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Parallel projected subgradient method for solving split system of fixed point set constraint equilibrium problems in Hilbert spaces

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Abstract. In this paper, we propose two strongly convergent algorithms which combine the Mann iterative scheme, the diagonal subgradient method, the projection method and the proximal method for solving split system of fixed point set constrained equilibrium problems in real Hilbert spaces. The computation of the first algorithm requires prior knowledge of operator norm. The problem of finding, or at least estimating the norm of an operator, in general, is not an easy task in Hilbert spaces. Based on the first algorithm, we propose another algorithm with a way of selecting the step-sizes such that its implementation does not need any prior information about the operator norm. The strong convergence properties of the algorithms are established under mild assumptions on equilibrium bifunctions.

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1. Introduction

Split Inverse Problem (SIP) is an archetypal model presented in [6, Sect. 2] given by

$$\begin{cases} \text{find } x^* \in X \text{ that solves IP1} \\ \text{such that} \\ y^* = Ax^* \in Y \text{ and solves IP2} \end{cases} \tag{1.1}$$

where A is a bounded linear operator from a space X to another space Y and IP1 and IP2 are two inverse problems installed in X and Y, respectively. Real-world inverse problems can be cast into this framework by making different choices of the spaces X and Y (including the case $X = Y$), and by choosing

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appropriate inverse problems for IP1 and IP2. Different choices for IP1 and IP2 are proposed in many research works and literature, such as minimization problems, equilibrum problems and inclusion problems, see for example [4, 15, 16, 22, 11]. The split feasibility problem is the first instance of an SIP where the two problems IP1 and IP2 are of the Convex Feasibility Problems type.

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For a bifunction $f : C \times C \to \mathbb{R}$, the problem

$$\text{find } z^* \in C \text{ such that } f(z^*, z) \geq 0,\ \forall z \in C \tag{1.2}$$

is called the *equilibrium problem* (Fan inequality [12]) of f on C, denoted by $\mathrm{EP}(f, C)$. The set of all solutions of the $\mathrm{EP}(f, C)$ is denoted by $\mathrm{SEP}(f, C)$, i.e., $\mathrm{SEP}(f, C) = \{z^* \in C : f(z^*, z) \geq 0,\ \forall z \in C\}$. If $f(x, y) = \langle Ax, y - x\rangle$ for every $x, y \in C$ where A is a mapping from C into H, then the equilibrium problem becomes the classical variational inequality problem studied in [5, 7, 9, 26, 30], i.e., finding a point $x^* \in C$ such that $\langle Ax^*, y - x^*\rangle \geq 0,\ \ \forall y \in C$.

For a given bifunction $f : C \times C \to \mathbb{R}$ and with its suitable assumption Combettes and Hirstoaga [8] proved that the resolvent operator $T_r^f : H \to C$ (for $r > 0$) given by

$$T_r^f(u) = \{w \in C : g(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0,\ \forall v \in C\} \tag{1.3}$$

is single-valued and firmly nonexpansive. The resolvent operator is used to approximate the solution of $\mathrm{EP}(f, C)$ by many authors (see for instance, [1, 8, 24, 25]). Proximal operator is also used as a method of solving the equilibrium problem. It is well known that if $h : C \to \mathbb{R}$ is convex and lower semicontinuous, $\lambda > 0$, then the proximity operator of h, defined by $\mathrm{prox}_{\lambda h} : C \to C$ given by

$$\mathrm{prox}_{\lambda h}(x) = \arg\min\{\lambda h(y) + \frac{1}{2}\|x - y\|^2 : y \in C\} \tag{1.4}$$

is single valued. The proximal-like method has also been called the extragradient method. Extragradient method is a widely used method of solving the equilibrium problem (1.2) (see, for instance, [13]). Santos and Scheimberg [27] used projection method and subgradient method for solving the equilibrium problem (1.2) in a finite-dimensional space. They proposed the following:

$$\begin{cases} x_1 \in C, \ \ w_n \in \partial_{\epsilon_n} f(x_n, .)(x_n), \quad \eta_n = \max\{\rho_n, \|w_n\|\}, \\ \alpha_n = \frac{\beta_n}{\eta_n}, \quad x_{n+1} = P_C(x_n - \alpha_n w_n), \end{cases}$$

where $\{\rho_n\}$, $\{\beta_n\}$ and $\{\epsilon_n\}$ are nonnegative real sequences such that

$$\rho_n \geq \rho > 0,\ \beta_n \geq 0,\ \epsilon_n \geq 0,\ \sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty,\ \sum_{n=1}^{\infty} \frac{\beta_n \epsilon_n}{\rho_n} < +\infty,\ \sum_{n=1}^{\infty} \beta_n^2 < +\infty.$$

Under suitable assumptions on the bifunction f, the sequence $\{x_n\}$ generated by the algorithm strongly converges to $x^* \in \mathrm{SEP}(f, C)$. The algorithm uses only one projection and does not require any Lipschitz condition for the bifunction.

Later on, many iterative algorithms were considered to find the point $\bar{x} \in FixU \bigcap \mathrm{SEP}(f, C)$ where $U : C \to C$ is a nonexpansive mapping; see [2, 3, 25, 28, 29]. A mapping $U : C \to C$ is said to be *nonexpansive* if $\|U(x) - U(y)\| \leq \|x - y\|, \quad \forall x, y \in C$. The set of fixed points of U is denoted by $FixU$ and is given by $FixU = \{x \in C : Ux = x\}$. Most of the existing algorithms for this problem are based on the proximal point method applying to equilibrium problem $EP(C, f)$ combining with a Mann's iteration to the problem of finding a fixed point of U. In 2013, Anh and Muu [3] propose a strongly convergent algorithm for finding a point in $FixU \bigcap \mathrm{SEP}(f, C)$.

In this paper, we propose iterative algorithm solving the combination of equilibrium problems and fixed point problems in the framework of SIP, the so called, *split system of fixed point set constraint equilibrium problem* (SSFPSCEP). Let $\Phi = \{1, ..., N\}$, $\Psi = \{1, ..., M\}$, $\Phi' = \{1, ..., N'\}$, $\Psi' = \{1, ..., M'\}$, and $A : H_1 \to H_2$ be a nonzero bounded linear operator. Suppose C is a nonempty closed convex subset of H_1 and $U_{i'} : C \to C$ are nonexpansive operators for $i' \in \Phi'$, and D is a nonempty closed convex subset of H_2 and $V_{j'} : D \to D$ are nonexpansive operators for $j' \in \Psi'$. Given bifunctions $f_i : C \times C \to \mathbb{R}$ for $i \in \Phi$, and $g_j : D \times D \to \mathbb{R}$ for $j \in \Psi$, the SSFPSCEP is finding a point $x^* \in H_1$ with the property that

$$x^* \in \Omega_1 = \Big(\bigcap_{i' \in \Phi'} FixU_{i'} \Big) \bigcap \Big(\bigcap_{i \in \Phi} \mathrm{SEP}(f_i, C) \Big)$$
$$\text{such that} \tag{1.5}$$
$$Ax^* \in \Omega_2 = \Big(\bigcap_{j' \in \Psi'} FixV_{j'} \Big) \bigcap \Big(\bigcap_{j \in \Psi} \mathrm{SEP}(g_j, D) \Big).$$

Let Γ be the solution set of SSFPSCEP (1.5), i.e., $\Gamma = \{x^* \in \Omega_1 : Ax^* \in \Omega_2\}$. For a subset Q of a real Hilbert space H, Id_Q is a mapping from Q onto Q given by $Id_Q(x) = x$ for all $x \in Q$. Thus, if $U_{i'} = Id_C$ for all $i' \in \Phi'$ and $V_{j'} = Id_D$ for all $j' \in \Psi'$, then problem (1.5) is reduced to *split system equilibrium problem* (SSEP).

The ongoing researches are directed toward reducing the computational difficulty by imposing a weaker condition on each f_i and establish relatively simple algorithms for a wide class of problems. In 2012, He [18] introduced iterative algorithm solving SSEP imposing the same conditions on f_i and $g_j = g$ for all i and j so that the bifunctions f_i and g are treated the same way using regularization technique (the resolvent operator) (1.3). However, regularization technique (1.3) is not computationally easier, and if each bifunction is more general monotone, for instance pseudomonotone, then problem (1.3) in general is not strongly monotone. So, the unique solvability of problem (1.3) is not guaranteed, even its solution set might not be. Hence, employing (1.3) for each f_i and $g_j = g$ in yields a computationally difficult algorithm. For this reason, the authors in [19] proposed algorithms for solving SSEP using the extragradient method (proximal operator for f_i) replacing problem (1.3) by the following two strongly convex programs;

$$\begin{cases} y_n^i = \arg\min\{\lambda_n f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \quad i \in \Phi, \\ z_n^i = \arg\min\{\lambda_n f_i(y_n^i, y) + \frac{1}{2}\|y_n^i - y\|^2 : y \in C\}, \quad i \in \Phi, \end{cases} \tag{1.6}$$

where λ_n is a suitable parameter and each f_i satisfy a certain Lipschitz-type condition. The advantage of the extragradient method is that two optimization programs are solved at each iteration which seems to be computed easily by the Matlab Optimization Toolbox. However, this might still be costly and affects the efficiency of the used method if the structure of feasible set and equilibrium bifunction are complex. Moreover, Lipschitz-type condition depends on two positive parameters c_1 and c_2 which are unknown in some cases, or difficult to approximate.

We designed algorithms for solving (1.5) that require only one projection rather than two strongly convex programs (1.6) for each $i \in \Phi$. The algorithms combine the well-known Mann iterative scheme for fixed point [23] and two methods including the projection method and the diagonal subgradient method on the foundation of projected subgradient algorithms proposed by Hieu [20], which generate a sequence $\{x_n\}$ by

$$\tag{1.7} \begin{cases} x_1 \in C, \;\; \epsilon_n \in (0,\infty), \\ w_n^i \in \partial_{\epsilon_n} f_i(x_n,.)(x_n), \;\; \alpha_n^i = \frac{\beta_n}{\eta_n^i}, \;\; \eta_n^i = \max\{\rho_n, \|w_n^i\|\}, \;\; i \in \Phi, \\ y_n^i = P_C(x_n - \alpha_n^i w_n^i), \;\; i \in \Phi, \\ x_{n+1} = \sum\limits_{i\in\Phi} \xi_n^i y_n^i, \end{cases}$$

and $\{x_n\}$ strongly converges to a common element of the set of solution of the system of equilibrium problems $\bigcap_{i\in\Phi} EP(f_i, C)$ for pseudomonotone bifunctions f_i. Another advantage of our algorithms is that, as a result of projected subgradient method the convergence of our algorithms are proved under pseudomonotone assumptions of the bifunction and without Lipschitz-type condition of each f_i. Comparing with the algorithms in [19], our proposed algorithms solve a wide class of problems and have a simple structure, and the metric projection, in general, is simpler than solving strongly convex optimization subproblems on the same feasible set and finding shrinking projections. Furthermore, the results in [14, 21] are particular cases of our problem and hence our algorithms are more general.

