ||\Theta_1\mu_n - \mu_n|| = 0, \tag{3.27}$$

and

$$\lim_{n\to\infty} ||\Psi_{M-2}\nu_n - \Psi_{M-3}\nu_n|| = \cdots = \lim_{n\to\infty} ||\Psi_1\nu_n - \nu_n|| = 0. \tag{3.28}$$

Hence it follows that

$$\lim_{n\to\infty} ||\Theta_i\mu_n - \Theta_{i-1}\mu_n|| = 0, i = 1, 2, \cdots, N, \tag{3.29}$$

and

$$\lim_{n\to\infty} ||u_n - \mu_n|| = 0.$$

Moreover,

$$\lim_{n\to\infty} ||\Psi_j\nu_n - \Psi_{j-1}\nu_n|| = 0, j = 1, 2, \cdots, M, \tag{3.30}$$

and

$$\lim_{n\to\infty} ||v_n - \nu_n|| = 0.$$

Again, we obtain from the definition of μ_n that

$$\begin{aligned} 0 &\le ||J_p^{E_1}\mu_n - J_p^{E_1}x_n|| \\ &\le t_n||A^*||||J_p^{E_2}(Ax_n - By_n)|| \\ &\le \Big(\frac{q}{C_q\|A\|^q + D_q\|B\|^q}\Big)^{\frac{1}{q-1}}||A^*||||(Ax_n - By_n)|| \to 0, n \to \infty. \end{aligned}$$

Since $J_q^{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we have that

$$\lim_{n\to\infty}||\mu_n - x_n|| = \lim_{n\to\infty}||J_q^{E_1^*}J_p^{E_1}v_n - J_q^{E_1^*}J_p^{E_1}u_n|| \to 0, n \to \infty. \tag{3.31}$$

Thus, from (3.18) and (3.31), we have

$$||x_n - u_n|| \le ||x_n - \mu_n|| + ||\mu_v - u_n|| \to 0, n \to \infty.$$

Similarly, we have $\lim_{n\to\infty} ||\nu_n - y_n|| = 0$ and $||y_n - v_n|| \to 0, n \to \infty.$

Thus $Ax^* - By^* \in w_w(Ax_n - By_n)$ and since the norm is weakly lower semicontinuous, we obtain

$$||Ax^* - By^*|| \le \liminf_{n\to\infty} ||Ax_n - By_n|| = 0.$$

We next show that $(x^*, y^*) \in \cap_{i=1}^N EP(G_1^i) \times \cap_{j=1}^M EP(G_2^j)$.

Now since $u_{n_k} \rightharpoonup x^*$ and $\lim_{n\to\infty} ||u_n - \mu_n|| = 0$, we have that $\mu_{n_k} \rightharpoonup x^*$. Also from (3.18),(3.23), (3.27) and $\mu_{n_k} \rightharpoonup x^*$, we have that $\Theta_i \mu_{n_k} \rightharpoonup x^*, k \to \infty$, for each $i = 1, 2, \cdots, N$. Again using (3.29), we get that

$$\lim_{n\to\infty} ||J_p^{E_1}(\Theta_i \mu_n) - J_p^{E_1}(\Theta_{i-1}\mu_n)|| = 0, i = 1, 2, \cdots, N. \tag{3.32}$$

Therefore by (2.7), we have that for each $i = 1, 2, \cdots, N$,

$$G_1^i(\Theta_i \mu_{n_k}, z) + \langle z - \Theta_i \mu_{n_k}, J_p^{E_1}(\Theta_i \mu_{n_k}) - J_p^{E_1}(\Theta_{i-1}\mu_{n_k})\rangle \ge 0, \quad \forall z \in C.$$

Again using (A2), we obtain

$$\langle z - \Theta_i \mu_{n_k}, J_p^{E_1}(\Theta_i \mu_{n_k}) - J_p^{E_1}(\Theta_{i-1} y_{n_k})\rangle \ge G_1^i(z, \Theta_i \mu_{n_k}). \tag{3.33}$$

Thus, a combination of (A4), (3.32), (3.33) and $\Theta_i \mu_{n_k} \rightharpoonup x^*, k \to \infty$, gives us that for each $i = 1, 2, \cdots, N$,

$$G_1^i(z, x^*) \le 0, \quad \forall z \in C.$$

Then for fixed $z \in C$, let $a_{t,z} := tz + (1-t)x^*$ for all $t \in (0, 1]$. This implies that $a_{t,z} \in C$ and further yields that $G_1^i(z_{t,y}, x^*) \le 0$. It then follows from (A1) and (A4) that

$$\begin{aligned} 0 &= G_1^i(a_{t,z}, a_{t,z}) \\ &\le tG_1^i(a_{t,z}, y) + (1-t)G_1^i(a_{t,z}, x^*) \\ &\le tG_1^i(a_{t,z}, z), \end{aligned}$$

and hence, from condition (A3), we obtain $G_1^i(x^*, z) \ge 0, \quad \forall z \in C$, which implies that

$$x^* \in \cap_{i=1}^N EP(G_1^i).$$

Similarly, we have

$$y^* \in \cap_{j=1}^M EP(G_2^j).$$

Next, we show that $(\{x_n\}, \{y_n\})$ converges strongly to (x^*, y^*).

