Novi Sad J. Math. Vol. 50, No. 2, 2020, 51-59
https://doi.org/10.30755/NSJOM.09638

Commutative weakly nil-neat rings

Peter Danchev[1,2] **and Mahdi Samiei**[3]

Abstract. We introduce and explore the notion of commutative *weakly nil-neat* rings as those rings whose proper homomorphic images are weakly nil-clean. Our characterization theorem completely gives a description of this class of rings and extends results due to Danchev-McGovern [3] and Samiei [7].

AMS Mathematics Subject Classification (2010): 16U99; 16E50; 13B99

Key words and phrases: nil-clean rings; nil-neat rings; weakly nil-clean rings; weakly nil-neat rings

1. Introduction and background

Throughout this article we shall assume that all rings are commutative, possessing identity. The letters $Nil(R)$, $Id(R)$ and $U(R)$ will stand for the set of nilpotents, the set of idempotents and the set of units of R, respectively. As it is well-known in the commutative case, $Nil(R)$ forms an ideal which we shall denote hereafter by $N(R)$. Denoting by $J(R)$ the Jacobson radical of R, it is well known that $N(R) \subseteq J(R)$. Besides, $U(R)$ is a group which properly contains $1 + J(R) \supseteq 1 + N(R)$. In that way, the set of all unipotent elements $1 + N(R)$ of R is denoted by $Uni(R)$.

In [5], the important concept of a *neat* ring was defined and investigated: A ring R is said to be *neat* if all its proper (i.e., those not equal to R) homomorphic images are *clean* in the sense that they are of the type $Id(R) + U(R)$ (see [6]). Besides the class of clean rings, some valuable examples of such rings are as follows: the ring (domain) $\mathbb{Z}$ of integers and any nonlocal PID; FGC-domains; h-local domains, which are all known to be non clean; etc. as other non-trivial constructions are provided in [5, Propositions 2.1, 2.4]. However, there is no satisfactory description of the algebraic structure of these rings yet.

Imitating this idea, the more restricted class of so-called nil-neat rings was introduced and studied in [7]. They are those rings whose proper (that is, different from the former ring) homomorphic images are *nil-clean* in the sense that they are of the kind $Id(R) + N(R)$ (see [4]). As nil-clean rings are always clean, it is easily seen that nil-neat rings are themselves neat. A complete characterizing result for nil-neatness is obtained in the next useful form in [7, Corollary 2.12]: *Let R be a ring. Then R is nil-neat $\iff$ either R is a field,*

[1]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria, e-mail: danchev@math.bas.bg, pvdanchev@yahoo.com
[2]Corresponding author
[3]Department of Mathematics, Velayat University, Iranshahr, Iran, e-mail: m.samiei@velayat.ac.ir, m.samiei91@yahoo.com

or $R/J(R)$ is boolean (thus being isomorphic to a subring of a direct product of copies of the field $\mathbb{Z}_2$) and, moreover, every non-zero prime ideal of R is maximal.

In the other vein, the notion of *weakly nil-clean* rings was defined in [3] as those rings R which are of the sort $\pm Id(R) + N(R)$. Note that $\mathbb{Z}_3$ is weakly nil-clean but, unfortunately, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is *not*.

That is why it is rather natural to combine the above concepts, and so we come to our basic point of view.

Definition 1.1. We shall say that a ring is *weakly nil-neat* if each its proper (i.e., different from the whole ring) homomorphic image is weakly nil-clean.

Besides weakly nil-clean rings and nil-neat rings, direct examples of such rings are the direct products $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, whereas the triple direct products $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ are definitely *not* so as they contain the homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$ which is *not* weakly nil-clean as already noticed above.

The motivation for writing up this paper is to considerably enlarge the results for nil-neat rings obtained in [7] to this new point of view. The paper is structured as follows: In the subsequent section, we state and prove our preliminaries and our central results. Our main theorem (Theorem 2.8) is actually a criterion establishing when a ring is weakly nil-neat, thereby generalizing the main theorem in [7] and its corollaries. We finish off the work with some discussion and an open problem.