We formulated two iterative algorithms to find a solution for SSFPSCEPs (1.5) and we proved the strong convergence for the algorithms. In the first algorithm, $N + 1$ projections on the feasible set need to be computed per each iteration and the prior knowledge of operator norm (or at least an estimate of the operator norm) is needed. However, to employ the second algorithm, one does not need any prior information about the norm of the bounded linear operator A. Moreover, only one projection is performed on the feasible set while the first N parallel projections over C are replaced by N parallel projections onto a tangent plane to C in order to reduce the number of optimization subproblems to be solved.

This paper is organized in the following way. In Section 2, we recall some basic definitions and lemmas that are useful for our main result. In Section 3, we present two algorithms solving *split system of fixed point set constraint equilibrium problem* (1.5) and we analyze the convergence result of our proposed algorithms. Finally, we give some conclusions.

2. Preliminaries

In order to state and prove our main results, we recall some notations, definitions, and some useful results which will be needed in the sequel. The symbols " $\rightharpoonup$ " and " $\to$ " denote weak and strong convergence, respectively.

Let H be a real Hilbert space and C a nonempty closed convex subset of H. The *metric projection* on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Properties: Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be a metric projection on C. Since C is nonempty, closed and convex, $P_C(x)$ exists and is unique. From the definition of P_C, it is easy to show that P_C has the following characteristic properties:

(i) For all $y \in C$, $\|P_C(x) - x\| \leq \|x - y\|$.

(ii) For all $x, y \in H$, $\|P_C(x) - P_C(x)\|^2 \leq \langle P_C(x) - P_C(x), x - y\rangle$.

(iii) For all $x \in C$, $y \in H$, $\|x - P_C(y)\|^2 + \|P_C(y) - y\|^2 \leq \|x - y\|^2$.

(iv) $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$.

Lemma 2.1. *[17] Suppose C is closed convex subset of a Hilbert space H and $U : C \to C$ is a nonexpansive mapping. Then*

(i) If U has a fixed point, then $FixU$ is a closed convex subset of H.

(ii) $Id_C - U$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(Id_C - U)x_n\}$ strongly converges to some y, it follows that $(Id_C - U)x = y$.

Lemma 2.2. *[31] If $\{a_n\}$ and $\{b_n\}$ are two nonnegative real sequences such that*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq n_0,$$

with $\sum b_n < \infty$, then $\{a_n\}$ converges.

Lemma 2.3. *[10, Proposition 4.34] Suppose C is closed convex subset of a Hilbert space H and $U_i : C \to C$ be nonexpansive mappings for $i \in R = \{1, 2, ..., q\}$ such that $\bigcap_{i=1}^{q} FixU_i \neq \emptyset$. Let $U(x) := \sum_{i=1}^{q} \theta_i U_i(x)$ with $0 < \theta_i \leq 1$ for every $i \in R$ and $\sum_{i=1}^{q} \theta_i = 1$. Then U is nonexpansive and $FixU = \bigcap_{i=1}^{q} FixU_i$.*

Lemma 2.4. *(Opial's condition) For any sequence $\{x_n\}$ in the Hilbert space H with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to+\infty} \|x_n - x\| < \liminf_{n\to+\infty} \|x_n - y\|$$

holds for each $y \in H$ with $y \neq x$.

Definition 2.5. Let H be a Hilbert space and $f : C \times C \to \mathbb{R}$ be a bifunction where $f(x, .)$ is a convex function for each x in C. Then for $\epsilon \geq 0$ the ϵ-subdifferential (ϵ-diagonal subdifferential) of f at x, denoted by $\partial_\epsilon f(x, .)(x)$ or $\partial_\epsilon f(x, x)$, is given by

$$\partial_\epsilon f(x, .)(x) = \{w \in H : f(x, y) - f(x, x) + \epsilon \geq \langle w, y - x\rangle, \quad \forall y \in C\}.$$

If $\partial_\epsilon f(x, .)(x) \neq \emptyset$, $f(x, .)$ is said to be ϵ-subdifferentiable (subdifferentiable) on C at x.

Let C be a subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ be a bifunction. Then, f is said to be

(i) *strongly monotone* on C, if there is $M > 0$ (M-strongly monotone on C) iff

$$f(x, y) + f(y, x) \leq -M\|y - x\|^2, \ \forall x, y \in C.$$

(ii) *monotone* on C iff $f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C$.

(iii) *pseudomonotone* on C with respect to $x \in C$ iff $f(x, y) \geq 0$ implies $f(y, x) \leq 0, \ \forall y \in C$.

(iv) *Lipschitz-type continous* on C if there exist positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \ \forall x, y, z \in C.$$

We say that f is pseudomonotone on C with respect to $S \subset C$ if it is pseudomonotone on C with respect to every $x \in S$, i.e., if $x \in S$ and $y \in C$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

When $S = C$, f is called pseudomonotone on C. Clearly, $(i) \Rightarrow (ii) \Rightarrow (iii)$. It is clear that monotone bifunction is pseudomonotone.

Let $SEP(f_{\leq}, C)$ represent the solution of the problem

$$\text{find } x^* \in C \text{ such that } f(y, x^*) \leq 0, \ \forall y \in C, \tag{2.1}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction on a closed convex subset of a Hilbert space H. When f is a pseudomonotone bifunction on C, it holds that $SEP(f, C) \subset SEP(f_{\leq}, C)$. Moreover, this inclusion is also valid for monotone bifunctions.

Let C be a closed, convex subset of a real Hilbert space H. Then the bifunction $f : C \times C \to \mathbb{R}$ is said to satisfy **Condition I** on C if the following six conditions are satisfied:

(B1) $f(x, x) = 0$ for all $x \in C$;

(B2) $SEP(f, C) \subset SEP(f_{\leq}, C)$;

(B3) f satisfies the strict paramonotonicity property; i.e.,

$$x \in SEP(f, C), \;\; y \in C, \;\; f(y, x) = 0 \Rightarrow y \in SEP(f, C);$$

(B4) $f(., y)$ is weakly sequencially upper semicontinuous on C with every fixed $y \in C$, i.e., $\limsup_{n\to\infty} f(x_n, y) \leq f(x, y)$ for each sequence $\{x_n\}$ in C converging weakly to x;

(B5) if $\{x_n\}$ is a bounded sequence in C, then the sequence $\{w_n\}$ with $w_n \in \partial_{\epsilon_n} f(x_n, .)(x_n)$ is bounded;

(B6) $f(x, .)$ is convex, lower semicontinous and subdifferentiable on C, for all $x \in C$.

The assumption (B2) is pseudomonotonicity of f on C with respect to the solution set SEP(f, C). The following example shows that assumption (B2) of Condition I is weaker than the pseudomonotonicity assumption of f on C.

Example 2.6. Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, given by

$$f(x, y) = 2y|x|(y - x) + xy|y - x|.$$

The bifunction f is not pseudomonotone on $C = [-1, 1]$. But, $SEP(f, C) = \{0\}$ and $f(y, x^*) = f(y, 0) = 0$ for all $y \in [-1, 1]$. Hence, (B2) holds. In fact, we have $f(-0.5, 0.5) = f(0.5, -0.5) = 0.25 > 0$.

Let D be a closed, convex subset of a real Hilbert space H and $g : D\times D \to \mathbb{R}$ be a bifunction. Then we say that g satisfies **Condition II** on D if the following four assumptions are satisfied:

(A1) $g(u, u) = 0$, for all $u \in D$;

(A2) g is monotone on D, i.e., $g(u, v) + g(v, u) \leq 0$, for all $u, v \in D$;

(A3) for each $u, v, w \in D$

$$\limsup_{\alpha\downarrow 0} g(\alpha w + (1 - \alpha)u, v) \leq g(u, v);$$

(A4) $g(u, .)$ is convex and lower semicontinuous on D for each $u \in D$.

We introduce the following results from equilibrium programming in Hilbert Spaces which are useful in the discussion of solving equlibrum problem.

Lemma 2.7. *[8] Let g satisfy Condition CO on D. Then, for each $r > 0$ and $u \in H_2$, there exists $w \in D$ such that*

$$g(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0, \; \forall v \in D.$$

Lemma 2.8. *[8] Let g satisfy Condition CO on D. Then, for each $r > 0$ and $u \in H_2$, define a mapping (called resolvant of g), given by*

$$T_r^g(u) = \left\{w \in D : g(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0, \ \forall v \in D\right\}.$$

Then the following hold:

(i) *T_r^g is single-valued;*

(ii) *T_r^g is firmly nonexpansive, i.e., for all $u, v \in H$,*

$$\|T_r^g(u) - T_r^g(v)\|^2 \leq \langle T_r^g(u) - T_r^g(v), u - v\rangle;$$

(iii) *$Fix(T_r^g) = SEP(g, D)$, where $Fix(T_r^g)$ is the fixed point set of T_r^g;*

(iv) *$SEP(g, D)$ is closed and convex.*

Lemma 2.9. *[8] For $r, s > 0$ and $u, v \in H$. Under the assumptions of Lemma 2.8,*

$$\|T_r^g(u) - T_s^g(v)\| \leq \|u - v\| + \frac{|s - r|}{s}\|T_s^g(v) - v\|.$$

3. Main result

In this section, we propose two algorithms for solving SSFPSCEP (1.5) with assumptions that each f_i satisfies **Condition I** on C for all $i \in \Phi$, each g_j satisfies **Condition II** on D for all $j \in \Psi$, and Γ is nonempty.