Now, we observe that

$$\Delta_p(x_{n+1}, u_n) + \Delta_p(y_{n+1}, v_n)$$

$$
\begin{aligned}
&= \Delta_p(J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], u_n) \\
&\quad + \Delta_p(J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(u) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_2}(S(v_n))\Big], v_n) \\
&\leq \alpha_n \Delta_p(u, u_n) + \beta_n \Delta_p(u_n, u_n) + \gamma_n \Delta_p(T(u_n), u_n) \\
&\quad + \alpha_n \Delta_p(u, v_n) + \beta_n \Delta_p(v_n, v_n) \\
&\quad + \gamma_n \Delta_p(S(v_n), v_n) \to 0, n \to \infty.
\end{aligned}
$$

Hence,

$$||x_{n+1} - u_n|| \to 0, \quad n \to \infty, \quad \text{and} \quad ||y_{n+1} - v_n|| \to 0, \quad n \to \infty.$$

Thus

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0, n \to \infty,$$

and

$$||y_{n+1} - y_n|| \leq ||y_{n+1} - v_n|| + ||v_n - y_n|| \to 0, n \to \infty.$$

From (3.7) and (3.8), we obtain

$$
\begin{aligned}
&\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*) \\
&\quad \leq (1 - \alpha_n)(\Delta_p(x_n, x)^* + \Delta_p(y_n, y^*)) \\
&\quad\quad + \alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^* \rangle \\
&\quad\quad + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^* \rangle). \qquad (3.34)
\end{aligned}
$$

Therefore, by Lemma 2.5, we conclude that $\Delta_p(x_n, x^*) + \Delta_p(y_n, x^*) \to 0, \quad n \to \infty$, that is, $||x_n - x^*|| \to 0, \quad n \to \infty$ and $||y_n - x^*|| \to 0, \quad n \to \infty$. Therefore, $x_n \to x^*$ and $y_n \to y^*$.

Case 2. Suppose that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\Delta_p(x_{n_k,x}) + \Delta_p(y_{n_k,y}) < \Delta_p(x_{n_k+1}, x) + \Delta_p(y_{n_k+1}, y)$ for all $k \in \mathbb{N}$. Then, by Lemma 2.6 there exists a nondecreasing sequence $\{m_\tau\} \subseteq \mathbb{N}$ such that $m_\tau \to \infty$.

$$\Delta_p(x_{m_\tau}, x) + \Delta_p(y_{m_\tau}, y) \leq \Delta_p(x_{m_\tau+1}, x) + \Delta_p(y_{m_\tau+1}, y),$$

and

$$\Delta_p(x_k, x) \leq \Delta_p(x_{m_k+1}, x).$$

Using the same line of arguments as in (3.10),(3.11),(3.12),(3.13) and noting that $\Delta_p(x_{m_\tau}, x) + \Delta_p(y_{m_\tau}, y) \leq \Delta_p(x_{m_\tau+1}, x) + \Delta_p(y_{m_\tau+1}, y)$, we can show that

$$\lim_{\tau \to \infty} ||Tu_{m_\tau} - u_{m_\tau}|| = 0, \quad \text{and} \quad \lim_{\tau \to \infty} ||Sv_{m_\tau} - v_{m_\tau}|| = 0.$$

Again from (3.7) and (3.8), we have

$$
\begin{aligned}
&\Delta_p(x_{m_\tau+1}, x^*) + \Delta_p(y_{m_\tau+1}, x^*) \\
&\quad \leq (1 - \alpha_{m_\tau})(\Delta_p(x_{m_k}, x^*) + \Delta_p(y_{m_\tau}, x^*))
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_{m_\tau}(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{m_\tau+1} - x^*\rangle \\
&+\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{m_\tau+1} - y^*\rangle),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\alpha_{m_\tau}(\Delta_p(x_{m_\tau}, x^*) + \Delta_p(y_{m_\tau}, x^*)) \\
&\quad \leq \quad (\Delta_p(x_{m_\tau}, x^*) + \Delta_p(y_{m_\tau}, y^*)) \\
&\qquad -(\Delta_p(x_{m_\tau+1}, x^*) + \Delta_p(y_{m_\tau+1}, y^*) \\
&\qquad +\alpha_{m_\tau}(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{m_\tau+1} - x^*\rangle \\
&\qquad +\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{m_\tau+1} - y^*\rangle).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\Delta_p(x_{m_\tau}, x^*) + \Delta_p(y_{m_\tau}, y^*) \\
&\quad \leq \quad \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{m_\tau+1} - x^*\rangle \\
&\qquad +\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{m_\tau+1} - y^*\rangle).
\end{aligned}
$$