2. Preliminary and main results

We begin this section with a few preliminaries, starting with the following.

Proposition 2.1. *A homomorphic image of a weakly nil-neat ring is again a weakly nil-neat ring.*

Proof. It is straightforward by employing Definition 1.1, which says that a weakly nil-neat ring is the one for which every proper homomorphic image is a weakly nil-clean ring and taking into account that the latter rings are closed under homomorphisms. □

Proposition 2.2. *If R is a weakly nil-neat ring which is not weakly nil-clean, then R is reduced.*

Proof. Suppose, on the contrary, that R is a weakly nil-neat ring which is not weakly nil-clean and $N(R) \neq 0$. Thus, by definition, $R/N(R)$ is weakly nil-clean, and hence in accordance with [3, Proposition 1.9 (i)] the ring R is also weakly nil-clean, which is the desired contradiction. □

Proposition 2.3. *Let R be a ring. Then $Uni(R) \cap -Uni(R) \neq \emptyset$ if, and only if, $2 \in N(R)$.*

Proof. "**Necessity.**" By assumption there is $1+q = -1+t$ for some $q, t \in N(R)$. Thus $2 = t - q \in N(R)$, as stated.

"**Sufficiency.**" Given 2 is a nilpotent in R, we write $2 = q$ for some $q \in N(R)$ and so $1 = -(1+(-q)) \neq 0$ lies in the intersection $[1+N(R)] \cap -[1+N(R)]$ showing that it is manifestly nonempty, as required. □

Imitating [2], let us recall that a ring R is said to be *weakly UU*, and henceforth abbreviated for short as *WUU*, if every unit can be presented as $q + 1$ or $q - 1$, where $q \in N(R)$.

Recall also that a ring R is *local*, provided $R/J(R)$ is a field, that is, $J(R)$ is a maximal ideal of R. Likewise, a ring R is *indecomposable*, provided that $Id(R) = \{0, 1\}$.

The following is valid:

Lemma 2.4. *Let R be a ring. Then the following statements are equivalent:*

(1) R is a local weakly nil-clean ring.

(2) R is an indecomposable weakly nil-clean ring.

(3) For all $x \in R$, either $x \in N(R)$, or $x \in Uni(R)$ or $x \in -Uni(R)$.

(4) R is a WUU *ring and R has exactly one prime ideal.*

Proof. The implication (1) ⇒ (2) is clear.

The equivalence (2) ⇔ (3) is trivial by a direct use of definitions.

To prove that (2) ⇒ (4), suppose that R is an indecomposable weakly nil-clean ring. It is easy to see that, for every $x \in U(R)$, we have that $x \in Uni(R)$ or $x \in -Uni(R)$. Now, we shall show that the ring R has exactly one prime ideal. Letting R be an indecomposable weakly nil-clean domain, we conclude from [3, Proposition 1.9(iii)] that R is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and so we are done. Next, if P_1 and P_2 are two non-zero prime ideals of R, then $R/(P_1P_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R/(P_1P_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. However, this contradicts the fact, which we leave to the reader for verification, that every homomorphic image of an indecomposable weakly nil-clean ring is again an indecomposable weakly nil-clean ring.

Finally, both implications (4) ⇒ (3) and (4) ⇒ (1) are clearly true. □

We now arrive at our crucial tool needed for further applications.

Lemma 2.5. *The following conditions are equivalent for a ring R:*

(1) R is weakly nil-neat.

(2) R/aR is weakly nil-clean for every non-zero $a \in R$.

(3) For any collection of non-zero prime ideals $\{P_j\}_{j \in J}$ of R with $I = \cap_{j \in J} P_j$ different than 0, the factor-ring R/I is weakly nil-clean.

(4) R/aR is weakly nil-neat for every $a \in R$.

(5) R/I is weakly nil-clean for every non-zero semi-prime ideal I.

(6) $R/I = Id(R/I) \cup -Id(R/I)$ for every non-zero semi-prime ideal I.