By (B1), (B4) and (B2) of Condition I, the set $SEP(f_i, C)$ is closed and convex. Thus, the set Ω_1 is closed and convex. Moreover, from Lemma 2.8 *(iv)*, $SEP(g_j, D)$ is closed and convex. Hence, Ω_2 is closed and convex in H_2. Therefore, by linear property of the operator A, the solution set Γ is a closed and convex subset of H_1. Hence, P_Γ is well defined.

3.1. Algorithm requiring prior knowledge of operator norm

In order to design the algorithm, we consider parameter sequences satisfying the following conditions.

Condition 1

(C1) $\rho_n \geq \rho > 0$, $\beta_n \geq 0$, $\epsilon_n \geq 0$, $0 < \sigma_1 \leq \delta_n \leq \sigma_2 < 1$.

(C2) $r_n \geq r > 0$, $0 < \gamma_1 \leq \mu_n \leq \gamma_2 < \frac{1}{\sigma^2}$ for some $\sigma \in [\|A\|, +\infty)$.

(C3) $0 < \xi \leq \xi_n^i \leq 1$, $(i \in \Phi)$ such that $\sum_{i \in \Phi} \xi_n^i = 1$ for each $n \geq 1$.

(C4) $0 < \theta \leq \theta_n^{j'} \leq 1$, $(j' \in \Psi')$ such that $\sum_{j' \in \Psi'} \theta_n^{j'} = 1$ for each $n \geq 1$.

(C5) $\sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty, \quad \sum_{n=1}^{\infty} \frac{\beta_n \epsilon_n}{\rho_n} < +\infty, \quad \sum_{n=1}^{\infty} \beta_n^2 < +\infty.$

In the formulation of the following algorithm, we need a real number σ such that either $\sigma = \|A\|$ or at least $\sigma > \|A\|$. Hence, it requires prior knowledge or estimated value of operator norm $\|A\|$. The algorithm involves the evaluation of $N+1$ projections on the feasible set C where the first N projections are computed in parallel.

Algorithm 3.1
Initialization: Choose $x_1 \in C$ and the parameter sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, $\{r_n\}$, $\{\delta_n\}$, $\{\xi_n^i\}$ $(i \in \Phi)$, $\{\theta_n^{j'}\}$ $(j' \in \Psi')$ and $\{\mu_n\}$ which satisfy Condition 1.

Step 1. For each $i \in \Phi$, take $w_n^i \in H_1$ such that $w_n^i \in \partial_{\epsilon_n} f_i(x_n, .)(x_n)$.

Step 2. For each $i \in \Phi$, calculate $\alpha_n^i = \frac{\beta_n}{\eta_n^i}, \quad \eta_n^i = \max\{\rho_n, \|w_n^i\|\}$ and

$$y_n^i = P_C(x_n - \alpha_n^i w_n^i).$$

Step 3. Evaluate $y_n = \sum_{i \in \Phi} \xi_n^i y_n^i$.

Step 4. For each $i' \in \Psi'$ find $t_n^{i'} = \delta_n x_n + (1-\delta_n) U_{i'}(y_n)$.

Step 5. Find among $t_n^{i'}$, $i' \in \Psi'$, the farthest element from x_n, i.e.,

$$t_n = \arg\max\{\|v - x_n\| : v \in \{t_n^{i'} : i' \in \Phi'\}\}.$$

Step 6. For each $j \in \Psi$ find $u_n^j = T_{r_n}^{g_j}(At_n), \quad j \in \Psi$.

Step 7. Find among u_n^j, $j \in \Psi$, the farthest element from At_n, i.e.,

$$u_n = \arg\max\{\|v - At_n\| : v \in \{u_n^j : j \in \Psi\}\}.$$

Step 8. Evaluate $x_{n+1} = P_C\Big(t_n + \mu_n A^*\Big(\sum_{j' \in \Psi'} \theta_n^{j'} V_{j'}(u_n) - At_n\Big)\Big)$.

Step 9. Set $n := n+1$ and go to Step 1.

Remark 3.1. Each $f_i(x, .)$ is a lower semicontinuous convex function and $C \subset dom f_i(x, .)$ for each $x \in C$, and thus ϵ_n-diagonal subdifferential $\partial_{\epsilon_n} f_i(x_n, .)(x_n)$ is nonempty for every $\epsilon_n > 0$. Moreover, by Combettes and Hirstoaga in [8], for each r_n the problems in Step 6 are uniquely solvable, and since C is a nonempty closed convex set, the projection in Step 8 exists and is unique. Therefore, all steps in Algorithm 3.1 are defined with no ambiguity and Algorithm 3.1 is well defined.

For the sake of simplicity, we define

(1a) $W_n = \sum_{j' \in \Psi'} \theta_n^{j'} V_{j'}$ for $n \geq 1$,

(2a) $\{i'_n\}_{n=1}^{\infty}$ is a sequence where for each n, $i'_n \in \Phi'$ such that

$$t_n = t_n^{i'_n} = \arg\max\{\|v - x_n\| : v \in \{t_n^{i'} : i' \in \Phi'\}\},$$

i.e., $t_n = t_n^{i'_n} = \delta_n x_n + (1 - \delta_n) U_{i'_n}(y_n)$,

(3a) $\{j_n\}_{n=1}^{\infty}$ is a sequence where for each n, $j_n \in \Psi$ such that

$$u_n = T_{r_n}^{g_{j_n}} A t_n = u_n^{j_n} = \arg\max\{\|v - At_n\| : v \in \{u_n^j : j \in \Psi\}\}.$$

Remark 3.2. By Lemma 2.3 each W_n is a nonexpansive mapping and $FixW_n = \bigcap_{j' \in \Psi'} FixV_{j'}$ for all $n \geq 1$. Hence, it is easy to see that

$$FixW_1 = FixW_2 = \ldots = FixW_n = \ldots.$$

To establish the convergence of Algorithm 3.1, we need the following Lemmas.

Lemma 3.3. *For sequences $\{y_n^i\}$ $(i \in \Phi)$, $\{y_n\}$, $\{t_n\}$ and $\{x_n\}$ generated by Algorithm 3.1, we have*

$$\|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2(1 - \delta_n) \sum_{i \in \Phi} \xi_n^i \alpha_n^i f_i(x_n, x^*) - L_n + \vartheta_n, \quad \forall x^* \in \Gamma$$

where $L_n = (1 - \delta_n) \sum_{i \in \Phi} \xi_n^i \|x_n - y_n^i\|^2 + \delta_n (1 - \delta_n) \|U_{i'_n}(y_n) - x_n\|^2$ *and* $\vartheta_n = 2(1 - \delta_n) \frac{\beta_n \epsilon_n}{\rho_n} + 2(1 - \delta_n) \beta_n^2$.

Proof. Let $x^* \in \Gamma$. From $y_n^i = P_C(x_n - \alpha_n^i w_n^i)$ and $x^* \in \Gamma$, we have

$$\langle x_n - \alpha_n w_n^i - y_n^i, y_n^i - x^* \rangle \geq 0,$$

implying that

$$\begin{aligned} \langle x^* - y_n^i, x_n - y_n^i \rangle &\leq \alpha_n^i \langle w_n^i, x^* - y_n^i \rangle \\ &= \alpha_n^i \langle w_n^i, x^* - x_n \rangle + \alpha_n^i \langle w_n^i, x_n - y_n^i \rangle \\ &\leq \alpha_n^i \langle w_n^i, x^* - x_n \rangle + \alpha_n^i \|w_n^i\| \|x_n - y_n^i\|. \end{aligned} \tag{3.1}$$

But also $x_n \in C$. Thus,

$$\langle x_n - \alpha_n^i w_n^i - y_n^i, y_n^i - x_n \rangle \geq 0,$$

and this together with (3.1) gives us

$$\langle x_n - y_n^i, x_n - y_n^i \rangle = \|x_n - y_n^i\|^2 \leq \alpha_n^i \langle w_n^i, x_n - y_n^i \rangle \leq \alpha_n^i \|w_n^i\| \|x_n - y_n^i\|.$$

That is, $\|x_n - y_n^i\| \le \alpha_n^i \|w_n^i\|$. Thus,

$$\alpha_n^i \|w_n^i\| \|x_n - y_n^i\| \le (\alpha_n^i \|w_n^i\|)^2 = \Big(\frac{\beta_n \|w_n^i\|}{\eta_n^i}\Big)^2$$

$$= \beta_n^2 \Big(\frac{\|w_n^i\|}{\max\{\rho_n, \|w_n^i\|\}}\Big)^2 \le \beta_n^2. \tag{3.2}$$

Since $x_n \in C$ and $w_n \in \partial_{\epsilon_n} f_i(x_n, .)(x_n)$, we have

$$f_i(x_n, x^*) + \epsilon_n = f_i(x_n, x^*) - f_i(x_n, x_n) + \epsilon_n \ge \langle w_n, x^* - x_n\rangle. \tag{3.3}$$

Using definition of α_n^i and η_n^i, we obtain

$$\alpha_n^i = \frac{\beta_n}{\eta_n^i} = \frac{\beta_n}{\max\{\rho_n, \|w_n^i\|\}} \le \frac{\beta_n}{\rho_n}. \tag{3.4}$$

From (3.1)-(3.4), we have

$$\langle x^* - y_n^i, x_n - y_n^i\rangle \le \alpha_n^i f_i(x_n, x^*) + \tfrac{\beta_n \epsilon_n}{\rho_n} + \beta_n^2. \tag{3.5}$$

However,

$$2\langle x^* - y_n^i, x_n - y_n^i\rangle = \|y_n^i - x^*\|^2 + \|x_n - y_n^i\|^2 - \|x_n - x^*\|^2. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\|y_n^i - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - y_n^i\|^2 + 2\alpha_n^i f_i(x_n, x^*) + 2\tfrac{\beta_n \epsilon_n}{\rho_n} + 2\beta_n^2. \tag{3.7}$$