Therefore

$$
\lim_{\tau\to\infty}(\Delta_p(x_{m_\tau}, x^*) + \Delta_p(y_{m_\tau}, y^*)) = 0,
$$

and since

$$
\Delta_p(x_\tau, x^*) + \Delta_p(y_\tau, y^*) \leq \Delta_p(x_{m_\tau+1}, x^*) + \Delta_p(y_{m_\tau+1}, y^*), \quad \text{for all} \quad \tau \in \mathbb{N},
$$

we conclude that

$$
x_\tau \to x^* \quad \text{and} \quad y_\tau \to y^*, \quad \tau \to \infty.
$$

□

Corollary 3.2. *Let E_1, E_2 and E_3 be three real Banach spaces which are $p-$uniformly convex and uniformly smooth and C, Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$ the adjoint of A and B, respectively. Let $g_1^i : C \times C \to \mathbb{R}$ $(i = 1, 2, ..., N)$ and $g_2^j : Q \times Q \to \mathbb{R}$ $(j = 1, 2, ..., M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\varphi_1^i : C \to \mathbb{R} \cup \{+\infty\}$ $(i = 1, 2, ..., N)$ and $\varphi_2^j : Q \to \mathbb{R} \cup \{+\infty\}$ $(j = 1, 2, ..., M)$ be two finite families of proper lower semicontinuous and convex functions. Let $T : C \to C$ and $S : Q \to Q$ be left Bregman strongly nonexpansive mappings such that $\Omega_\varphi \neq \emptyset$ and let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For a fixed $u \in E_1$ and a fixed $v \in E_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by*

$x_0 \in E_1$ and $y_0 \in E_2$:

$$(3.35)\begin{cases} u_n = Res^f_{G_1^N} \circ Res^f_{G_1^{N-1}} \circ ... \circ Res^f_{G_1^2} \\ \circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], \\ v_n = Res^f_{G_2^M} \circ Res^f_{G_2^{M-1}} \circ ... \circ Res^f_{G_2^2} \\ \circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], \\ y_{n+1} = J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_1}(S(v_n))\Big], \end{cases}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n, \gamma_n \leq d < 1$;
(iv) $0 < t \leq t_n \leq k < \Big(\dfrac{q}{C_q\|A\|^q + D_q\|B\|^q}\Big)^{\frac{1}{q-1}}$;
$G_\iota(x,y) := g_\iota(x,y) + \varphi_\iota(y) - \varphi_\iota(x), \quad (\iota = 1,2)$. *Then, $(\{x_n\},\{x_n\})$ converges strongly to $(x^*,y^*) \in \Omega_\varphi$, where*

$$\begin{aligned}\Omega_\varphi = \{(\bar{x},\bar{y}) :& \bar{x} \in F(T) \cap (\cap_{i=1}^N GMEP(g_1^i,\varphi_1^i)), \\ & \bar{y} \in F(S) \cap (\cap_{j=1}^M GMEP(g_2^j,\varphi_2^j)) : A\bar{x} = B\bar{y}\}.\end{aligned}$$

Corollary 3.3. *Let E_1, E_2 and E_3 be three real Banach spaces which are $p-$uniformly convex and uniformly smooth and C, Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$ the adjoint of A and B, respectively. Let $g_1^i : C \times C \to \mathbb{R}$ $(i = 1,2,...,N)$ and $g_2^j : Q \times Q \to \mathbb{R}$ $(j = 1,2,...,M)$ be two finite families of bifunctions satisfying conditions $(A1) - (A4)$. Let $\Phi_1^i : C \to E_1^*$ $(i = 1,2,...,N)$ and $\Phi_2^j : Q \to E_2^*$ $(j = 1,2,...,M)$ be two finite families of continuous and monotone mappings. Let $T : C \to C$ and $S : Q \to Q$ be left Bregman strongly nonexpansive mappings such that $\Omega_\Phi \neq \emptyset$ and let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For a fixed $u \in E_1$ and a fixed $v \in E_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by $x_0 \in E_1$ and $y_0 \in E_2$:*

$$(3.36)\begin{cases} u_n = Res^f_{G_1^N} \circ Res^f_{G_1^{N-1}} \circ ... \circ Res^f_{G_1^2} \\ \circ Res^f_{G_1^1} J_q^{E_1^*}[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)], \\ v_n = Res^f_{G_2^M} \circ Res^f_{G_2^{M-1}} \circ ... \circ Res^f_{G_2^2} \\ \circ Res^f_{G_2^1} J_q^{E_2^*}[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)], \\ x_{n+1} = J_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(T(u_n))\Big], \\ y_{n+1} = J_q^{E_2^*}\Big[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_1}(S(v_n))\Big], \end{cases}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n, \gamma_n \leq d < 1$;
(iv) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q + D_q\|B\|^q}\right)^{\frac{1}{q-1}}$;
$G_\iota(x,y) := g_\iota(x,y) + \langle \Phi_\iota x, y - x\rangle, \quad (\iota = 1,2)$. *Then,* $(\{x_n\},\{x_n\})$ *converges strongly to* $(x^*, y^*) \in \Omega_\Phi$, *where*

$$\Omega_\Phi = \{(\bar{x},\bar{y}) : \bar{x} \in F(T) \cap (\cap_{i=1}^N GMEP(g_1^i, \Phi_1^i)),\\ \bar{y} \in F(S) \cap (\cap_{j=1}^M GMEP(g_2^j, \Phi_2^j)) : A\bar{x} = B\bar{y}\}.$$

4. Numerical Example

In this section, we present two numerical examples of our algorithm on the real line and in an infinite dimensional Hilbert space, to show its efficiency.