(7) R/I is either boolean, or is isomorphic to $\mathbb{Z}_3$, or is isomorphic to $B \times \mathbb{Z}_3$ for some boolean ring B, for every non-zero semi-prime ideal I.

(8) For every non-zero semi-prime ideal I of R, the factor-ring R/I is zero-dimensional and $R/P \cong \mathbb{Z}_3$ for at most one maximal ideal P containing I, while $R/Q \cong \mathbb{Z}_2$ for all other maximal ideals Q containing I.

(9) For every non-zero semi-prime ideal I of R it must be that $J(R/I) = 0$ and R/I is isomorphic to either a boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is clear by using the simple fact that a homomorphic image of a weakly nil-clean ring is also weakly nil-clean.

The equivalence (3) $\Leftrightarrow$ (5) is immediate since any semi-prime ideal is the intersection of some family of prime ideals and since a homomorphic image of a weakly nil-clean ring is again a weakly nil-clean ring.

The implication (2) $\Rightarrow$ (4) is evident by virtue of [3, Proposition 1.9(i)].

The implication (4) $\Rightarrow$ (1) is true by choosing $a = 0$.

The implication (1) $\Rightarrow$ (5) is apparent.

As for the implication (5) $\Rightarrow$ (1), assume that I is a non-zero ideal of R. Thus, by our assumption, $R/\sqrt{I}$ is a weakly nil-clean ring. However, it follows from [3, Corollary 1.18] that R/I is a weakly nil-clean ring.

The double implications (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are self-evident applying [3, Theorem 1.13].

The double implications (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) are obvious by making use of [3, Theorem 1.17]. $\square$

The following comments could be useful in shedding up some more light on the proof of the previous statement.

Remark 2.6. The implication (5) $\Rightarrow$ (8) is trivial employing [3, Theorem 1.17].

Besides, concerning the implication (8) $\Rightarrow$ (6), suppose that I is a non-zero semi-prime ideal of R. Then, by an easy verification, we can show that R/I is a subring of the ring $\mathbb{Z}_3 \times \prod \mathbb{Z}_2$, thus concluding that $R/I = Id(R/I) \cup -Id(R/I)$, as required.

As an immediate consequence, we obtain:

Corollary 2.7. *A ring R is weakly nil-neat if, and only if,*

(i) Every non-zero prime ideal of R is maximal and

(ii) For any non-zero semi-prime ideal I of R it must be that $R/M \cong \mathbb{Z}_3$ for at most one maximal ideal M containing I, while $R/N \cong \mathbb{Z}_2$ for all other maximal ideals N containing I.

Proof. Suppose that R is a weakly nil-neat ring. Point (i) can be obtained by applying [3, Proposition 1.9] to any non-zero prime ideal of R, and point (ii) is clear by employing Lemma 2.5 (8).

The converse implication is pretty obvious by using Lemma 2.5 (8). $\square$

Before stating and proving our key result, it is worthwhile to mention the following equivalent facts, namely: A ring R is zero-dimensional (abbreviated as $\dim(R) = 0$) if, and only if, R is π-regular (that is, for each $r \in R$, there is $n \in \mathbb{N}$ such that $r^n \in r^{n+1}R$) if, and only if, every non-zero prime ideal of R is maximal. We are now ready to proceed by proving our main characterization theorem, which asserts the following:

Theorem 2.8. *A ring R is weakly nil-neat if, and only if, exactly one of the following is true:*

(1) R is a field (in particular, R could be isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_3$),

or

(2) $J(R) \neq 0$ and $R/J(R)$ is isomorphic to either a boolean ring (i.e., to a subring of a direct product of copies of $\mathbb{Z}_2$), or to $\mathbb{Z}_3$, or to the direct product of two such rings,

or

(3) $J(R) = 0$, R is not a field, and R is isomorphic to either a boolean ring B (i.e., to a subring of a direct product of copies of $\mathbb{Z}_2$), or to $B \times \mathbb{Z}_3$, or to $\mathbb{Z}_3 \times \mathbb{Z}_3$ and, moreover, in all cases every non-zero prime ideal of R is maximal.