Then, by definition of t_n and by convexity of $\|.\|^2$, we have

$$\begin{aligned}
&\|t_n - x^*\|^2 = \|\delta_n x_n + (1-\delta_n) U_{i'_n}(y_n) - x^*\|^2 \\
&= \|\delta_n (x_n - x^*) + (1-\delta_n)(U_{i'_n}(y_n) - x^*)\|^2 \\
&= \delta_n \|x_n - x^*\|^2 + (1-\delta_n)\|U_{i'_n}(y_n) - x^*\|^2 - \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2 \\
&= \delta_n \|x_n - x^*\|^2 + (1-\delta_n)\|U_{i'_n}(\sum_{i\in\Phi} \xi_i^n y_n^i) - U_{i'_n}(x^*)\|^2 \\
&\qquad\qquad - \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2 \\
&\le \delta_n \|x_n - x^*\|^2 + (1-\delta_n)\|\sum_{i\in\Phi} \xi_n^i y_n^i - x^*\|^2 - \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2 \\
&\le \delta_n \|x_n - x^*\|^2 + (1-\delta_n)\sum_{i\in\Phi} \xi_i^n \|y_n^i - x^*\|^2 - \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2.
\end{aligned}$$

The last result together with (3.7) proves the lemma. □

Lemma 3.4. *For sequences $\{y_n^i\}$ ($i \in \Phi$), $\{y_n\}$, $\{u_n\}$, $\{t_n\}$ and $\{x_n\}$ generated by Algorithm 3.1, we have*

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 + 2(1-\delta_n)\sum_{i\in\Phi} \xi_n^i \alpha_n^i f_i(x_n, x^*) + \vartheta_n - K_n - L_n, \quad \forall x^* \in \Gamma,$$

where $K_n = \mu_n(1 - \mu_n \sigma^2)\|W_n(u_n) - At_n\|^2 + \mu_n \|u_n - At_n\|^2$,

$$L_n = (1-\delta_n)\sum_{i\in\Phi} \xi_n^i \|x_n - y_n^i\|^2 + \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2$$

and $\vartheta_n = 2(1-\delta_n)\frac{\beta_n \epsilon_n}{\rho_n} + 2(1-\delta_n)\beta_n^2$.

Proof. Let $x^* \in \Gamma$. By (ii) and (iii) of Lemma 2.8, we have

$$\begin{aligned}\|u_n - Ax^*\|^2 &= \|T_{r_n}^{g_{j_n}} At_n - Ax^*\|^2 = \|T_{r_n}^{g_{j_n}} At_n - T_{r_n}^{g_{j_n}} Ax^*\|^2 \\ &\leq \langle T_{r_n}^{g_{j_n}} At_n - T_{r_n}^{g_{j_n}} Ax^*, At_n - Ax^*\rangle \\ &= \langle T_{r_n}^{g_{j_n}} At_n - Ax^*, At_n - Ax^*\rangle \\ &= \tfrac{1}{2}(\|T_{r_n}^{g_{j_n}} At_n - Ax^*\|^2 + \|At_n - Ax^*\|^2 - \|T_{r_n}^{g_{j_n}} At_n - At_n\|^2) \\ &= \tfrac{1}{2}(\|u_n - Ax^*\|^2 + \|At_n - Ax^*\|^2 - \|u_n - At_n\|^2).\end{aligned}$$

This yields

$$\|u_n - Ax^*\|^2 \leq \|At_n - Ax^*\|^2 - \|u_n - At_n\|^2. \tag{3.8}$$

Using the nonexpansive property of V and (3.8), we have

$$\begin{aligned}\|W_n(u_n) - Ax^*\|^2 = \|W_n(u_n) - W_n(Ax^*)\|^2 &\leq \|u_n - Ax^*\|^2 \\ &\leq \|At_n - Ax^*\|^2 - \|u_n - At_n\|^2.\end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned}&\langle A(t_n - x^*), W_n(u_n) - At_n\rangle \\ &= \langle A(t_n - x^*) + W_n(u_n) - At_n - W_n(u_n) + At_n, W_n(u_n) - At_n\rangle \\ &= \langle W_n(u_n) - Ax^*, W_n(u_n) - At_n\rangle - \|W_n(u_n) - At_n\|^2 \\ &= \tfrac{1}{2}\big(\|W_n(u_n) - Ax^*\|^2 + \|W_n(u_n) - At_n\|^2 - \|At_n - Ax^*\|^2\big) \\ &\qquad -\|W_n(u_n) - At_n\|^2 \\ &= \tfrac{1}{2}\big(\|W_n(u_n) - Ax^*\|^2 - \|W_n(u_n) - At_n\|^2 - \|At_n - Ax^*\|^2\big).\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\langle A(t_n - x^*), W_n(u_n) - At_n\rangle \leq -\tfrac{1}{2}(\|u_n - At_n\|^2 + \|W_n(u_n) - At_n\|^2). \tag{3.11}$$

Then, using (3.11) and *(C2)* of Condition 1, we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_C(t_n + \mu_n A^*(W_n(u_n) - At_n)) - P_C(x^*)\|^2 \\ &\leq \|(t_n - x^*) + \mu_n(W_n(u_n) - At_n)\|^2 \\ &= \|t_n - x^*\|^2 + \|\mu_n A^*(W_n(u_n) - At^n)\|^2 \\ &\qquad +2\mu_n\langle t_n - x^*, A^*(W_n(u_n) - At_n)\rangle \\ &\leq \|t_n - x^*\|^2 + \mu_n^2\|A^*\|^2\|W_n(u_n) - At_n\|^2 \\ &\qquad +2\mu_n\langle A(t_n - x^*), W_n(u_n) - At_n\rangle \\ &\leq \|t_n - x^*\|^2 + \mu_n^2\|A\|^2\|W_n(u_n) - At_n\|^2 \\ &\qquad -\mu_n\big(\|u_n - At_n\|^2 + \|W_n(u_n) - At_n\|^2\big) \\ &= \|t_n - x^*\|^2 - \mu_n(1 - \mu_n\|A\|^2)\|W_n(u_n) - At_n\|^2 - \mu_n\|u_n - At_n\|^2 \\ &= \|t_n - x^*\|^2 - \mu_n(1 - \mu_n\sigma^2)\|W_n(u_n) - At_n\|^2 - \mu_n\|u_n - At_n\|^2.\end{aligned} \tag{3.12}$$

Therefore, the result follows from Lemma 3.3 and from (3.12). □

Lemma 3.5. *Let $\{y_n^i\}$ $(i \in \Phi)$, $\{y_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{x_n\}$ be sequences generated by Algorithm 3.1. Then,*

(a) *for* $x^* \in \Gamma$, $\lim_{n\to\infty} \|x_n - x^*\|^2$ *exists (and* $\{x_n\}$ *is bounded),*

(b) *for each* $i \in \Phi$, $\limsup_{n\to\infty} f_i(x_n, x) = 0$ *for all* $x \in \Gamma$,

(c) *for each* $i \in \Phi$ *and* $i' \in \Phi'$, *we have*

$$\begin{aligned}\lim_{n\to\infty} \|W_n(u_n) - At_n\| &= \lim_{n\to\infty} \|u_n - At_n\| \\ &= \lim_{n\to\infty} \|x_n - y_n^i\| = \lim_{n\to\infty} \|U_{i'_n}(y_n) - x_n\| = 0,\end{aligned}$$

(d) *for all* $i' \in \Phi'$ *and* $j \in \Psi$, *we have* $\lim_{n\to\infty} \|u_n^j - At_n\| = \lim_{n\to\infty} \|t_n - x_n\| = \lim_{n\to\infty} \|U_{i'}(x_n) - x_n\| = \lim_{n\to\infty} \|W_n(u_n) - u_n\| = 0$.

Proof. (a). Let $x^* \in \Gamma$. Since $f_i(x_n, x^*) \le 0$, $K_n \ge 0$ and $L_n \ge 0$, from Lemma 3.4, we have

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 + \vartheta_n.$$

Observing that $\vartheta_n = 2(1-\delta_n)\frac{\beta_n \epsilon_n}{\rho_n} + 2(1-\delta_n)\beta_n^2 \le 2\frac{\beta_n\epsilon_n}{\rho_n} + 2\beta_n^2$ and using *(C5)* of Condition 1, we can see that $\sum_{n=1}^{\infty} \vartheta_n < \infty$. Therefore, by Lemma 2.2, $\lim_{n\to\infty} \|x_n - x^*\|^2$ exists and this implies that the sequence $\{x_n\}$ is bounded.
(b). From Lemma 3.4, we have

$$\begin{aligned}&K_n + L_n - 2(1-\delta_n) \sum_{i\in\Phi} \xi_i^n \alpha_n^i f_i(x_n, x^*) \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \vartheta_n \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\tfrac{\beta_n}{\rho_n}\epsilon_n + 2\beta_n^2.\end{aligned}$$

Summing up the above inequalities for every k, we obtain

$$\begin{aligned}0 &\le \sum_{n=1}^{k} \Big(K_n + L_n + 2(1-\delta_n) \sum_{i\in\Phi} \xi_i^n \alpha_i^n [-f_i(x_n, x^*)]\Big) \\ &\le \sum_{n=1}^{k} \Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\tfrac{\beta_n}{\rho_n}\epsilon_n + 2\beta_n^2\Big).\end{aligned}$$

This will yield

$$\begin{aligned}0 &\le \sum_{n=1}^{k} (K_n + L_n) + \sum_{n=1}^{k} \Big(2(1-\delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)]\Big) \\ &\le \|x_1 - x^*\|^2 - \|x_{k+1} - x^*\|^2 + 2\sum_{n=1}^{n} \tfrac{\beta_n}{\rho_n}\epsilon_n + 2\sum_{n=1}^{n} \beta_n^2.\end{aligned}$$

Letting $k \to +\infty$, we have

$$0 \le \sum_{n=1}^{\infty} K_n + \sum_{n=1}^{\infty} L_n + \sum_{n=1}^{\infty} \Big(2(1-\delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)]\Big) < +\infty.$$

Hence,

$$\sum_{n=1}^{\infty} K_n < +\infty, \quad \sum_{n=1}^{\infty} L_n < +\infty \tag{3.13}$$

and

$$\sum_{n=1}^{\infty}\Big(2(1-\delta_n)\sum_{i\in\Phi}\xi_n^i\alpha_n^i[-f_i(x_n,x^*)]\Big) < +\infty.$$