Throughout this section, we shall take $\alpha_n = \frac{2}{n+2}$, $\beta_n = \frac{n+1}{2(n+2)}$ and $\gamma_n = \frac{n+1}{2(n+2)}$.

Example 4.1. Let $E_1 = E_2 = E_3 = \mathbb{R}$ and $C = Q = [-1,1]$. Take $g_1^i(x,y) := -9ix^2 + xy + (9i-1)y^2$, $\Phi_1^i(x) = (9i-3)x$, $\varphi_1^i(x) := (9i-6)x$, $i = 1,2,3,\cdots,M$, we have $Res_{G_1^i}^f(x) = \dfrac{x}{5(9i-3)}$. Also, we take $g_2^j(x,y) := -7ix^2 + xy + (7i-1)y^2$, $\Phi_2^j(x) = (7i-3)x$, $\varphi_2^j(x) := (7i-6)x$, $j = 1,2,3,\cdots,N$, and obtain $Res_{G_2^j}^f(x) = \dfrac{x}{5(7i-3)}$. Furthermore, let $Ax := 2x$, $Bx := 3x$ and $T(x) = S(x) = \Pi_C(x) = \Pi_Q(x) = P_C(x)$, with

$$P_C(x) = P_Q(x) = \begin{cases} -1, & x < -1, \\ x, & x \in [-1,1], \\ 1, & x > 1. \end{cases}$$

Let $M = N = 5$, then the iteration scheme (3.1) becomes:

$$\begin{cases} u_n = \Pi_{i=1}^5 \dfrac{1}{5(9i-3)}[x_n - 2t_n(2x_n - 3y_n)], \\ v_n = \Pi_{j=1}^5 \dfrac{1}{5(7j-3)}[y_n - 3t_n(2x_n - 3y_n)], \\ x_{n+1} = \dfrac{2}{n+1}u + \dfrac{n+1}{2(n+2)}(u_n) + \dfrac{n+1}{2(n+2)}(P_C(u_n)), \\ y_{n+1} = \dfrac{1}{n+1}v + \dfrac{n+1}{2(n+2)}(v_n) + \dfrac{n+1}{2(n+2)}(P_Q(v_n)). \end{cases} \tag{4.1}$$

Case I

(a) Take $u = 1$, $v = \frac{1}{2}$, $x_0 = 0.1$, $y_0 = 0.22$ and $t_n = 0.0000032$.

(b) Take $u = 1$, $v = \frac{1}{2}$, $x_0 = 0.1$, $y_0 = 0.22$ and $t_n = 0.00000051$.

Case II

(a) Take $u = 2$, $v = 0.1$, $x_0 = 0.3$, $y_0 = 0.02$ and $t_n = 0.00018$.

(b) Take $u = 2$, $v = 0.1$, $x_0 = 0.3$, $y_0 = 0.02$ and $t_n = 0.00000071$.

Case III

(a) Take $u = 1$, $v = 1$, $x_0 = 0.1$, $y_0 = 0.1$ and $t_n = 0.00008$.

(b) Take $u = 1$, $v = 1$, $x_0 = 0.1$, $y_0 = 0.1$ and $t_n = 0.00000011$.

Example 4.2. Let $E_1 = E_2 = E_3 = L_2([0,1])$ be endowed with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \ \forall\, x, y \in L_2([0,1])$$

and norm

$$||x|| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}} \ \ \forall\, x, y \in L_2([0,1]).$$

Let $C = Q = \{x \in L_2([0,1]) : \langle y, x\rangle \leq a\}$, where $y = 2t^3$ and $a = 3$. Then we define $g_1 : C \times C \to \mathbb{R}$ and $g_2 : Q \times Q \to \mathbb{R}$ by $g_1(x,y) = \langle L_1 x, y - x\rangle$ and $g_2(x,y) = \langle L_2 x, y - x\rangle$, where $L_1 x(t) = \frac{x(t)}{2}$ and $L_2 x(t) = \frac{x(t)}{5}$. Thus, it is easy to check that g_1 and g_2 satisfy conditions (A1)-(A4). Also, define $\Phi_1 : C \to L_2([0,1])$ and $\Phi_2 : Q \to L_2([0,1])$ by $\Phi_1(x) = \max\{0, x(t)\}$ and $\Phi_2(x) = \int_0^1 \Big(x(t) - \Big(\frac{2tse^{t+s}}{e\sqrt{e^2-1}}\Big)\cos x(s)\Big)ds + \frac{2te^t}{e\sqrt{e^2-1}}$, $t \in [0,1]$. Then, Φ_1 and Φ_2 are monotone and continuous (see [7]). Let $\varphi_1 = 0 = \varphi_2$.