Proof. Firstly, assume that R is a weakly nil-neat ring. If R is a field, we are done, so we shall assume hereafter that R is a weakly nil-neat ring which is *not* a field.

Firstly, let $J(R) \neq 0$. Then $J(R)$ is a non-zero semi-prime ideal of R and so, by virtue of Lemma 2.5, the quotient $R/J(R)$ is either boolean, or is isomorphic to $\mathbb{Z}_3$, or is isomorphic to $B \times \mathbb{Z}_3$ for some boolean ring B, as asserted.

Secondly, suppose $J(R) = 0$ and $\mathrm{Max}(R) = \{M_i\}_{i \in T}$ for some index set T. It is clear that $M_i \neq 0$, because R is not a field. This fact shows that R has at least two maximal ideals. If $T = \{1, 2\}$, then in accordance with Lemma 2.5 (9) we will have that either $R/M_i \cong \mathbb{Z}_2$, or $R/M_i \cong \mathbb{Z}_3$ for $i \in T$. Thus R is isomorphic to a subring of either $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$. For the case when $|T| > 2$, and hence $i > 2$, we set $I_k := \bigcap_{i \neq k} M_i$ and let $R/M_k \cong \mathbb{Z}_3$. We, therefore, can see by routine check that I_k is a non-zero semi-prime ideal of R. We now claim that $R/M_s \ncong \mathbb{Z}_3$, for all maximal ideals $M_s \in \{M_i\}_{i \neq k}$. Otherwise, with Corollary 2.7 at hand, there is a maximal ideal $M_l \in \{M_i\}_{i \neq k}$ such that $R/M_l \cong \mathbb{Z}_3$. Consequently, we deduce that R is isomorphic to a subring of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \prod \mathbb{Z}_2$, that is a contradiction (for example, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ has a homomorphic image isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, which is surely *not* weakly nil-clean). Letting now $R/M_k \ncong \mathbb{Z}_3$. By using Lemma 2.5 (9), we have that $R/M_k \cong \mathbb{Z}_2$ and also, by Corollary 2.7, that for at most one maximal ideal $M_l \in \{M_i\}_{i \neq k}$ the isomorphism $R/M_l \cong \mathbb{Z}_3$ holds, whereas for all other maximal ideals $M_s \in \{M_i\}_{i \neq k}$ we have that $R/M_l \cong \mathbb{Z}_2$. Since $R/M_k \cong \mathbb{Z}_2$, we conclude that R is isomorphic to a subring of $\mathbb{Z}_3 \times \prod \mathbb{Z}_2$, as claimed.

Conversely, assume the validity of precisely one of points (1), (2) and (3). First, it is clear that R has to be a weakly nil-neat ring whenever R is a field. Now, assume that R is *not* a field. If foremost $J(R) \neq 0$ and I is a non-zero semi-prime ideal of R, by our hypothesis, the quotient $R/J(R)$ is isomorphic to either a boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings. Moreover, it is elementarily seen that $J(R) \subset I$ and so R/I is also isomorphic to either a boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings, as required by Lemma 2.5 (7).

Letting now $J(R) = 0$, we shall distinguish between two basic cases:

Case 1. *Assume that R is isomorphic to a subring of a direct product of copies of $\mathbb{Z}_2$ and at most one copy of $\mathbb{Z}_3$.*