Since the sequence $\{x_n\}$ is bounded by Condition I (B6) the sequence $\{w_n^i\}$ is also bounded. Thus, there is a real number $w_i \geq \rho$ such that $\|w_n^i\| \leq w_i$. Thus, for $w = \max\{w_i : i \in I\}$, we have

$$\alpha_n^i = \frac{\beta_n}{\eta_n^i} = \frac{\beta_n}{\max\{\rho_n, \|w_n^i\|\}} = \frac{\beta_n}{\rho_n \max\{1, \frac{\|w_n^i\|}{\rho_n}\}} \geq \frac{\beta_n \rho}{\rho_n w_i} \geq \frac{\beta_n \rho}{\rho_n w}. \tag{3.14}$$

Using $0 < \sigma_1 \leq \delta_n \leq \sigma_2 < 1$ from *(C1)* of Condition 1, we have

$$\begin{aligned} 0 &\leq 2(1-\sigma_2)\sum_{n=1}^{\infty}\Big(\sum_{i\in\Phi}\xi_i^n\alpha_n^i[-f_i(x_n,x^*)]\Big) \\ &\leq \sum_{n=1}^{\infty}\Big(2(1-\delta_n)\sum_{i\in\Phi}\xi_n^i\alpha_n^i[-f_i(x_n,x^*)]\Big) < +\infty. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we have

$$\begin{aligned} 0 &\leq 2(1-\sigma_2)\sum_{n=1}^{\infty}\Big(\sum_{i\in\Phi}\xi_n^i\tfrac{\beta_n\rho}{\rho_n w}[-f_i(x_n,x^*)]\Big) \\ &\leq 2(1-\sigma_2)\sum_{n=1}^{\infty}\Big(\sum_{i\in\Phi}\xi_n^i\alpha_n^i[-f_i(x_n,x^*)]\Big) < +\infty. \end{aligned}$$

Using $0 < \xi \leq \xi_n^i \leq 1$ from *(C3)* of Condition 1 , we have

$$\begin{aligned} 0 &\leq \frac{2\rho\xi(1-\sigma_2)}{w}\sum_{n=1}^{\infty}\Big(\frac{\beta_n}{\rho_n}\sum_{i\in\Phi}[-f_i(x_n,x^*)]\Big) \\ &\leq 2(1-\sigma_2)\sum_{n=1}^{\infty}\Big(\sum_{i\in\Phi}\xi_n^i\frac{\beta_n\rho}{\rho_n w}[-f_i(x_n,x^*)]\Big) < +\infty. \end{aligned} \tag{3.16}$$

Since $\sum_{n=1}^{\infty}\frac{\beta_n}{\rho_n} = +\infty$ and $\sum_{i\in\Phi}[-f_i(x_n,x^*)] \leq 0$, from (3.16) we can conclude that $\liminf_{n\to\infty}\big[-f_i(x_n,x^*)\big] = 0, \ \forall x^* \in \Gamma$. Hence, the result follows.

(c). From (3.13) and *(C1)*-*(C3)* of Condition 1, we have

$$\begin{aligned} \lim_{n\to\infty}\|W_n(u_n) - At_n\|^2 = \lim_{n\to\infty}\|u_n - At_n\|^2 &= \lim_{n\to\infty}\|x_n - y_n^i\|^2 \\ &= \lim_{n\to\infty}\|U_{i'_n}(y_n) - x_n\|^2 = 0. \end{aligned}$$

Hence, the result follows.
(d). By definition of u_n and using $\|u_n - At_n\| \to 0$ from (c) above we have $\|u_j - At_n\| \to 0$. Using definition of t_n, we have $\|t_n - x_n\| = (1-\delta_n)\|x_n - U_{i'_n}(y_n)\|$. This together with $\|x_n - U_{i'_n}(y_n)\| \to 0$ from (c) above gives $\|t_n - x_n\| \to 0$. Hence, $\|t_n^{i'} - x_n\| \to 0$ follows from $\|t_n^{i'} - x_n\| \leq \|t_n - x_n\|$.
Similarly, using $\|W_n(u_n) - u_n\| \leq \|W_n(u_n) - At_n\| + \|u_n - At_n\|$ and using results in (c) above, yields $\|W_n(u_n) - u_n\| \to 0$. By definition of $t_n^{i'}$, we have

$$\begin{aligned}
\|t_n^{i'} - x_n\| &= \|\delta_n x_n + (1-\delta_n)U_{i'}(y_n) - x_n\| = (1-\delta_n)\|x_n - U_{i'}(y_n)\| \\
&= (1-\delta_n)\|x_n - U_{i'}(x_n) + U_{i'}(x_n) - U_{i'}(y_n)\| \\
&\geq (1-\delta_n)\big(\|x_n - U_{i'}(x_n)\| - \|U_{i'}(x_n) - U_{i'}(y_n)\|\big) \\
&\geq (1-\delta_n)\big(\|x_n - U_{i'}(x_n)\| - \|x_n - y_n\|\big)
\end{aligned}$$

That is, for all $i' \in \Phi'$, we have

$$\|x_n - U_{i'}(x_n)\| \leq \|x_n - y_n\| + \frac{1}{1-\delta_n}\|t_n^{i'} - x_n\|.$$

Combining the last inequality together with *(C1)* of Condition 1, $\|t_n^{i'} - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$, we have $\|x_n - U_{i'}(x_n)\| \to 0$ for all $i' \in \Phi'$. □

Now we state the first main theorem for convergence of Algorithm 3.1.

Theorem 3.6. *Let $\{y_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{x_n\}$ be the sequences generated by Algorithm 3.1. Then the sequences $\{y_n\}$, $\{t_n\}$ and $\{x_n\}$ converge strongly to a point p in Γ and $\{u_n\}$ converges strongly to a point $Ap \in \Omega_2$ where $p = \lim_{n\to+\infty} P_\Gamma(x_n)$.*

Proof. Let $x^* \in \Gamma$. From Lemma 3.5 *(a)*, we have seen that the sequence $\{x_n\}$ is bounded. There exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightharpoonup p$ as $l \to +\infty$, where $p \in C$ and

$$\limsup_{l\to+\infty} f_i(x_{n_l}, x^*) = \lim_{n\to+\infty} f_i(x_n, x^*).$$

But by the weak upper semicontiniuty of each $f_i(., x^*)$ and by Lemma 3.5 (b), we have

$$f_i(p, x^*) \geq \limsup_{l\to+\infty} f_i(x_{n_l}, x^*) = \lim_{l\to+\infty} f_i(x_{n_l}, x^*) = \limsup_{n\to+\infty} f_i(x_n, x^*) = 0.$$

Since $x^* \in \Omega$ and $p \in C$ we have $f_i(x^*, p) \geq 0$. By (B2) of Condition I, we have $f_i(p, x^*) \leq 0$. This together with the above fact gives $f_i(x^*, p) = 0$. Hence, by (B3) of Condition I, we have $p \in SEP(f_i, C)$.
Since $\|x_n - U_{i'}(x_n)\| \to 0$ from Lemma 3.5 (c) and since $x_{n_l} \rightharpoonup p$, by demiclosedness of $Id_C - U_{i'}$, we have $p \in FixU_{i'}$ for all $i' \in \Phi'$. Thus, $p \in \bigcap_{i'\in\Phi'} FixU_{i'}$.
Hence, $p \in \Omega_1$.
Since $\langle y_{n_l}, h\rangle = \langle y_{n_l} - x_{n_l}, h\rangle + \langle x_{n_l}, h\rangle$, $\forall h \in H_1$, and using $\lim_{k\to+\infty} \|x_n -$

$y_n\| = 0$ from Lemma 3.5, we have $y_{n_l} \rightharpoonup p$ as $l \to +\infty$. Therefore, $Ay_{n_l} \rightharpoonup Ap$ as $l \to +\infty$. Similarly, we can have $t_{n_l} \rightharpoonup p$ as $j \to +\infty$ and hence $At_{n_l} \rightharpoonup Ap$ as $l \to +\infty$. Since $\lim_{n\to+\infty} \|u_n - At_n\| = 0$ and $\langle u_{n_l}, u\rangle = \langle u_{n_l} - At_{n_l}, u\rangle + \langle At_{n_l}, u\rangle$, $\forall u \in H_2$, we have $u_{n_l} \rightharpoonup Ap$ as $l \to +\infty$. If there exists $n_0 \in \mathbb{N}$ such that $Ap \in FixW_{n_0}$, then in view of Remark 3.2 it is easy to concude that $Ap \in \bigcap_{j'\in\Psi'} FixV_{j'}$. Thus, let us assume that $Ap \notin FixW_n$ for all $n \geq 1$. Using Opial's condition and Lemma 3.5 (d), we have

$$\begin{aligned}
\liminf_{l\to+\infty} \|u_{n_l} - Ap\| &< \liminf_{l\to+\infty} \|u_{n_l} - W_{n_l}(Ap)\| \\
&= \liminf_{l\to+\infty} \|u_{n_l} - W_{n_l}(u_{n_l}) + W_{n_l}(u_{n_l}) - W_{n_l}(Ap)\| \\
&\leq \liminf_{l\to+\infty} (\|u_{n_l} - W_{n_l}(u_{n_l})\| + \|W_{n_l}(u_{n_l}) - W_{n_l}(Ap)\|) \\
&= \liminf_{l\to+\infty} \|W_{n_l}(u_{n_l}) - W_{n_l}(Ap)\| \leq \liminf_{l\to+\infty} \|u_{n_l} - Ap\|,
\end{aligned}$$

which is a contradiction. It must be the case that $Ap \in FixW_n$ for some $n \geq 1$ (implying that $Ap \in FixW_n$ for all $n \geq 1$ using Remark 3.2). Hence, $Ap \in \bigcap_{j'\in\Psi'} FixV_{j'}$.