Furthermore, let $A, B : L_2([0,1]) \to L_2([0,1])$ be defined by $Ax(t) = \frac{2x(t)}{5}$ and $Bx(t) = \frac{x(t)}{2}$. Then, A and B are bounded linear operators. Also, let $T(x) = S(x) = \Pi_C(x) = \Pi_Q(x) = P_C(x)$, where

$$P_C(x) = P_Q(x) = \begin{cases} \frac{a - \langle y, x\rangle}{||y||^2_{L_2}} y + x, & \text{if } \langle y, x\rangle > a, \\ x, & \text{if } \langle y, x\rangle \leq a. \end{cases}$$

Then, T and S are left Bregman strongly nonexpansive mappings. Thus, by letting $M = N = 1$ in Theorem 3.1, iteration scheme (3.1) becomes:

$$(4.2)\quad \begin{cases} u_n = Res^f_{G_1}[x_n - \frac{2}{5}t_n(\frac{2}{5}x_n - \frac{1}{2}y_n)], \\ v_n = Res^f_{G_2}[y_n - \frac{1}{2}t_n(\frac{2}{5}x_n - \frac{1}{2}y_n)], \\ \\ x_{n+1} = \dfrac{2}{n+1}u + \dfrac{n+1}{2(n+2)}(u_n) + \dfrac{n+1}{2(n+2)}(P_C(u_n)), \\ y_{n+1} = \dfrac{1}{n+1}v + \dfrac{n+1}{2(n+2)}(v_n) + \dfrac{n+1}{2(n+2)}(P_Q(v_n)). \end{cases}$$

Case 1

(a) Take $u = \sin t$, $v = \cos t$, $x_0 = 3\cos t$, $y_0 = \sin 2t$ and $t_n = 0.0000032$.

(b) Take $u = \sin t$, $v = \cos t$, $x_0 = 3\cos t$, $y_0 = \sin 2t$ and $t_n = 0.00000051$.

Case 2

(a) Take $u = 2t$, $v = t + 1$, $x_0 = t^2$, $y_0 = t^2 + 1$ and $t_n = 0.00018$.

(b) Take $u = 2t$, $v = t + 1$, $x_0 = t^2$, $y_0 = t^2 + 1$ and $t_n = 0.00000071$.

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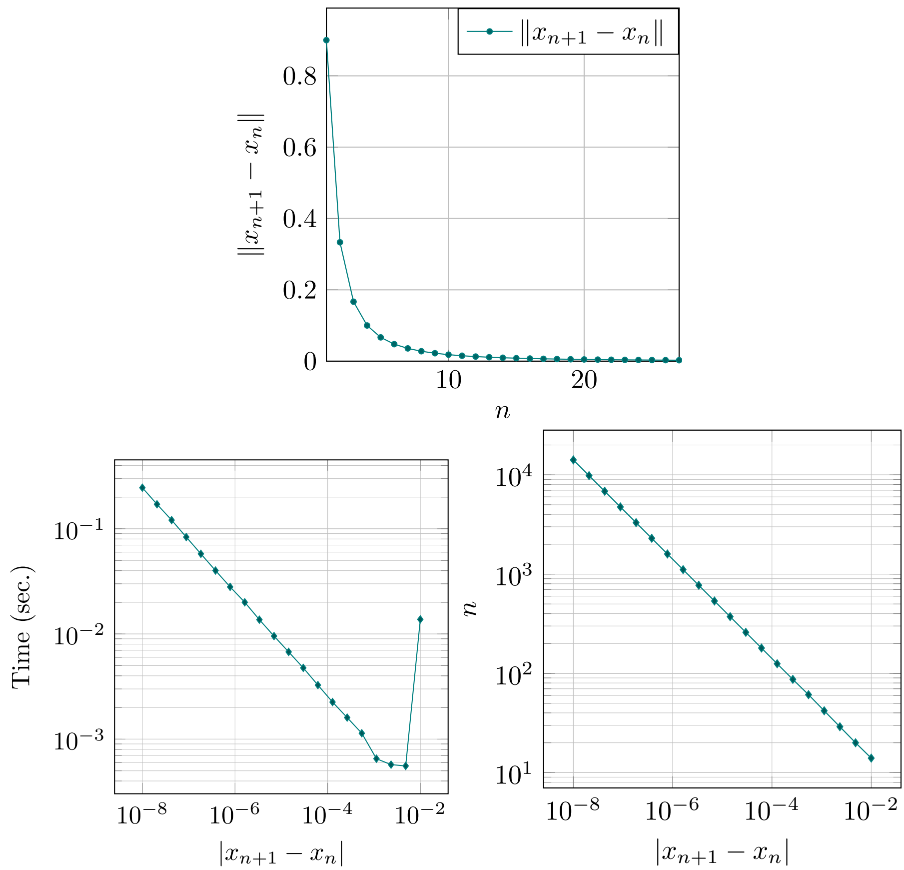

Figure 1: **Example 4.1, Case I** (a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

