Novi Sad J. Math. Vol. 50, No. 2, 2020, 41-50
https://doi.org/10.30755/NSJOM.09413

Lifting modules with respect to images of a fully invariant submodule

Tayyebeh Amouzegar[1] **and Ali Reza Moniri Hamzekolaee**[23]

Abstract. Lifting modules as a main concept in module theory have been studied and investigated extensively in recent decades. The first author in [1] tried to consider and investigate this concept with a homological approach. Let R be a ring and M be a right R-module. Then M is called $\mathcal{I}$-lifting if image of every endomorphism of M lies above a direct summand of M. In this paper, we are interested in studying modules M with the property that $\varphi(F)/D \ll M/D$ for every endomorphism φ of M and for some direct summands D of M, where F is a fixed fully invariant submodule of M. We call such modules $\mathcal{I}_F$-lifting. We provide some examples of $\mathcal{I}_F$-lifting modules as a proper generalization of lifting modules. Some characterizations of $\mathcal{I}_F$-lifting modules are given. We also define relative $\mathcal{I}_F$-lifting modules to study direct summands and finite direct sums of $\mathcal{I}_F$-lifting modules.

AMS Mathematics Subject Classification (2010): 16D10; 16D80

Key words and phrases: lifting module; $\mathcal{I}$-lifting module; $\mathcal{I}_F$-lifting module; dual Rickart module; endomorphisms ring.

1. Introduction

All rings considered in this paper are associative with an identity element and all modules are unitary right modules, unless otherwise stated. Let R be a ring and M an R-module. Then $S = End_R(M)$ will denote the ring of all R-endomorphisms of M. We use the notation $N \ll M$ to indicate that N is small in M (i.e., $\forall L \lneq M, L + N \neq M$). A module M is called *hollow* if every proper submodule of M is small in M. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M),\ \phi(N) \subseteq N$). $Rad(M)$ and $Soc(M)$ denote the radical and the socle of a module M, respectively.

We recall that L is a *cosmall submodule of* K *in* M (denoted by $L \overset{cs}{\hookrightarrow} K$ in M) if $K/L \ll M/L$. Recall that a submodule L of M is called *coclosed* in M, if L has no proper cosmall submodule. It is clear that every direct summand of M is a coclosed submodule of M. Let $L \subseteq K \leq M$. We say that K *lies above* L if $K/L \ll M/L$. A module M is called *lifting* if every submodule A of M lies above a direct summand D of M ([2]).

[1]Department of Mathematics, Faculty of Sciences, Quchan University of Technology, Quchan, Iran, e-mail: t.amoozegar@yahoo.com, t.amouzgar@qiet.ac.ir
[2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran, e-mail: a.monirih@umz.ac.ir
[3]Corresponding author

A number of results concerning lifting modules have appeared in the literature in the recent years and many generalizations of the concept of lifting modules have been introduced and studied by several authors (see [4], [5], [6], [9]).

Recently, Lee, Rizvi and Roman introduced the notion of Rickart modules and dual Rickart modules in [7, 8]. Let M be a module. Then M is called a *(dual) Rickart* if, for every endomorphism φ of M, $(Im\varphi)$ $Ker\varphi$ is a direct summand of M. (Dual) Rickart modules determine the importance of idempotents in the ring of all endomorphisms of a module. In particular, as a famous result, a module M is Rickart and dual Rickart if and only if $End_R(M)$ is a von Neumann regular ring. Recently, the first author in [1] introduced a generalization of both lifting modules and dual Rickart modules namely $\mathcal{I}$-lifting modules. A module M is called *$\mathcal{I}$-lifting* if image of every endomorphism of M lies above a direct summand of M. The author showed that a projective $\mathcal{I}$-lifting module is a direct sum of cyclic modules. She also present a characterization of $\mathcal{I}$-lifting rings in terms of finitely supplemented modules. Since the class of $\mathcal{I}$-lifting modules is larger than the class of dual Rickart modules, the study of them seem to be more difficult.

Motivating by mentioned works on lifting modules via a homological approach, we are interested in studying $\mathcal{I}$-lifting modules via images of fully invariant submodules. In fact, in the definition of an $\mathcal{I}$-lifting module M one can replace the image of M by the image of a fully invariant submodule of M. We say that a module M is *$\mathcal{I}_F$-lifting* provided that, for every endomorphism φ of M, there is a direct summand D of M such that $\varphi(F)/D \ll M/D$. In what follows by F we mean a fully invariant submodule of M.