Let $r > 0$. Assume $Ap \notin Fix(T_r^{g_{j_0}})$ for some $j_0 \in \Psi$ and for some $r > 0$. Thus, $T_r^{g_{j_0}}(Ap) \neq Ap$. That is, $Ap \notin \bigcap_{j\in\Psi} Fix(T_r^{g_j})$. Thus, using Opial's condition, Lemma 2.9 and $\lim_{n\to\infty} \|u_n^j - At_n\| = 0$, $\forall j \in \Psi$, we get

$$\begin{aligned}
\liminf_{l\to+\infty} \|At_{n_l} - Ap\| &< \liminf_{l\to+\infty} \|At_{n_l} - T_r^{g_{j_0}}(Ap)\| \\
&= \liminf_{l\to+\infty} \|At_{n_l} - u_{n_l}^{j_0} + u_{n_l}^{j_0} - T_r^{g_{j_0}}(Ap)\| \\
&\leq \liminf_{l\to+\infty} (\|At_{n_l} - u_{n_l}^{j_0}\| + \|u_{n_l}^{j_0} - T_r^{g_{j_0}}(Ap)\|) \\
&= \liminf_{l\to+\infty} \|u_{n_l}^{j_0} - T_r^{g_{j_0}}(Ap)\| \\
&= \liminf_{l\to+\infty} \|T_{r_{n_l}}^{g_{j_0}}(At_{n_l}) - T_r^{g_{j_0}}(Ap)\| \\
&\leq \liminf_{l\to+\infty} (\|At_{n_l} - Ap\| + \tfrac{|r_{n_l} - r|}{r_{n_l}} \|T_{r_{n_l}}^{g_{j_0}}(At_{n_l}) - At_{n_l}\|) \\
&= \liminf_{l\to+\infty} (\|At_{n_l} - Ap\| + \tfrac{|r_{n_l} - r|}{r_{n_l}} \|u_{n_l}^{j_0} - At_{n_l}\|) \\
&= \liminf_{l\to+\infty} \|At_{n_l} - Ap\|
\end{aligned}$$

which is a contradiction. Hence, it must be the case that $Ap \in Fix(T_r^{g_j})$ for all $j \in \Psi$ and $r > 0$. By Lemma 2.8 $((iii))$, $Fix(T_r^{g_j}) = SEP(g_i, D)$. Therefore, $Ap \in \bigcap_{j\in\Psi} SEP(g_j, D)$. Therefore, $Ap \in \Omega_2$. That is, $p \in \Omega_1$ and $Ap \in \Omega_2$. Hence, $p \in \Gamma$ and p is weak cluster point of the sequence $\{x_n\}$. By Lemma 3.5, $\{\|x_n - p\|^2\}$ converges. Thus, we conclude that the sequence $\{x_n\}$ strongly converges to p. As a result of this it is easy to see that $t_n \to p$ and $y_n \to p$ as $n \to +\infty$. Moreover, $Ay_n \to Ap$, $At_n \to Ap$, and $Ax_n \to Ap$. From

$$\|u_n - Ap\| \leq \|u_n - At_n\| + \|At_n - Ap\|$$

we have $u_n \to Ap$. We will end the proof by showing $p = \lim_{n\to+\infty} P_\Gamma(x_n)$. From Lemma 3.4, we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \vartheta_n, \quad \forall x^* \in \Gamma. \tag{3.17}$$

Let $z_n = P_\Gamma(x_n)$. Since $P_\Gamma(x_n) \in \Gamma$ and using (3.17), we have

$$\|x_{n+1} - z_n\|^2 \leq \|x_n - z_n\|^2 + \vartheta_n. \tag{3.18}$$

But by the property of metric projection, we have

$$\|x_{n+1} - z_{n+1}\|^2 \leq \|x_{n+1} - x^*\|^2, \quad \forall x^* \in \Gamma.$$

Thus,

$$\|x_{n+1} - z_{n+1}\|^2 \leq \|x_{n+1} - z_n\|^2. \tag{3.19}$$

From (3.18) and (3.19), we have $\|x_{n+1} - z_{n+1}\|^2 \leq \|x_n - z_n\|^2 + \vartheta_n$. Since $\sum_{n=1}^{\infty} \vartheta_n < \infty$, by Lemma 2.2 we have that $\lim_{n\to+\infty} \|x_n - z_n\|^2$ exists. Using the definition of metric projection, we can conclude

$$\|P_\Gamma(x_k) - P_\Gamma(x_m)\|^2 + \|x_m - P_\Gamma(x_m)\|^2 \leq \|x_m - P_\Gamma(x_k)\|^2. \tag{3.20}$$

Let $m \geq k$. Then from (3.18-3.20), we get

$$\begin{aligned}
\|z_k - z_m\|^2 &= \|P_\Gamma(x_k) - P_\Gamma(x_m)\|^2 \leq \|x_m - P_\Gamma(x_k)\|^2 - \|x_m - P_\Gamma(x_m)\|^2 \\
&= \|x_m - z_k\|^2 - \|x_m - z_m\|^2 \\
&\leq \|x_{m-1} - z_k\|^2 + \vartheta_{m-1} - \|x_m - z_m\|^2 \\
&\leq \|x_k - z_k\|^2 + \textstyle\sum_{l=k}^{m-1} \vartheta_l - \|x_m - z_m\|^2.
\end{aligned}$$

As a result of $\sum_{n=1}^{\infty} \vartheta_n < \infty$ and since $\lim_{n\to+\infty} \|x_n - z_n\|^2$ exists if we let $m, k \to +\infty$ we can see that $\|z_k - z_m\|^2 \to 0$. This implies the sequence $\{z_n\}$ is a Cauchy sequence and hence it converges to some point z in Γ. Since $z_n = P_\Gamma(x_n)$, we have

$$\langle x_n - z_n, x^* - z_n \rangle \leq 0, \quad \forall x^* \in \Gamma.$$

Thus, $\langle x_n - z_n, p - z_n \rangle \leq 0$. This leads to

$$\|z - p\|^2 = \langle p - z, p - z \rangle = \lim_{n\to+\infty} \langle x_n - z_n, p - z_n \rangle \leq 0.$$

Therefore, $p = z$ and $\lim_{n\to+\infty} P_\Gamma(x_n) = p$. □

The following corollary is an immediate consequence of Theorem 3.6 obtained by setting $U_{i'} = Id_C$ for all $i' \in \Phi'$ and $V_{j'} = Id_D$ for all $j' \in \Psi'$.

Corollary 3.7. *Let $\{y_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{x_n\}$ be sequences generated by iterative algorithm*

$$\begin{cases} x_1 \in C \\ w_n^i \in \partial_{\epsilon_n} f_i(x_n,.)(x_n), \quad i \in \Phi, \\ \alpha_n^i = \frac{\beta_n}{\eta_n^i}, \quad \eta_n^i = \max\{\rho_n, \|w_n^i\|\}, \quad i \in \Phi, \\ y_n^i = P_C(x_n - \alpha_n^i w_n^i), \quad i \in I, \\ y_n = \sum\limits_{i\in\Phi} \xi_n^i y_n^i, \\ t_n = \delta_n x_n + (1-\delta_n) y_n, \\ u_n^j = T_{r_n}^{g_j}(At_n), \quad j \in \Psi, \\ u_n = \arg\max\{\|v - At_n\| : v \in \{u_n^j : j \in \Psi\}\}, \\ x_{n+1} = P_C(t_n + \mu_n A^*(u_n - At_n)). \end{cases}$$

Then $\{y_n\}$, $\{t_n\}$ and $\{x_n\}$ converge strongly to a point $p \in \big\{x \in \bigcap\limits_{i\in\Phi} SEP(f_i, C) : Ax \in \bigcap\limits_{j\in\Psi} SEP(g_j, D)\big\}$ and $\{u_n\}$ converge strongly to $Ap \in \bigcap\limits_{j\in\Psi} SEP(g_j, D)$. Moreover, $p = \lim\limits_{n\to+\infty} P_\Omega(x_n)$, where

$$\Omega = \big\{x \in \bigcap_{i\in\Phi} SEP(f_i, C) : Ax \in \bigcap_{j\in\Psi} SEP(g_j, D)\big\}.$$

Note that when $\Phi = \Psi = \Phi' = \Psi' = \{1\}$, the Algorithm 3.1 coincides with Algorithms 3.1 in [14].

3.2. Algorithm without prior knowledge of the operator norm

In practice, to estimate the norm of an operator is not always easy. Next, we modify Algorithm 3.1 where the implementation of the algorithm does not need any prior information regarding the operator norm if it is not easy to estimate the norm of an operator.
Take the parameter sequences in Algorithm 3.2 satisfying the conditions:

Condition 2

(C1) $\rho_n \geq \rho > 0$, $\beta_n \geq 0$, $\epsilon_n \geq 0$, $0 < \sigma_1 \leq \delta_n \leq \sigma_2 < 1$.

(C2) $r > 0$, $0 < \eta < 4$, $0 < \eta \leq \eta_n \leq 4 - \eta$.

(C3) $0 < \xi \leq \xi_n^i \leq 1$, $(i \in \Phi)$ such that $\sum\limits_{i\in\Phi} \xi_n^i = 1$ for each $n \geq 1$.

(C4) $0 < \theta \leq \theta^{j'} \leq 1$, $(j' \in \Psi')$ such that $\sum\limits_{j'\in\Psi'} \theta^{j'} = 1$.

(C5) $\sum\limits_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty$, $\sum\limits_{n=1}^{\infty} \frac{\beta_n \epsilon_n}{\rho_n} < +\infty$, $\sum\limits_{n=1}^{\infty} \beta_n^2 < +\infty$.

We shall now introduce the setting used as a stepsize that can be controlled and help us eliminate the requirement of the operator norm. Let $\alpha > 0$ and $x \in H_1$. Then $h_\alpha^{g_j}(x)$, $l_\alpha^{g_j}(x)$, $h_\alpha(x)$ and $l_\alpha(x)$ are defined as follows:

(I) for each $j \in \Psi$,

$$h_\alpha^{g_j}(x) = \frac{1}{2}\Big\| \sum_{j' \in \Psi'} \theta^{j'} V_{j'}\big(T_\alpha^{g_j} A(x)\big) - A(x)\Big\|^2$$

and

$$l_\alpha^{g_j}(x) = A^*\Big(\sum_{j' \in \Psi'} \theta^{j'} V_{j'}\big(T_\alpha^{g_j} A(x)\big) - A(x)\Big).$$

(II) $h_\alpha(x) = h_\alpha^{g_{j_0}}(x)$ and $l_\alpha(x) = l_\alpha^{g_{j_0}}(x)$, where j_0 is in Ψ such that

$$T_\alpha^{g_{j_0}} A(x) = \arg\max\{\|v - Ax\| : v \in \{T_\alpha^{g_j}(Ax) : j \in \Psi\}\},$$

that is,

$$h_\alpha(x) = \frac{1}{2}\Big\| \sum_{j' \in \Psi'} \theta^{j'} V_{j'}\big(T_\alpha^{g_{j_0}} A(x)\big) - A(x)\Big\|^2$$

and so

$$l_\alpha(x) = A^*\Big(\sum_{j' \in \Psi'} \theta^{j'} V_{j'}\big(T_\alpha^{g_{j_0}} A(x)\big) - A(x)\Big)$$

where $j_0 \in \Psi$ such that

$$T_\alpha^{g_{j_0}} A(x) = \arg\max\{\|v - Ax\| : v \in \{T_\alpha^{g_j}(Ax) : j \in \Psi\}\}.$$

Using $h_\alpha^{g_j}$, $l_\alpha^{g_j}$, h_α and l_α given in **(I)** and **(II)** above, we are now in a position to introduce our algorithm.