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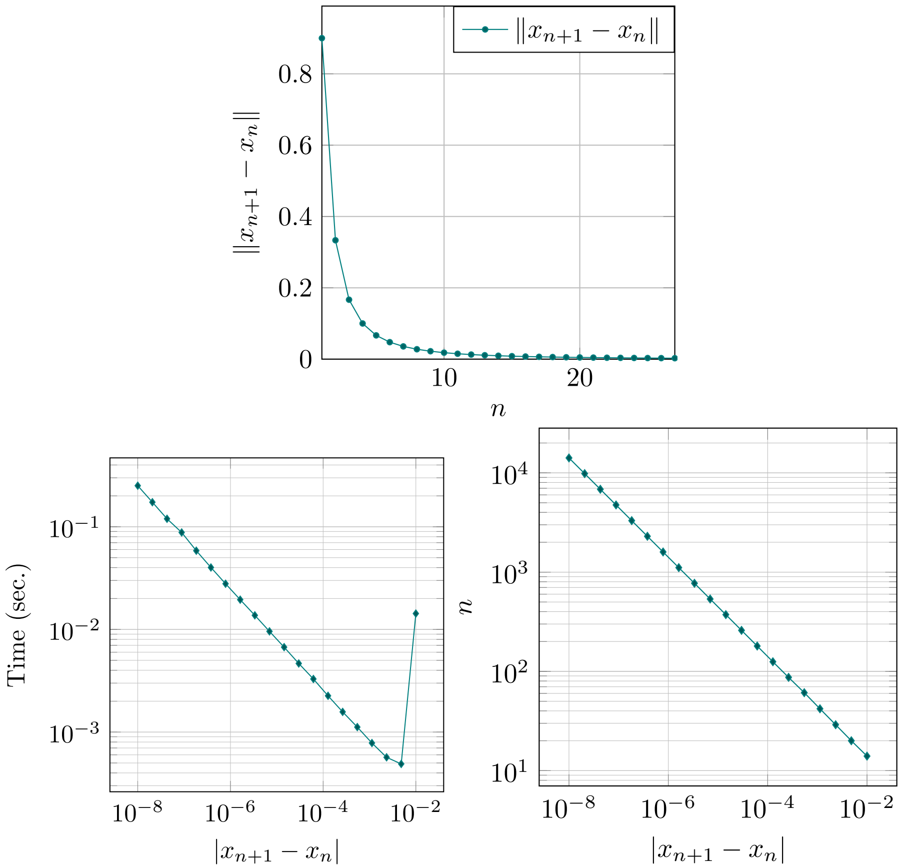

Figure 2: **Example 4.1, Case I** (b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

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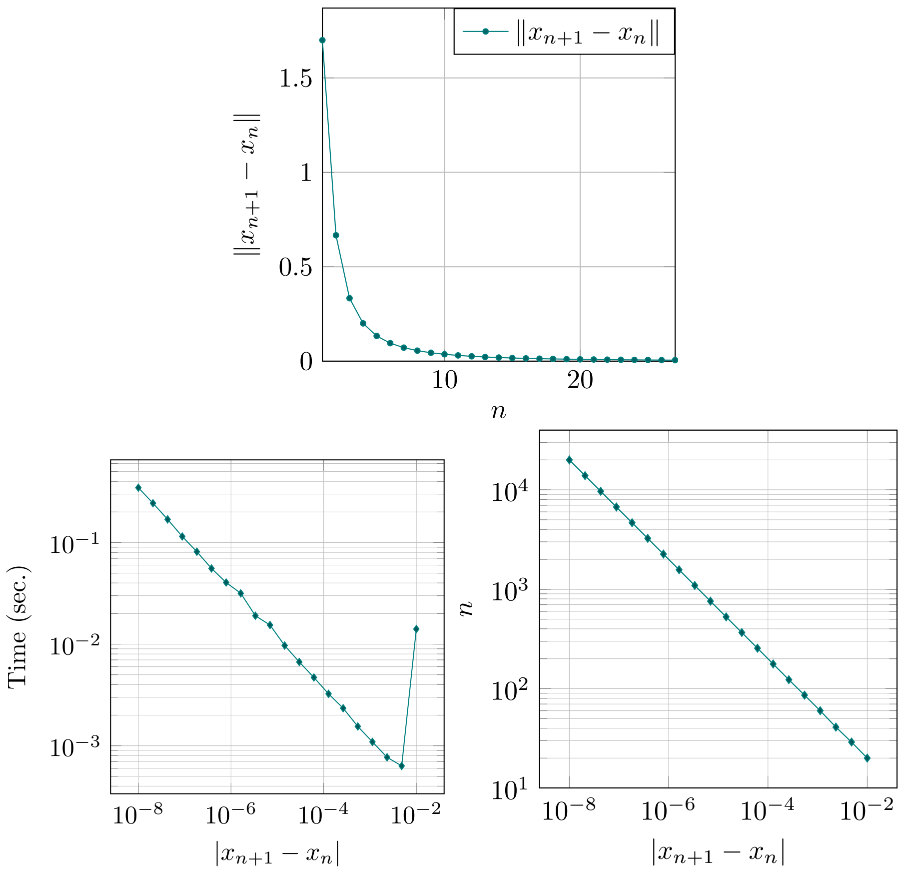