In Section 2, we present some examples of $\mathcal{I}_F$-lifting modules and study some properties of these modules. We show that a module M is $\mathcal{I}_F$-lifting if and only if F is $\mathcal{I}$-lifting for a fully invariant direct summand F of M. Some characterizations of $\mathcal{I}_F$-lifting modules are provided. We also discuss the homomorphic images of $\mathcal{I}_F$-lifting modules. In Section 3, we introduce relative $\mathcal{I}_F$-lifting modules and we use this concept to study direct summands and direct sums of $\mathcal{I}_F$-lifting modules.

2. $\mathcal{I}_F$-lifting modules

In this section, we introduce and study a form of $\mathcal{I}$-lifting modules via image of fully invariant submodules.

Definition 2.1. Let M be a module and F be a fully invariant submodule of M. We say M is *$\mathcal{I}_F$-lifting* if for every $\varphi \in End_R(M)$, the submodule $\varphi(F)$ lies above a direct summand of M.

It is clear that every lifting module is $\mathcal{I}_F$-lifting. Obviously, M is $\mathcal{I}_M$-lifting if and only if M is $\mathcal{I}$-lifting. Note that every module M is clearly $\mathcal{I}_0$-lifting.

The following contains some examples of $\mathcal{I}_F$-lifting modules.

Example 2.2. (1) Let F be a fully invariant submodule of a module M. If F is small in M, then M is $\mathcal{I}_F$-lifting. In particular, every hollow module M

is $\mathcal{I}_F$-lifting for every fully invariant submodule F of M. For example, the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ is $\mathcal{I}_{(1/p+\mathbb{Z})}$-lifting. Note that $Soc(M) = (1/p + \mathbb{Z})$.

(2) Let p be a prime number. Then the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^2}$ is not a dual Rickart module. Now, $Rad(M) = (p) \neq 0$. Since M is a hollow module, M is $\mathcal{I}_{Rad(M)}$-lifting.

Let K and N be submodules of M. Then K is called a *supplement* of N in M if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K+N$ and $K \cap N \ll K$. A module M is called *supplemented*, if every submodule of M has a supplement in M.

Example 2.2(1), provides a rich source of $\mathcal{I}_F$-lifting modules which are lifting. The following examples show that an $\mathcal{I}_F$-lifting module need not be lifting in general.

Example 2.3. (1) Let M be a non-supplemented finitely generated module. Then by Example 2.2, M is $\mathcal{I}_{Rad(M)}$-lifting while M is not lifting. For instance, we may consider a non-semiperfect ring R which is $\mathcal{I}_{J(R)}$-lifting.

(2) Let p be any prime integer. Then the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not lifting by [3, Example 10]. Note that M is finitely generated so that $Rad(M) \ll M$. Hence M is $\mathcal{I}_{Rad(M)}$-lifting by Example 2.2(1).

The following provides an easy characterization of $\mathcal{I}_F$-lifting modules which is completely similar to lifting modules.

Proposition 2.4. *Let M be a module and let F be a fully invariant submodule of M. Then the following conditions are equivalent:*

(1) *M is $\mathcal{I}_F$-lifting;*

(2) *For every $\varphi \in S$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \varphi(F)$ and $M_2 \cap \varphi(F) \ll M_2$;*

(3) *For every $\varphi \in S$, $\varphi(F)$ can be written as $\varphi(F) = N \oplus S$ such that $N \leq^{\oplus} M$ and $S \ll M$.*

Proof. It follows from [2, 22.1]. □

Next, we shall present a characterization of $\mathcal{I}_F$-lifting modules when F is a direct summand of M.

Theorem 2.5. *Let M be a module and let F be a fully invariant direct summand of M. Then M is $\mathcal{I}_F$-lifting if and only if F is $\mathcal{I}$-lifting.*

Proof. ($\Rightarrow$) Let $g : F \to F$ be an endomorphism of F and $F \oplus F' = M$ for a submodule F' of M. Then $h = j \circ g \circ \pi_F : M \to M$ is an endomorphism of M where $j : F \to M$ is the inclusion and $\pi_K : M \to F$ is the projection map on F. It is straightforward that $h(F) = g(F)$. Since M is $\mathcal{I}_F$-lifting, there exists a direct summand D of M such that $g(F)/D \ll M/D$ and hence $g(F)/D \ll F/D$. It is left to reader to verify that F/D is a direct summand of M/D.

($\Leftarrow$) Let F be $\mathcal{I}$-lifting and let f be an endomorphism of M. Consider $q = \pi_F \circ f \circ j : F \to F$, which is an endomorphism of F, where $j : F \to M$

is the inclusion and $\pi_F : M \to F$ is the projection on F. F being a fully invariant submodule of M implies that $q(F) = f(F)$. As F is $\mathcal{I}$-lifting, there is a direct summand D of F (so that of M) such that $q(F)/D = f(F)/D \ll F/D$. Therefore, M is $\mathcal{I}_F$-lifting.