Then, we have a monomorphism $\varphi : R \to \mathbb{Z}_3 \times \prod \mathbb{Z}_2$. We know that the order of the element 1_R is equal to the order of the element $1_{\varphi(R)}$. This implies that $o(1_R)$ is either 2 or 3 or 6, because $\mathbb{Z}_3 \times \prod \mathbb{Z}_2$ has characteristic exactly 6. Now, let I be a non-zero semi-prime ideal of R, let M_j be a maximal ideal of R containing I, and consider the epimorphism $\pi_j : R \to R/M_j$. It is clear that $\pi_j(1_R) = 1_{R/M_j}$ and so, 2 or 3 divides the order of the element $1_{R/M_j}$. We infer that the field R/M_j has characteristic 2 or 3, and hence R/M_j is isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2$. It is easy to see that there do not exist two maximal ideals M and N of R containing I such that $R/M \cong \mathbb{Z}_3 \cong R/N$, since by our assumption R is isomorphic to a subring of the product $\mathbb{Z}_3 \times \prod \mathbb{Z}_2$, which will lead to a contradiction. Applying now Corollary 2.7, we see that R is weakly nil-neat.

Case 2. *Assume that R is isomorphic to a subring of $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

So, the ring R embeds into a 9-element ring, so R has either 1, 3 or 9 elements and this definitely suffices. In fact, any such 3-element ring R has no zero divisors (which is trivial to be shown by using the existence of the identity element), and thus R is an integral domain with no nontrivial ideals, and hence it is surely isomorphic to the field $\mathbb{Z}_3$. The one-element case is trivial, and this leaves the case when $|R| = 9$, so $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. This surely is a weakly nil-neat ring (as it was discussed earlier in the paper). □

Remark 2.9. As an alternative proof of Case 2 above, which is parallel to that of Case 1 and which could be of some interest to the reader as a valuable material for a further development of the investigated theme, we may suggest the following: We have a monomorphism $\psi : R \to \mathbb{Z}_3 \times \mathbb{Z}_3$ and hence $o(1_R)$ is 3, because $\mathbb{Z}_3 \times \mathbb{Z}_3$ has characteristic 3. Now, let I be a non-zero semi-prime ideal of R, let M_j be a maximal ideal of R containing I, and consider the epimorphism $\pi_j : R \to R/M_j$. It is clear that $\pi_j(1_R) = 1_{R/M_j}$, whence 3 divides the order of the element $1_{R/M_j}$. We deduce that the field R/M_j has characteristic 3 and hence R/M_j is isomorphic to $\mathbb{Z}_3$. It remains to show that if K is a non-zero semi-prime ideal of R, then there will not exist two maximal ideals M and N of R containing K such that $R/M \cong \mathbb{Z}_3 \cong R/N$. In fact, to show that, suppose the contrary, namely that this double isomorphism holds. Since $J(R) = 0$, we conclude that there exists a non-zero maximal ideal P of R which is not containing K. We, however, know by the above argument that $R/P \cong \mathbb{Z}_3$, which is a contradiction to the assumption that R is isomorphic to a subring of the product $\mathbb{Z}_3 \times \mathbb{Z}_3$, as needed.

Let us recall that a ring R is said to be *uniquely clean* if each its element is uniquely written as the sum of a unit and an idempotent, and is said to be *uniquely nil-clean* if every its element is uniquely written as the sum of a nilpotent and an idempotent.

The next consequence discovers the more complicated structure of weakly nil-neat rings than that of nil-neat rings.

Corollary 2.10. *Let R be a ring such that $J(R) \neq 0$ and let R be not a domain. Consider the following statements:*

(1) R is a weakly nil-clean ring.

(2) R is a weakly nil-neat ring.

(3) R is a clean WUU *ring.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$*. Moreover, if $2 \in N(R)$, then the above three statements are equivalent and also equivalent to:*

(4) R is a clean UU *ring.*

(5) R is a nil-clean ring;

(6) R is a uniquely nil-clean ring;

(7) R is a uniquely clean ring such that every prime ideal of R is maximal;

(8) $J(R)$ is a nil ideal, and $R/J(R)$ is a Boolean ring;

(9) R is an exchange UU *ring.*

(10) R is a nil-neat ring.

Proof. The implication (1) $\Rightarrow$ (2) is clearly true. The reverse implication (2) $\Rightarrow$ (1) follows directly by using Proposition 2.3.