Figure 3: **Example 4.1, Case II** (a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

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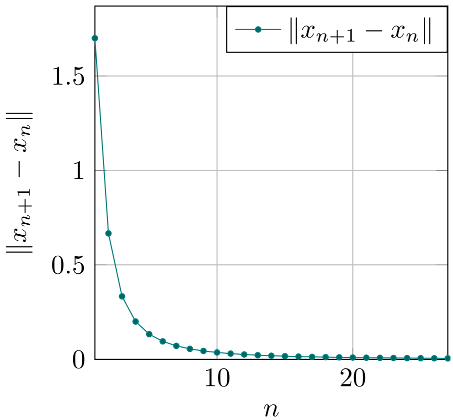

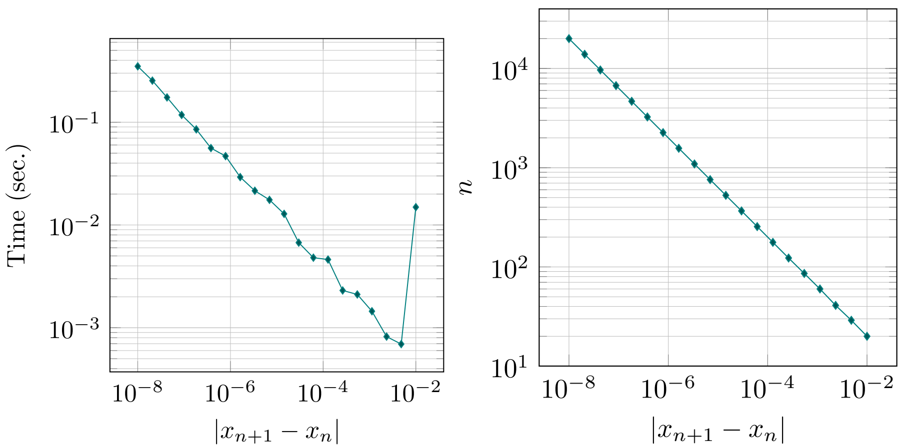

Figure 4: **Example 4.1, Case II** (b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

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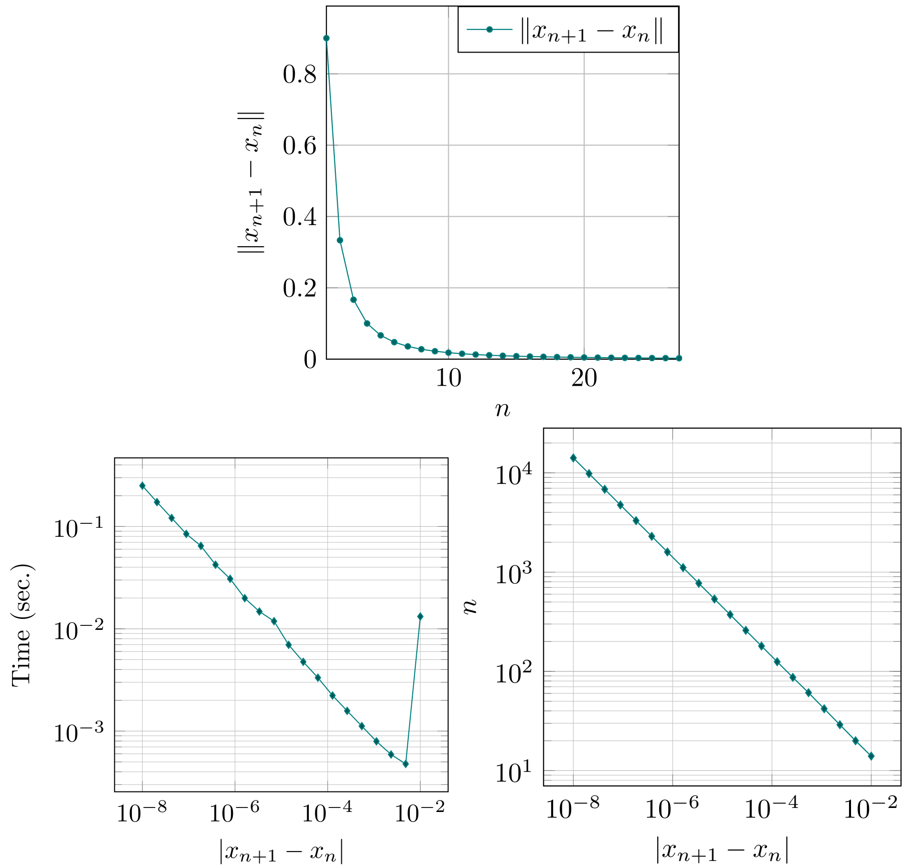

Figure 5: **Example 4.1, Case III** (a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