□

Recall from [10] that an R-module M is *noncosingular (cosingular)* provided $\overline{Z}(M) = M$ ($\overline{Z}(M) = 0$) where $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U\}$ for all small R-modules U.

Corollary 2.6. (1) *Let M be a module such that $\overline{Z}(M)$ is a direct summand of M. Then M is $\mathcal{I}_{\overline{Z}(M)}$-lifting if and only if $\overline{Z}(M)$ is dual Rickart.*

(2) *Let M be a module such that $Soc(M)$ is a direct summand of M. Then M is $\mathcal{I}_{Soc(M)}$-lifting.*

Proof. (1) Let M be $\mathcal{I}_{\overline{Z}(M)}$-lifting. Then by Theorem 2.5, the submodule $\overline{Z}(M)$ is $\mathcal{I}$-lifting. Note that since $\overline{Z}(M)$ is a direct summand of M, it is noncosinsgular. The result follows from [1, Lemma 3.1].

(2) It is clear as $Soc(M)$ is semisimple. □

We shall present a characterization of $\mathcal{I}_F$-lifting modules with no nonzero small submodules.

Theorem 2.7. *Let M be a module with $Rad(M) = 0$ and let $F \leq M$ be fully invariant. Then the following statements are equivalent:*

(1) *M is $\mathcal{I}_F$-lifting;*

(2) *F is a dual Rickart direct summand of M.*

Proof. (1) ⇒ (2) Let φ be an arbitrary endomorphism of M. Then there exists a direct summand D of M such that $\varphi(F)/D \ll M/D$. Since $Rad(M) = 0$ we conclude that $Rad(M/D) = 0$ as D is a direct summand of M. Therefore, $\varphi(F) = D$ is a direct summand of M. It follows that F is a direct summand of M. From Theorem 2.5, F is $\mathcal{I}$-lifting. Since $Rad(M) = 0$, we conclude from [1, Proposition 3.1] that F is a dual Rickart module.

(2) ⇒ (1) It follows directly from Theorem 2.5 and the fact that every dual Rickart module is $\mathcal{I}$-lifting. □

From the previous Theorem, we conclude that over a right V-ring R, a right R-module M is $\mathcal{I}_F$-lifting if and only if F is a dual Rickart direct summand of M.

Example 2.8. (1) Let F be a field and $R = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for each $i \in \mathbb{N}$. Then R is a von Neumann regular V-ring. Take $M = R$ and let F be any finitely generated ideal of R. Therefore F is a direct summand of M. It is well-known that M is a dual Rickart module (see [8, Remark 2.2]) and hence F as a direct summand is also dual Rickart (see [8, Proposition 2.8]). Hence, M is $\mathcal{I}_F$-lifting module by Theorem 2.7.

(2) Let L be a V-ring and K be a field. Then $S = K \times L$ is a V-ring, as well. Consider the central idempotent $e = (1, 0)$ of S. Then $Se = eS \cong K$,

as both a left S-module and a right S-module. Let R be the ring $M_n(S)$ (the ring of all $n \times n$ matrices with entries from S). As R is Morita-equivalent to S, it should also be a V-ring. Now, R has a central idempotent, $f = eI$ where I is the identity matrix of R. Then $fR = Rf$ is isomorphic to $M_n(Se)$ so that $fR = Rf \cong M_n(K)$. Note that $F = Rf$ is a two-sided ideal of R and F is also a direct summand of R. K being a field implies that $M_n(K)$, and hence also F is semisimple and so F is dual Rickart. It follows from Theorem 2.7 that R is an $\mathcal{I}_F$-lifting module.

The following contains some properties of fully invariant submodules of modules. The same results may be found in related works. We shall present them here for the sake of completeness.

Lemma 2.9. *Let M be a module. Then the following assertions hold:*

(1) *Let $F \leq M$ be fully invariant and let K be a direct summand of M. Then $F \cap K$ is a fully invariant submodule of K.*

(2) *Let $F \leq M$ be fully invariant and K a direct summand of M contained in F. Then F/K is a fully invariant submodule of M/K.*

Proof. (1) Let F be a fully invariant submodule of M and $K \leq^{\oplus} M$. Consider a decomposition $M = K \oplus K'$ and an arbitrary endomorphism g of K. Now $h = j \circ g \circ \pi_K$ is an endomorphism of M where $j : K \to M$ is the inclusion and $\pi_K : M \to K$ is the canonical projection. Since F is fully invariant in M, we have $h(F) \subseteq F$. It is easy to verify that $h(F) = g(F \cap K)$ (note that $F = (F \cap K) \oplus (F \cap K')$ implies $\pi_K(F) = F \cap K$). Therefore, $g(F \cap K)$ is contained in $F \cap K$ and so $F \cap K$ is a fully invariant submodule of K.