Regarding the implication (2) $\Rightarrow$ (3), suppose that R is a weakly nil neat ring. It follows from Theorem 2.8 that $J(R)$ is nil and $R/J(R)$ is isomorphic to either a boolean ring, or $\mathbb{Z}_3$, or the direct product of two such rings. We, therefore, deduce from [2, Corollary 2.14] that R is a clean WUU ring.

Now, let $2 \in J(R)$. To show the validity of (3) $\Rightarrow$ (2), let $x \in R$. Then $-x + 1 = u + e$ for some $u \in U(R)$ and $e \in Id(R)$. Since R is a WUU ring, we can write $-x = n + e$ or $-x = (2 + n) + e$ for some $n \in N(R)$, and hence $x = n + e$ or $x = m - e$ for some $m \in N(R)$, because $2 \in J(R)$. Thus R is a weakly nil-clean ring, as promised.

The implication (3) $\Rightarrow$ (5) is immediate by knowing [3, Proposition 1.10].

The series of implications (4-10) $\Rightarrow$ one of (1-3) are trivial; e.g. (5) $\Rightarrow$ (1), or (4) $\Rightarrow$ (3), are immediate, so we leave them to the interested reader for a direct check.

Finally, the equivalences (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10) are immediately true by taking into account [7, Theorem 2.13]. □

3. Discussion and open question

The next comments shed some more light on the discussed theme.

Remark 3.1. The class of weakly nil-clean rings (*not* necessarily commutative) was completely described in [2] and, independently, in [8] (see also [1]). In fact, it was proved that for such a ring R the direct decomposition $R \cong R_1 \times R_2$ holds, where either $R_1 = \{0\}$ or R_1 is a nil-clean ring, and either $R_2 = \{0\}$ or R_2 is a ring for which $R_2/J(R_2) \cong \mathbb{Z}_3$. Therefore, in the case when R_1 and R_2 are both non-zero, then there is a proper epimorphism of the weakly nil-clean ring R to the nil-clean ring R_1.

In that way, combining Theorem 2.8 with [7, Corollary 2.12], one concludes that a similar direct decomposition (and hence a corresponding epimorphism) of a commutative weakly nil-neat ring into a commutative nil-neat ring and an additional direct factor which can be visualized explicitly.

We finish off our work with the following intriguing and quite difficult problem.

Problem 3.2. Develop the theory of non-commutative nil-neat and weakly nil-neat rings.

Acknowledgement

We would like to thank the referee for his/her careful reading of the manuscript and for the several comments and suggestions in hopes of making this article more acceptable. The authors are also very grateful to the handling editor, Professor Petar Marković, for his valuable insightful remarks and friendly help in preparing the style of the presentation.

Funding

The work of the corresponding author (P. V. Danchev) is partially supported by the Bulgarian National Science Fund under Grant KP-06 N 32/1 of Dec. 07, 2019.

References

[1] Breaz, S., Danchev, P., and Zhou, Y. Rings in which every element is either a sum or a difference of a nilpotent and an idempotent. *J. Algebra Appl. 15*, 8 (2016), Art. No. 1650148, 11 pages.

[2] Danchev, P. V. Weakly UU rings. *Tsukuba J. Math. 40*, 1 (2016), 101–118.

[3] Danchev, P. V., and McGovern, W. Wm. Commutative weakly nil clean unital rings. *J. Algebra 425* (2015), 410–422.

[4] Diesl, A. J. Nil clean rings. *J. Algebra 383* (2013), 197–211.

[5] McGovern, W. Wm. Neat rings. *J. Pure Appl. Algebra 205*, 2 (2006), 243–265.

[6] Nicholson, W. K. Lifting idempotents and exchange rings. *Trans. Amer. Math. Soc. 229* (1977), 269–278.

[7] Samiei, M. Commutative rings whose proper homomorphic images are nil clean. *Novi Sad J. Math. 50*, 1 (2020), 37–44.

[8] Šter, J. Nil-clean quadratic elements. *J. Algebra Appl. 16*, 10 (2017), Art. No. 1750197, 8 pages.

Received by the editors June 14, 2019
First published online September 30, 2019