Algorithm 3.2
Initialization : Choose $x_1 \in C$. Let the real sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, $\{r_n\}$, $\{\delta_n\}$, $\{\xi_n^i\}$ $(i \in \Phi)$, $\{\eta_n\}$ and the real numbers r and $\theta^{j'}$ $(j' \in \Psi')$ satisfy Condition 2.

Step 1. For each $i \in \Phi$, find $w_n \in H_1$ such that $w_n \in \partial_{\epsilon_n} f(x_n, .)(x_n)$.

Step 2. For each $i \in \Phi$, calculate $\alpha_n^i = \frac{\beta_n}{\eta_n^i}$, $\quad \eta_n^i = \max\{\rho_n, \|w_n^i\|\}$ and $y_n^i = P_{T_n}(x_n - \alpha_n^i w_n^i)$, where

$$T_n = \begin{cases} C & \text{if} \quad n = 1, \\ \{z \in H_1 : \langle t_{n-1} + \mu_{n-1} l_r(t_{n-1}) - x_n, z - x_n\rangle \le 0\}, & \text{otherwise.} \end{cases}$$

Step 3. Evaluate $y_n = \sum_{i \in \Phi} \xi_n^i y_n^i$.

Step 4. For each $i' \in I'$ find $t_n^{i'} = \delta_n x_n + (1 - \delta_n) U_{i'}(y_n)$.

Step 5. Find among $t_n^{i'}$, $i' \in \Phi'$, the farthest element from x_n, i.e.,

$$t_n = \arg\max\{\|v - x_n\| : v \in \{t_n^{i'} : i' \in \Phi'\}\}.$$

Step 6. For each $j \in \Psi$ find $u_n^j = T_r^{g_j}(At_n), \quad j \in \Psi$.

Step 7. Find among u_n^j, $j \in \Psi$, the farthest element from At_n, i.e.,

$$u_n = \arg\max\{\|v - At_n\| : v \in \{u_n^j : j \in \Psi\}\}.$$

Step 8. Evaluate $x_{n+1} = P_C(t_n + \mu_n l_r(t_n))$, where

$$\mu_n = \begin{cases} 0 \text{ if } l_r(t_n) = 0, \\ \frac{\eta_n h_r(t_n)}{\|l_r(t_n))\|^2}, \text{ otherwise.} \end{cases}$$

Step 9. Set $n = n + 1$ and go to Step 1.

Remark 3.8. By definition of T_n, we see that T_n is either half-space or the whole space H_1. Therefore, for each n, T_n is closed and convex set, and the computation of projection $y_n^i = P_{T_n}(x_n - \alpha_n^i w_n^i)$ in Step 2 of Algorithm 3.2 is explicit and easier than the computation of projection $y_n^i = P_C(x_n - \alpha_n^i w_n^i)$ in Step 2 of Algorithm 3.2 when C has a complex structure. Moreover, by a similar reasoning as for Algorithm 3.1, Algorithm 3.2 is well defined.

Similarly, define

(1b) $W = \sum_{j' \in \Psi'} \theta^{j'} V_{j'}$,

(2b) $\{i'_n\}_{n=1}^{+\infty}$ is a sequence where for each n, $i'_n \in \Phi'$ such that

$$t_n = t_n^{i'_n} = \arg\max\{\|v - x_n\| : v \in \{t_n^{i'} : i' \in \Phi'\}\} = \delta_n x_n + (1-\delta_n) U_{i'_n}(y_n),$$

(3b) $\{j_n\}_{n=1}^{+\infty}$ is a sequence where for each n, $j_n \in \Psi$ such that

$$u_n = u_n^{j_n} = \arg\max\{\|v - At_n\| : v \in \{u_n^j : j \in \Psi\}\}.$$

Remark 3.9. **(1b)** above has a form of **(1a)** defined in Algorithm 3.1, i.e., by setting $W_n = W$ for all $n \geq 1$, where $W = \sum_{j' \in \Psi'} \theta^{j'} V_{j'}$.

Remark 3.10. In Algorithm 3.2, $h_r(t_n)$ and $l_r(t_n)$ are simply given as

$$h_r(t_n) = \frac{1}{2}\|WT_r^{g_{j_n}} At_n - At_n\|^2 = \frac{1}{2}\|Wu_n^{j_n} - At_n\|^2$$

and $l_r(t_n) = A^*\big(WT_r^{g_{j_n}} At_n - At_n\big) = A^*\big(Wu_n^{j_n} - At_n\big)$.

Lemma 3.11. *Let $\{y_n^i\}$ $(i \in \Phi)$, $\{y_n\}$, $\{t_n\}$ and $\{x_n\}$ be sequences generated by Algorithm 3.2. Then*

(a) $C \subset T_n$ *for all* $n \geq 1$.

(b) *For* $x^* \in \Gamma$,

$$\|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2(1-\delta_n)\sum_{i\in\Phi}\xi_n^i\alpha_n^i f_i(x_n, x^*) - L_n + \vartheta_n,$$

where $L_n = (1-\delta_n)\sum_{i\in\Phi}\xi_n^i\|x_n - y_n^i\|^2 + \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2$ *and* $\vartheta_n = 2(1-\delta_n)\frac{\beta_n\epsilon_n}{\rho_n} + 2(1-\delta_n)\beta_n^2$.

Proof. (a). From $x_n = P_C(t_{n-1} + \mu_{n-1}l_r(t_{n-1}))$ and by the property of metric projection we have

$$\langle t_{n-1} + \mu_{n-1}l_r(t_{n-1}) - x_n, z - x_n\rangle, \quad \forall z \in C$$

which, together with the definition of T_n, implies that $C \subset T_n$.
(b). Let $x^* \in \Gamma$. From $y_n^i = P_{T_n}(x_n - \frac{\beta_n}{\eta_n^i}w_n^i)$ and $x^*, x_n \in C \subset T_n$, we have

$$\langle x_n - \alpha_n^i w_n^i - y_n^i, y_n^i - x^*\rangle \geq 0.$$

The result follows by a similar proof as we used in Lemma 3.3. □

Lemma 3.12. *Let* $\{y_n^i\}$ $(i \in \Phi)$, $\{y_n\}$, $\{u_n\}$, *and* $\{x_n\}$ *be a sequences generated by Algorithm 3.2. Then, for all* $x^* \in \Gamma$, *we have*

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + 2(1-\delta_n)\sum_{i\in\Phi}\xi_n^i\alpha_n^i f_i(x_n, x^*) + \vartheta_n - K_n - \varphi_n,$$

where $K_n = (1-\delta_n)\sum_{i\in\Phi}\xi_n^i\|x_n - y_n^i\|^2 + \delta_n(1-\delta_n)\|U_{i'_n}(y_n) - x_n\|^2 - \|u_n - At_n\|^2$, $\vartheta_n = 2(1-\delta_n)\frac{\beta_n\epsilon_n}{\rho_n} + 2(1-\delta_n)\beta_n^2$ *and* $\varphi_n = 4\mu_n h_r(t_n) - \mu_n^2\|l_r(t_n)\|^2$.

Proof. Let $x^* \in \Gamma$. Using (3.9) in Lemma 3.4, we have

$$\begin{aligned}
&\langle t_n - x^*, l_r(t_n)\rangle = \langle t_n - x^*, A^*(W(u_n) - At_n)\rangle \\
&= \langle A(t_n - x^*), W(u_n) - At_n\rangle \\
&= \Big\langle A(t_n - x^*) + W(u_n) - At_n - W(u_n) + At_n, W(u_n) - At_n\Big\rangle \\
&= \Big\langle W(u_n) - Ax^*, W(u_n) - At_n\Big\rangle - \|W(u_n) - At_n\|^2 \\
&= \frac{1}{2}\Big(\|W(u_n) - Ax^*\|^2 + \|W(u_n) - At_n\|^2 - \|At_n - Ax^*\|^2\Big) \\
&\quad -\|W(u_n) - At_n\|^2 \\
&= \frac{1}{2}\Big(\|W(u_n) - Ax^*\|^2 - \|W(u_n) - At_n\|^2 - \|At_n - Ax^*\|^2\Big) \\
&\leq -\frac{1}{2}\Big(\|u_n - At_n\|^2 + \|W(u_n) - At_n\|^2\Big) \\
&= -\frac{1}{2}\Big(\|u_n - At_n\|^2 + 2h_r(t_n)\Big). \qquad (3.21)
\end{aligned}$$

Using (3.21), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|P_C(t_n + \mu_n l_r(t_n)) - P_C(x^*)\|^2 \leq \|t_n + \mu_n l_r(t_n) - x^*\|^2 \\
&= \|t_n - x^*\|^2 + \mu_n^2 \|l_r(t_n)\|^2 - 2\mu_n \langle l_r(t_n), t_n - x^*\rangle \\
&\leq \|t_n - x^*\|^2 + \mu_n^2 \|l_r(t_n)\|^2 - 4\mu_n h_r(t_n) - \|u_n - At_n\|^2 \\
&= \|t_n - x^*\|^2 - \|u_n - At_n\|^2 \\
&\quad -[4\mu_n h_r(t_n) - \mu_n^2 \|l_r(t_n)\|^2]. \qquad (3.22)
\end{aligned}$$

Therefore, (3.22) and Lemma 3.11 give the result. □

Note that by the definition of μ_n, we have

$$\varphi_n = \begin{cases} 0, & \text{if } l_r(t_n) = 0 \\ \eta_n(4 - \eta_n)\frac{h_r(t_n)^2}{\|l_r(t_n)\|^2}, & \text{otherwise.} \end{cases}$$