(2) Suppose that $g : M/K \to M/K$ is an endomorphism of M/K. Then $f = j \circ h \circ g \circ \pi : M \to M$ is an endomorphism of M. Here we should note that $\pi : M \to M/K$ is the canonical projection, $h : M/K \to K'$ is the isomorphism induced by the decomposition $M = K \oplus K'$ and $j : K' \to M$ is the inclusion map. Let $g(F/K) = T/K$ for a submodule T of M containing K. Now, $f(F) = T \cap K' \subseteq F$. Suppose that $t + K \in T/K$. Since $K \oplus (T \cap K') = T$, we conclude that $t + K = x + K$ for some $x \in T \cap K'$. It follows that $x \in F$ and so $g(F/K) = T/K \subseteq F/K$. □

Proposition 2.10. *Let F be a fully invariant submodule of a module M and let K be a fully invariant direct summand of M contained in F. If M is $\mathcal{I}_F$-lifting, then M/K is $\mathcal{I}_{F/K}$-lifting.*

Proof. Let $g : M/K \to M/K$ be an endomorphism of M/K and $M = K \oplus K'$. Then $f = j \circ h \circ g \circ \pi : M \to M$ is an endomorphism of M. Note that $\pi : M \to M/K$ is the canonical projection, $h : M/K \to K'$ is the isomorphism induced by the decomposition $M = K \oplus K'$ and $j : K' \to M$ is the inclusion map. By assuming $g(F/K) = T/K$, one can see that $f(F) = T \cap K'$. Since M is $\mathcal{I}_F$-lifting, there is a direct summand D of M such that $(T \cap K')/D \ll M/D$. Set $M = D \oplus D'$. Then $M/K = (D + K)/K + (D' + K)/K$. As K is a fully invariant submodule of M, we have $K = (K \cap D) \oplus (K \cap D')$. Hence $(D + K) \cap (D' + K) = K$ which implies that $(D + K)/K$ is a direct summand of

M/K. We shall prove that $\frac{T/K}{(D+K)/K} \ll \frac{M/K}{(D+K)/K}$. To verify this assertion, let $\frac{T/K}{(D+K)/K} + \frac{L/K}{(D+K)/K} = \frac{M/K}{(D+K)/K}$. Then $T/K + L/K = M/K$. So $T + L = M$. By the modular law, $T = K \oplus (T \cap K')$. Therefore, $T \cap K' + L = M$. Now combining $(T \cap K')/D + L/D = M/D$ and $(T \cap K')/D \ll M/D$ implies that $L = M$, as required. □

The following presents a characterization of an $\mathcal{I}_F$-lifting module M in terms of finitely generated ideals of $End_R(M)$.

Proposition 2.11. *Let M be a module and let F be a fully invariant submodule of M. Then M is $\mathcal{I}_F$-lifting if and only if for every finitely generated right ideal I of $End_R(M)$, the submodule $\sum_{\varphi \in I} \varphi(F)$ of M lies above a direct summand of M.*

Proof. ($\Rightarrow$) Let M be $\mathcal{I}_F$-lifting and $I = \langle f_1, \ldots, f_k \rangle$ a finitely generated right ideal of S. It is easy to check that $\sum_{f \in I} f(F) = [f_1(F) + \ldots + f_k(F)] \subseteq F$. Set $f = f_1 + \ldots + f_k$. Then $f(F) = \sum_{\varphi \in I} \varphi(F)$. Since M is $\mathcal{I}_F$-lifting, there exists a direct summand D of M such that $\sum_{\varphi \in I} \varphi(F)/D \ll M/D$.

($\Leftarrow$) Let $f \in S$. Consider the cyclic right ideal $I = \langle f \rangle$ of S. By assumption there is a direct summand D of M such that $\frac{\sum_{\varphi \in I} \varphi(F)}{D} \ll \frac{M}{D}$. It is not hard to verify that $f(F) = \sum_{\varphi \in I} \varphi(F)$. Now the proof is completed.