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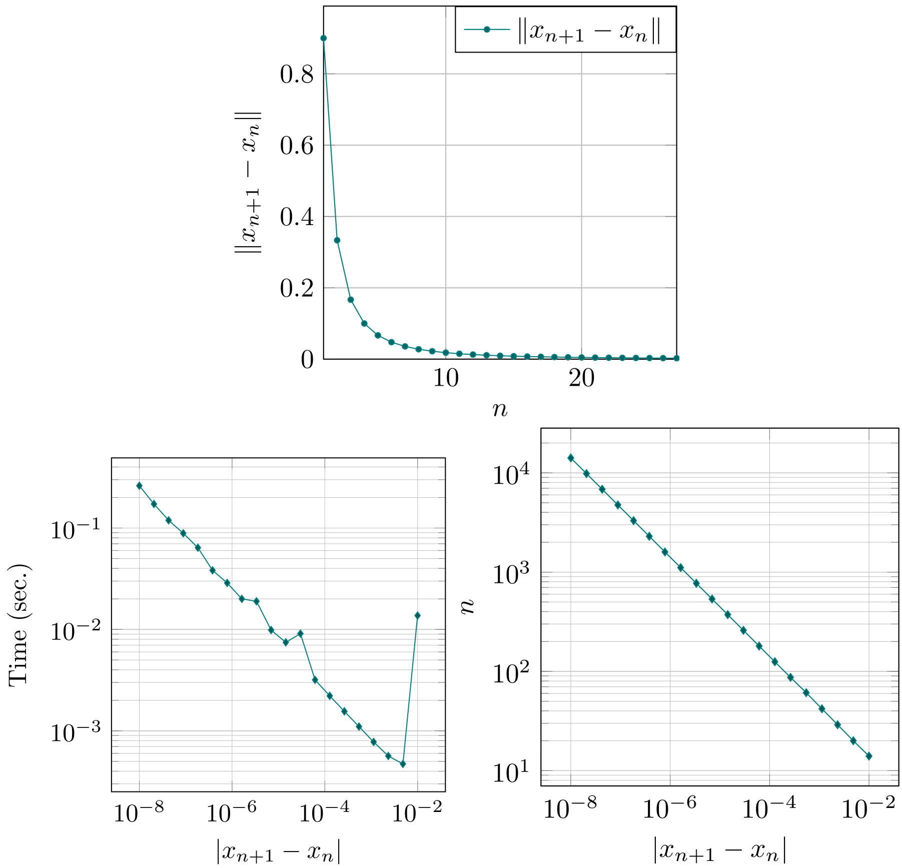

Figure 6: **Example 4.1, Case III** (b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

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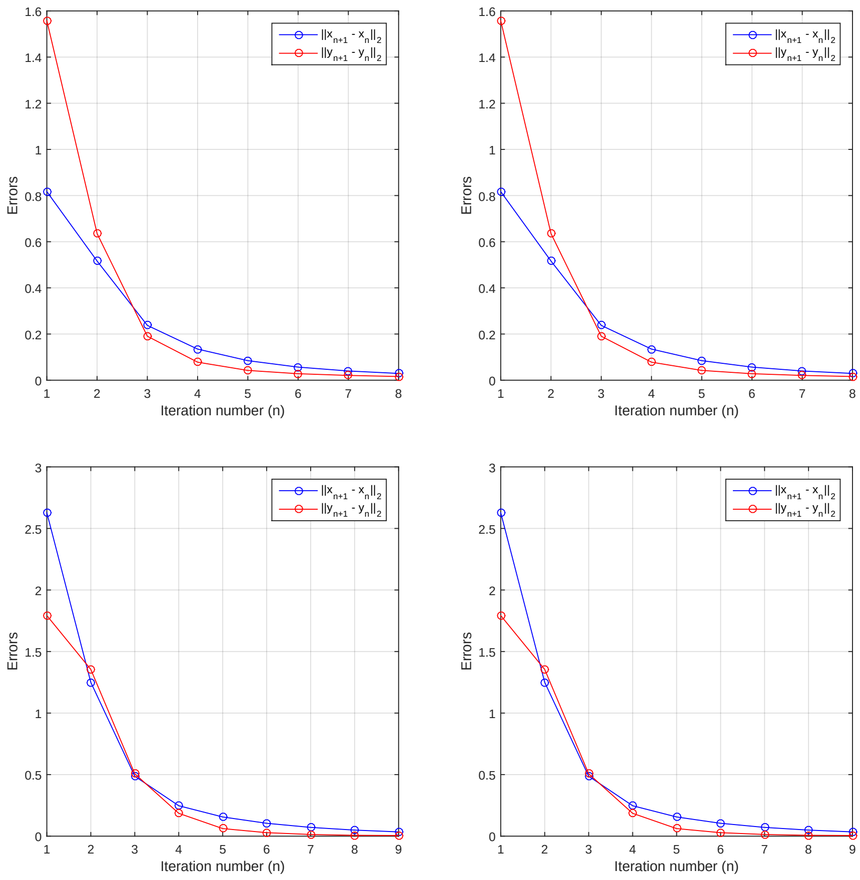

Figure 7: Errors vs Iteration numbers for **Example 4.2**: **Case 1** (a) (top left); **Case 1** (b) (top right); **Case 2** (a) (bottom left); **Case 2** (b) (bottom right).

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