Lemma 3.13. *Let $\{y_n^i\}$ $(i \in \Phi)$, $\{t_n\}$, $\{u_n\}$ and $\{x_n\}$ be the sequences generated by Algorithm 3.2. Then*

(a). *for $x^* \in \Gamma$, $\lim_{n\to+\infty} \|x_n - x^*\|^2$ exists (and $\{x_n\}$ is bounded),*

(b). $\limsup_{n\to\infty} f_i(x_n, x) = 0$ *for all $x \in \Gamma$,*

(c). $\lim_{n\to\infty} \|u_n - At_n\| = \lim_{n\to\infty} \|x_n - y_n^i\| = \lim_{n\to\infty} \|U_{i'_n}(y_n) - x_n\| = 0,$

(d). *for all $j \in \Psi$ and $i' \in \Phi'$, we have*

$$\begin{aligned}
\lim_{n\to\infty} \|u_n^j - At_n\| &= \lim_{n\to\infty} \|U_{i'}(x_n) - x_n\| = \lim_{n\to\infty} \|t_n - x_n\| \\
&= \lim_{n\to\infty} h_r(t_n) = \lim_{n\to\infty} \|W(u_n) - u_n\| = 0.
\end{aligned}$$

Proof. (a). Let $x^* \in \Gamma$. Since $f_i(x_n, x^*) \leq 0$, $K_n \geq 0$, $\varphi_n \geq 0$ from Lemma 3.12, we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \vartheta_n.$$

Therefore, the result follows.
(b). From Lemma 3.12, we have

$$\begin{aligned}
&\varphi_n + K_n + 2(1 - \delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)] \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \vartheta_n \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\tfrac{\beta_n}{\rho_n}\epsilon_n + 2\beta_n^2.
\end{aligned}$$

Summing up the above inequalities for every k, we obtain

$$\begin{aligned}
0 &\leq \sum_{n=1}^{k} \left(\varphi_n + K_n + 2(1 - \delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)]\right) \\
&\leq \sum_{n=1}^{k} \left(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\tfrac{\beta_n}{\rho_n}\epsilon_n + 2\beta_n^2\right).
\end{aligned}$$

This will yield

$$\begin{aligned}0 &\leq \sum_{n=1}^{k} \varphi_n + \sum_{n=1}^{k} K_n + \sum_{n=1}^{k} \left(2(1-\delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)] \right) \\ &\leq \|x_1 - x^*\|^2 - \|x_{k+1} - x^*\|^2 + 2 \sum_{n=1}^{k} \tfrac{\beta_n}{\rho_n} \epsilon_n + 2 \sum_{n=1}^{k} \beta_n^2.\end{aligned}$$

Letting $k \to +\infty$, we have

$$0 \leq \sum_{n=1}^{\infty} \varphi_n + \sum_{n=1}^{\infty} K_n + \sum_{n=1}^{\infty} \left(2(1-\delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)] \right) < +\infty.$$

Hence,

$$\begin{aligned}&\sum_{n=1}^{\infty} \varphi_n < +\infty, \quad \textstyle\sum_{n=1}^{\infty} K_n < +\infty, \\ &\sum_{n=1}^{\infty} \left(2(1-\delta_n) \sum_{i\in\Phi} \xi_n^i \alpha_n^i [-f_i(x_n, x^*)] \right) < +\infty.\end{aligned} \tag{3.23}$$

The result follows in the same way as in the proof of Lemma 3.5.
(c). From $\sum_{n=1}^{\infty} K_n < +\infty$ and Condition 2, we have

$$\lim_{n\to\infty} \|u_n - At_n\|^2 = \lim_{n\to\infty} \|x_n - y_n^i\|^2 = \lim_{n\to\infty} \|U_{i'_n}(y_n) - x_n\|^2 = 0.$$

(d). The proof of $\lim_{n\to\infty} \|u_n^j - At_n\| = \lim_{n\to\infty} \|U_{i'}(x_n) - x_n\| = \lim_{n\to\infty} \|t_n - x_n\|$ remains the same as in Lemma 3.5 *(d)*.
From (3.23) we have $\sum_{n=1}^{\infty} [4\mu_n h_r(t_n) - \mu_n^2 \|l_r(t_n)\|^2] < +\infty$. Without loss of generality, we can assume that $l_r(t_n) \neq 0$ for all n.
Thus, $\sum_{n=1}^{\infty} [4\mu_n h_r(t_n) - \mu_n^2 \|l_r(t_n)\|^2] < +\infty$ implies that

$$\sum_{n=1}^{\infty} \eta_n (4 - \eta_n) \frac{h_r(t_n)^2}{\|l_r(t_n)\|^2} < +\infty.$$

Since $0 < \eta \leq \eta_n \leq 4 - \eta$, we have

$$\sum_{n=1}^{\infty} \frac{h_r(t_n)^2}{\|l_r(t_n)\|^2} < +\infty. \tag{3.24}$$

Since $\lim_{n\to\infty} \|t_n - x_n\| = 0$ and $\{x_n\}$ is bounded, $\{t_n\}$ is also bounded. Moreover, since A is a bounded linear operator, W is a nonexpansive operator, and using the triangle inequality and Lemma 2.8 for $T_r^{g_j}$, it is easy to see that $l_r^{g_j}(.)$ is Lipschitz continuous for all $j \in \Psi$. Thus, from the Lipschitz continuity of $l_r^{g_j}(.)$ and from boundedness of the sequence $\{t_n\}$, it follows that $\{\|l_r^{g_j}(t_n)\|^2\}$ is bounded for each $j \in \Psi$ and hence $\{\|l_r(t_n)\|^2\}$ is bounded. This together with (3.24) implies that $\lim_{n\to\infty} h_r(t_n) = 0$. The inequality

$$\|W(u_n) - u_n\| \leq (2h_r(t_n))^{\frac{1}{2}} + \|u_n - At_n\|$$

gives $\lim_{n\to\infty} \|W(u_n) - u_n\| = 0$. □

The second main theorem is about the convergence of Algorithm 3.2.

Theorem 3.14. *Let $\{y_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{x_n\}$ be the sequences generated by Algorithm 3.2. Then the sequences $\{y_n\}$, $\{t_n\}$ and $\{x_n\}$ converge strongly to the point p in Γ and $\{u_n\}$ converges strongly to the point $Ap \in \Omega_2$ where $p = \lim_{n\to\infty} P_\Gamma(x_n)$.*

Proof. The proof of this Theorem is similar to the proof of Theorem 3.6 by taking $W_n = W$ for all $n \geq 1$. Therefore, the proof is omitted. □

Similarly, if we set $U_{i'} = Id_C$ for all $i' \in \Phi'$ and $V_{j'} = Id_D$ for all $j' \in \Psi'$, Theorem 3.14 solves

$$x \in \bigcap_{i\in\Phi} SEP(f_i, C) \text{ such that } Ax \in \bigcap_{j\in\Psi} SEP(g_j, D).$$

When $\Phi = \Psi = \Phi' = \Psi' = \{1\}$ the Algorithm 3.2 coincides with Algorithm 3.2 in [14].

Example 3.15. Let C and D be closed convex subsets of H_1 and H_2, respectively, where C and D contain the zero vector. Let the bifunctions $f_i : C \times C \to \mathbb{R}$ be defined by the Cournot-Nash equilibrium model

$$f_i(x, y) = \langle P_i x + Q_i y + s_i, y - x\rangle, \quad i \in \Phi = \{1, \dots, N\},$$

where P_i, Q_i are $p \times p$ matrices of order p such that Q_i is symmetric positive semidefinite and $Q_i - P_i$ is negative semidefinite, $s_i \in \mathbb{R}^p$, and let $g_j : D \times D \to \mathbb{R}$ be defined by

$$g_j(u, v) = G_j(v) - G_j(u), \ \ j \in \Psi = \{1, \dots, M\},$$

where $G_j(u) = \frac{1}{2} u^T \bar{H}_j u + \bar{B}_j^T u$, with $\bar{B}_j \in \mathbb{R}^p$ and $\bar{H}_j$ being a symmetric positive definite matrix of order q. Let $U_{i'} : C \to C$, $V_{j'} : D \to D$ given by

$$U_{i'}(x) = \frac{1}{i'} x, \quad i' \in \Phi' = \{1, \dots, N'\}$$

and

$$V_{j'}(u) = \frac{1}{j'} u, \ \ j' \in \Psi' = \{1, \dots, M'\}.$$

Let $A : \mathbb{R}^p \to \mathbb{R}^q$ where A is $q \times p$ nonzero matrix.
It is easy to show that each $U_{i'}$ and $V_{j'}$ is nonexpansive mapping, each f_i satisfy

Condition I on C, each g_j satisfy Condition II on D, $\Omega_1 = \{0\}$, $\Omega_2 = \{0\}$ and $A(0) = 0$. Therefore, $\Gamma = \{0\}$.

Note that in this case, the resolvent $T_r^{g_j}$ of the bifunction g_j coincides with the proximal mapping of the function G_j with the constant $r > 0$, that is,

$$T_r^{g_j}(u) = \arg\min\{rG_j(v) + \|u - v\|^2 : v \in D\}, \quad j \in \Psi = \{1, \dots, M\}$$

or the following convex quadratic problem

$$T_r^{g_j}(u) = \arg\min\big\{\frac{1}{2}v^T \hat{H}_j v + \hat{B}_j^T v : v \in D\big\}, \quad j \in \Psi = \{1, \dots, M\}$$

where $\hat{H}_j = 2(\bar{H}_j + \frac{1}{r}I_d)$ and $\hat{B}_j = \bar{B}_j - \frac{2}{r}u$ where I_d is $q \times q$ identity matrix. For each $j \in \Psi$, the convex quadratic problem $\arg\min\big\{\frac{1}{2}v^T \hat{H}_j v + \hat{B}_j^T v : v \in D\big\}$ can be effectively solved, for instance, by MATLAB Optimization Toolbox.

Conclusions

We proposed two algorithms and we proved that the proposed algorithms have strong convergence. The first algorithm is designed with $N + 1$ projections on the feasible set and with the prior knowledge of operator norm while the second algorithm is simpler in computations where only one projection on feasible set needs to be implemented and the information of operator norm is not necessary to construct solution approximations.

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