□

Proposition 2.12. *Let F be a fully invariant direct summand of a module M. Let M be an $\mathcal{I}_F$-lifting module and N be a submodule of M invariant under all maps $f \in End_R(M)$ with $f(F)$ a direct summand of M. Then N is a fully invariant submodule of M.*

Proof. Let $f \in End_R(M)$. Since M is $\mathcal{I}_F$-lifting, $M = M_1 \oplus M_2$ where $M_1 \subseteq f(F)$ and $M_2 \cap f(F) \ll M_2$. We have $f(F) = M_1 \oplus (M_2 \cap f(F))$. Consider the projection maps $\pi_{M_1} : M \to M$ and $\pi_{M_2} : M \to M$ of M onto M_1 and M_2, respectively. Note that $(\pi_{M_1} f)(F) = M_1$ is a direct summand of M. By hypothesis, $\pi_{M_1} f(N) \subseteq N$. As $(\pi_{M_2} f)(F) \ll M$ and F is a direct summand of M, we have $(1_M - \pi_{M_2} f)(F) = F$. By assumption, $(1_M - \pi_{M_2} f)(N) \subseteq N$. So $\pi_{M_2} f(N) \subseteq N$. Therefore $f(N) \subseteq N$. □

It is clear by definition that, if $F \ll M$, then M is clearly $\mathcal{I}_F$-lifting. The following consider the condition for the converse for a special fully invariant submodule of M.

Proposition 2.13. *Let M be an $\mathcal{I}_{Rad(M)}$-lifting projective module. Then $Rad(M)$ is small in M.*

Proof. Let $N \subseteq M$ be any submodule with $N + Rad(M) = M$. Then $Rad(M) \to M \to M/N$ is an epimorphism and there exists a homomorphism $f : M \to Rad(M)$ such that $f(Rad(M)) + N = Rad(M) + N = M$ (note that M is projective). Since M is $\mathcal{I}_{Rad(M)}$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq f(Rad(M))$ and $M_2 \cap f(Rad(M)) \ll M_2$. Note that $M_1 \subseteq f(Rad(M)) \subseteq Rad(M)$. By [11, 22.3], $M_1 = 0$ and so $f(Rad(M)) \ll M$. Hence $N = M$. Therefore $Rad(M) \ll M$. □

3. Relative $\mathcal{I}_F$-lifting modules

In this section we define relative $\mathcal{I}_F$-lifting modules and we apply this concept to study finite direct sums of $\mathcal{I}_F$-lifting modules.

Definition 3.1. Let M and N be R-modules and let F be a fully invariant submodule of M. We say that M is *N-$\mathcal{I}_F$-lifting* provided for every homomorphism $\phi : M \to N$, there exists $L \leq^{\oplus} N$ such that $L \subseteq \phi(F)$ and $\phi(F)/L \ll N/L$.

It is clear that a right module M is $\mathcal{I}_F$-lifting if and only if M is M-$\mathcal{I}_F$-lifting.

We provide an equivalent condition for relative $\mathcal{I}_F$-lifting modules.

Theorem 3.2. *Let M and N be right R-modules and let F be a fully invariant submodule of M. Then M is N-$\mathcal{I}_F$-lifting if and only if for all direct summands M' of M and coclosed submodules N' of N, M' is N'-$\mathcal{I}_{F\cap M'}$-lifting.*

Proof. Let $M' = eM$ for some $e^2 = e \in End_R(M)$, and let N' be a coclosed submodule of N. Assume that $\psi \in Hom(M', N')$. It is easy to check that $\psi eM = \psi M'$ and $\psi M' \subseteq N' \subseteq N$. Since M is N-$\mathcal{I}_F$-lifting, there exists a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq \psi e(F)$ and $N_2 \cap \psi e(F) \ll N_2$. As $N_1 \subseteq \psi e(F) \subseteq \psi(F) \subseteq N'$, we have $N' = N_1 \oplus (N_2 \cap N')$. It is easy to see that $e(F \cap M') = eF$ and $\psi e(F) = \psi(F \cap M')$. Thus $N_1 \subseteq \psi e(F) = \psi(F \cap M')$. By [2, 3.7(3)], $N_2 \cap N' \cap \psi e(F) \ll N'$. Hence $N_2 \cap N' \cap \psi(F \cap M') \ll N'$. Again, by [2, 3.7(3)], $N' \cap N_2 \cap \psi(F \cap M') \ll N_2 \cap N'$. Therefore, M' is N'-$\mathcal{I}_{F\cap M'}$-lifting. The converse is clear. □

Corollary 3.3. *The following conditions are equivalent for a module M and a fully invariant submodule F of M:*

(1) *M is an $\mathcal{I}_F$-lifting module;*

(2) *For any coclosed submodule N of M, every direct summand L of M is N-$\mathcal{I}_{F\cap L}$-lifting.*

Corollary 3.4. *Let F be a fully invariant submodule of a module M and let K be a direct summand of M. If M is $\mathcal{I}_F$-lifting, then K is $\mathcal{I}_{F\cap K}$-lifting.*

Let F be a fully invariant submodule of a module M. We say that M is *$\mathcal{I}_F$-supplemented* if for every $f \in S$, $f(F)$ has a supplement in M.

A module M is called *N-$\mathcal{I}_F$-supplemented* provided that for every homomorphism $\phi : M \to N$ there exists $L \leq N$ such that $\phi(F) + L = N$ and $\phi(F) \cap L \ll L$. It is clear that a right module M is $\mathcal{I}_F$-supplemented if and only if M is M-$\mathcal{I}_F$-supplemented. Note also that every $\mathcal{I}_F$-lifting module is $\mathcal{I}_F$-supplemented.

The following considers a finite direct sum of relative $\mathcal{I}_F$-supplemented modules.

Theorem 3.5. *Let M_1, M_2 and N be modules. Let F be a fully invariant submodule of N. If N is M_i-$\mathcal{I}_F$-supplemented for $i = 1, 2$, then N is $M_1 \oplus M_2$-$\mathcal{I}_F$-supplemented. The converse is true if $M_1 \oplus M_2$ is a duo module.*

Proof. Suppose N is M_i-$\mathcal{I}_F$-supplemented for $i = 1, 2$. We prove that N is $M_1 \oplus M_2$-$\mathcal{I}_F$-supplemented. Let $\phi = (\pi_1\phi, \pi_2\phi)$ be any homomorphism from N to $M_1 \oplus M_2$, where π_i is the projection map from $M_1 \oplus M_2$ to M_i for $i = 1, 2$. Since N is M_i-$\mathcal{I}_F$-supplemented, there exists a submodule K_i of M_i such that $\pi_i\phi F + K_i = M_i$ and $\pi_i\phi F \cap K_i \ll K_i$, for $i = 1, 2$. Let $K = K_1 \oplus K_2$. Then

$$M_1 \oplus M_2 = \pi_1\phi F + \pi_2\phi F + K_1 + K_2 = \phi F + K.$$

Since

$$\phi F \cap (K_1 + K_2) \leq (\phi F + K_1) \cap K_2 + (\phi F + K_2) \cap K_1,$$

we have

$$\phi F \cap (K_1 + K_2) \leq (\phi F + M_1) \cap K_2 + (\phi F + M_2) \cap K_1.$$

As $\phi F + M_1 = \pi_2\phi F \oplus M_1$ and $\phi F + M_2 = \pi_1\phi F \oplus M_2$, we conclude that $\phi F \cap K \subseteq (\pi_2\phi F \cap K_2) + (\pi_1\phi F \cap K_1)$. Since $\pi_i\phi F \cap K_i \ll K_i$ for $i = 1, 2$, $\phi F \cap K \ll K_1 + K_2 = K$. Hence N is $M_1 \oplus M_2$-$\mathcal{I}_F$-supplemented. Conversely, let N be $M_1 \oplus M_2$-$\mathcal{I}_F$-supplemented. Let ϕ be a homomorphism from N to M_1. Then $\phi\iota(F) = \phi(F)$, where ι is the canonical inclusion from M_1 to $M_1 \oplus M_2$. Since N is $M_1 \oplus M_2$-$\mathcal{I}_F$-supplemented, there exists $K \subseteq M_1 \oplus M_2$ such that $M_1 \oplus M_2 = \phi(F) + K$ and $\phi(F) \cap K \ll K$. Thus $M_1 = \phi(F) + (K \cap M_1)$ and $\phi(F) \cap K \cap M_1 = \phi(F) \cap K \ll K$. As $M_1 \oplus M_2$ is a duo module, $K \trianglelefteq M_1 \oplus M_2$ and so $K \cap M_1$ is a direct summand of K. Hence $\phi(F) \cap K \cap M_1 \ll (K \cap M_1)$. Therefore N is M_1-$\mathcal{I}_F$-supplemented. □

Recall from [11] that two submodules N and K of a module M are mutual supplements in M, provided that $N + K = M$, $N \cap K \ll N$ and $N \cap K \ll K$.

Corollary 3.6. *Suppose that $M = M_1 \oplus M_2$, F is a fully invariant submodule of M and M is M_i-$\mathcal{I}_F$-supplemented module for $i = 1, 2$. Then:*

(1) *M is $\mathcal{I}_F$-supplemented and for every $f \in S$, $f(F)$ has a supplement of the form $K_1 + K_2$ with $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$.*

(2) *Let $f \in S$ and $f(F)$ be a supplement submodule of M. Then K_1 and $f(F) + K_2$ are mutual supplements in M and the same is true for K_2 and $f(F) + K_1$.*

Proof. (1) It follows from the proof of Theorem 3.5.

(2) Let $f \in S$ and $f(F)$ be a supplement submodule of M. Consider the proof of Theorem 3.5. Then, by [2, 20.2(c)], since $f(F) \cap (K_1 + K_2) \ll M$ we have $f(F) \cap (K_1 + K_2) \ll f(F)$. Hence

$$K_1 \cap (f(F) + K_2) \subseteq [f(F) \cap (K_1 + K_2)] + [K_2 \cap (f(F) + K_1)] \ll f(F) + K_2.$$

Similarly, $K_2 \cap (f(F) + K_1) \ll f(F) + K_1$. □

A module K is said to be *generalized M-projective* if, for any epimorphism $g : M \to X$ and morphism $f : K \to X$, there exist two decompositions $K = K_1 \oplus K_2$, $M = M_1 \oplus M_2$, a morphism $h_1 : K_1 \to M_1$ and an epimorphism $h_2 : M_2 \to K_2$, such that $h_1 g = f|_{K_1}$ and $h_2 f = g|_{M_2}$ ([2]).

Lemma 3.7. *Assume that M is a module, F a fully invariant submodule of M, $N \leq^{\oplus} M$ and $N = K \oplus L$. Let M be an L-$\mathcal{I}_F$-lifting module. If K is generalized L-projective, then for every homomorphism $f : M \to N$ such that $N = f(F)+L$ and $(\pi f)(F) = f(F) \cap \Im\pi$, where π is an arbitrary projection map of N, there exist $X \overset{cs}{\hookrightarrow} f(F)$ in N, $K' \subseteq K$ and $L' \subseteq L$ such that $N = X \oplus K' \oplus L'$.*

Proof. Let $f : M \to N$ be a homomorphism such that $N = f(F)+L$. Consider the homomorphism $\pi_L f : M \to L$, where π_L is the projection map from N onto L. Note that $\pi_L f(F) = f(F) \cap L$ by hypothesis. Since M is L-$\mathcal{I}_F$-lifting, there exists a decomposition $L = L_1 \oplus L_2$ such that $L_2 \subseteq L \cap f(F)$ and $(L \cap f(F)) \cap L_1 = L_1 \cap f(F) \ll L_1$. Thus we get $N = f(F) + L = f(F) + L_1$ and $L_1 \cap f(F) \ll L_1$. As $L_2 \subseteq f(F)$, $f(F) = L_2 \oplus ((L_1 \oplus K) \cap f(F))$. Set $U = L_1 \oplus K$. Since $N = f(F)+L_1$, $U = (U \cap f(F))+L_1$. By [2, 4.43 and 4.42], there exists a decomposition $U = T \oplus K' \oplus L_1' = T + L_1$ with $T \subseteq U \cap f(F)$, $K' \subseteq K$ and $L_1' \subseteq L_1$. As $T \subseteq U \cap f(F)$ and $f(F) = L_2 + (U \cap f(F))$, we have $L_2 \oplus T \subseteq f(F)$. Since $N = (L_2 + T) + L_1$ and $f(F) \cap L_1 \ll N$, we have, by [2, 3.2(6)], that $(L_2 \oplus T) \overset{cs}{\hookrightarrow} f(F)$ in N. As $N = L_2 \oplus U$, the proof is completed. □

Theorem 3.8. *Suppose that $M = M_1 \oplus M_2$, F is a fully invariant submodule of M and M is M_i-$\mathcal{I}_F$-lifting for $i = 1, 2$. Let M_1 and M_2 be relatively generalized projective modules. Then for every $f \in S$, $f(F)$ is a direct summand of M if $f(F)$ is a coclosed submodule of M and $(\pi f)(F) = f(F) \cap \Im\pi$, where π is any projection map of M.*

Proof. Let $f \in S$ satisfy that $f(F)$ is a coclosed submodule of M. Since M is M_i-$\mathcal{I}_F$-supplemented, M is an $\mathcal{I}_F$-supplemented module and $f(F)$ has a supplement $M_1' \oplus M_2'$, where $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$ (see Corollary 3.6). As M is $\mathcal{I}_F$-supplemented, $f(F)$ is a supplement submodule and $f(F) + M_1'$ and $f(F) + M_2'$ are mutual supplement submodules of M (see Corollary 3.6 again). Since M is M_2-$\mathcal{I}_F$-lifting, M_1 is generalized M_2-projective and also $f(F) + M_1'$ is a supplement submodule, it follows that there exists a decomposition $M = (f(F) + M_1') \oplus M_1'' \oplus M_2''$, with $M_1'' \subseteq M_1$ and $M_2'' \subseteq M_2$ (see Lemma 3.7). Set $U = M_1 + f(F)$ and $N = M_1'' \oplus M_2''$. Then $M = U \oplus N$ and $M/f(F) = U/f(F) \oplus (N+f(F))/f(F)$. Hence $(N+f(F))/f(F)$ is a supplement submodule in $M/f(F)$. By [2, 20.5(2)], $N+f(F)$ is a supplement of M because $f(F)$ is a supplement submodule in M. Since we have $N + f(F) = f(F) \oplus M_1'' \oplus M_2''$, by [2, 20.5(1)], $f(F) \oplus M_2''$ is a supplement submodule in M and $M = f(F) + M_2'' + M_1' + M_1'' \subseteq (f(F) \oplus M_2'') + M_1$. Applying Lemma 3.7 again, we have $M = (f(F) \oplus M_2'') \oplus M_1^* \oplus M_2^*$, with $M_1^* \subseteq M_1$ and $M_2^* \subseteq M_2$. Hence $M = f(F) \oplus M_1^* \oplus (M_2^* \oplus M_2'')$, where $M_1^* \subseteq M_1$ and $M_2^* \oplus M_2'' \subseteq M_2$. Therefore $f(F) \leq^{\oplus} M$. □

Theorem 3.9. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a module, $F \trianglelefteq M$ and $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$. Then M is an $\mathcal{I}_F$-lifting module if and only if M_i is $\mathcal{I}_{F \cap M_i}$-lifting for all $i \in \{1, \ldots, n\}$.*

Proof. The necessity follows from Theorem 3.2. Conversely, let M_i be an $\mathcal{I}_{F\cap M_i}$-lifting module for all $i \in \{1, \ldots, n\}$. Since $F \trianglelefteq M$, $F = \bigoplus_{i=1}^{n}(F \cap M_i)$. Let $\phi = (\phi_{ij})_{i,j\in\{1,\ldots,n\}} \in End_R(M)$ be arbitrary, where $\phi_{ij} \in Hom(M_j, M_i)$. Since M_i is a fully invariant submodule of M for all $i \in \{1, \ldots, n\}$ and also $F = \bigoplus_{i=1}^{n}(F \cap M_i)$, we have $\phi(F) = \bigoplus_{i=1}^{n} \phi_{ii}(F \cap M_i)$. As M_i is $\mathcal{I}_{F\cap M_i}$-lifting, there exist a direct summand X_i of M_i and a submodule Y_i of M_i with $X_i \subseteq \phi_{ii}(F \cap M_i)$, $\phi_{ii}(F \cap M_i) = X_i + Y_i$ and $Y_i \ll M_i$. Set X to be $\bigoplus_{i=1}^{n} X_i$, then X is a direct summand of M. Moreover, we should note that $\phi(F) = \bigoplus_{i=1}^{n} \phi_{ii}(F\cap M_i) = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i$ and $\bigoplus_{i=1}^{n} Y_i \ll \bigoplus_{i=1}^{n} M_i = M$. Therefore M is $\mathcal{I}_F$-lifting. □

References

[1] Amouzegar, T. A generalization of lifting modules. *Ukrainian Math. J. 66*, 11 (2014), 1654–1664.

[2] Clark, J., Lomp, C., Vanaja, N., and Wisbauer, R. *Lifting Modules. Supplements and projectivity in module theory.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[3] Keskin, D. Finite direct sums of $(D1)$-modules. *Turk. J. Math. 22*, 1 (1998), 85–92.

[4] Keskin, D., and Xue, W. Generalizations of lifting modules. *Acta Math. Hungar. 91*, 3 (2001), 253–261.

[5] Keskín Tütüncü, D., and Orhan, N. CCSR-modules and weak lifting modules. *East-West J. Math. 5*, 1 (2003), 89–96.

[6] Kosan, M. T. δ-lifting and δ-supplemented modules. *Algebra Colloq. 14*, 1 (2007), 53–60.

[7] Lee, G., Rizvi, S. T., and Roman, C. Rickart modules. *Comm. Algebra 38*, 11 (2010), 4005–4027.

[8] Lee, G., Rizvi, S. T., and Roman, C. Dual Rickart modules. *Comm. Algebra 39*, 11 (2011), 4036–4058.

[9] Talebi, Y., and Amoozegar, T. Pjective modules and a generalization of lifting modules. *East-West J. Math. 12*, 1 (2010), 9–15.

[10] Talebi, Y., and Vanaja, N. The torsion theory cogenerated by M-small modules. *Comm. Algebra 30*, 3 (2002), 1449–1460.

[11] Wisbauer, R. *Foundations of module and ring theory*, german ed., vol. 3 of *Algebra, Logic and Applications.* Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

Received by the editors April 23, 2019
First published online March 1, 2020