Novi Sad J. Math. Vol. 50, No. 2, 2020, 35-50
https://doi.org/10.30755/NSJOM.09381

Cyclic Picard operator and simulation type functions[1]

Sumit Chandok[2]

Abstract. In this manuscrpt, we introduce generalized $(\alpha, \beta, \mathcal{Z}_\mathcal{G})-$ contraction using the concept of cyclic (α, β)-admissible mapping and prove the existence of a Picard operator for such class in the structure of metric spaces. Also we provide an example for the illustration of the same.

AMS Mathematics Subject Classification (2010): 47H10; 54H25; 46J10; 46J15

Key words and phrases: Picard operator, fixed point, simulation function, generalized $(\alpha, \beta, \mathcal{Z}_\mathcal{G})-$ contraction.

1. Introduction & Preliminaries

Let M be a nonempty set and $f : M \to M$. A sequence $\{u_n\}$ defined by $u_n = f^n u_0$ is called a *Picard sequence* based at the point $u_0 \in M$. An operator f is said to be a *Picard operator* if it has a unique fixed point $z \in M$ and $z = \lim_{n\to\infty} f^n u$ for all $u \in M$. An operator f is said to be a *weakly Picard operator* if it has a fixed point $z \in M$ and $z = \lim_{n\to\infty} f^n u$ for all $u \in M$. Various classes of Picard operators exist in the literature (see, for example, [4, 3, 5, 9, 14, 13]). Using the concept of cyclic (α, β)-admissible mapping, we introduce generalized $(\alpha, \beta, \mathcal{Z}_\mathcal{G})-$ contraction and prove the existence of a Picard operator for such class in the structure of metric spaces. Also we give an example for the illustration of the same.

A mapping $f : M \to M$ is continuous if and only if it is sequentially continuous, i.e., $\lim_{n\to\infty} d(fx_n, fx) = 0$ for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.

Now, we define a C-class function (see also [7, 10]) as

Definition 1.1. A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and $G(s, t) \le s$ for all $s, t \ge 0$.

Definition 1.2. A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ has the property C_G if there exists an $C_G \ge 0$ such that

(C_G1) $G(s, t) > C_G$ implies $s > t$;

(C_G2) $G(t, t) \le C_G$, for all $t \in [0, +\infty)$.

[1]This work has been supported by the AISTDF/DST research grant CRD/2018/000017.

[2]School of Mathematics, Thapar Institute of Engineering & Technology, Patiala-147004, Punjab, India, e-mail: sumit.chandok@thapar.edu

Some examples of C-class functions that have property C_G are as follows:

a) $G(s,t) = s - t$, $C_G = r$, $r \in [0, +\infty)$;

b) $G(s,t) = s - \frac{(2+t)t}{1+t}$, $C_G = 0$;

c) $G(s,t) = \frac{s}{1+kt}$, $k \geq 1$, $C_G = \frac{r}{1+k}$, $r \geq 2$.

For more examples of C-class functions that have property C_G see [2, 7].

Khojasteh et al. ([6]) (see also [12, 8]) introduced the concept of a simulation function.

Definition 1.3. (see [7]) We define $\mathcal{Z}_G$ to be the family of all C_G-simulation functions $\zeta : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following:

($\mathcal{Z}_G 1$) $\zeta(t,s) < G(s,t)$ for all $t, s > 0$, where $G : [0, +\infty)^2 \to \mathbb{R}$ is a C-class function;

($\mathcal{Z}_G 2$) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G$.

Some examples of simulation functions and C_G-simulation functions are:

d) $\zeta(t,s) = \frac{s}{s+1} - t$ for all $t, s \geq 0$.

e) $\zeta(t,s) = s - \varphi(s) - t$ for all $t, s \geq 0$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi continuous function and $\varphi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and C_G-simulation functions see [2, 12, 6, 7, 8, 15].

Each simulation function as in paper [6] is also a C_G-simulation function as in Definition 1.3, but the converse is not true. For this claim see Example 3.3 of [12] using the C-class function $G(s,t) = s - t$.

Alizadeh et al. [1] introduced the notion of a cyclic (α, β)-admissible mapping which is defined as follows:

Definition 1.4. Let M be a nonempty set, f be a self-mapping on M and $\alpha, \beta : M \to [0, \infty)$ be two mappings. We say that f is a cyclic (α, β)-admissible mapping if $x \in M$ with $\alpha(x) \geq 1$ implies $\beta(fx) \geq 1$ and $\beta(x) \geq 1$ implies $\alpha(fx) \geq 1$.

The following result will be required in the sequel.

Lemma 1.5. *(see [11, 10]) Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.1}$$

If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ when $k \to +\infty$:

$$d\left(x_{m(k)}, x_{n(k)}\right),\ d\left(x_{m(k)}, x_{n(k)+1}\right),\ d\left(x_{m(k)-1}, x_{n(k)}\right),$$
$$d\left(x_{m(k)-1}, x_{n(k)+1}\right),\ d\left(x_{m(k)+1}, x_{n(k)+1}\right). \tag{1.2}$$

2. Main results

Definition 2.1. Let (M, d) be a complete metric space, $f : M \to M$ be a mapping and $\alpha, \beta : \mathbb{R} \to [0, \infty)$ be two functions. Then f is said to be a generalized $(\alpha, \beta, \mathcal{Z}_{\mathcal{G}})-$ contraction mapping if f satisfies the following conditions:

(1) f is cyclic (α, β)-admissible;

(2) there exits a $\zeta \in \mathcal{Z}_{\mathcal{G}}$ such that for all $u, v \in M$, we have

$$\alpha(u)\beta(v) \geq 1, d(fu, fv) > 0 \Rightarrow \zeta(d(fu, fv), d(u, v)) \geq C_G. \tag{2.1}$$

Lemma 2.2. *Let M be a nonempty set and $f : M \to M$ be a cyclic $(\alpha, \beta)-$admissible mapping. Assume that there exists an element $x_0 \in M$ such that $\alpha(x_0) \geq 1 \implies \beta(x_1) \geq 1$ and $\beta(x_0) \geq 1 \implies \alpha(x_1) \geq 1$. Define a Picard sequence $\{x_n\} \subseteq M$ by $x_{n+1} = f^n x_0 = f x_n$. Then $\alpha(x_n) \geq 1 \implies \beta(x_m) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Assume that there exist $x_0 \in M$ such that $\alpha(x_0) \geq 1$. Define a Picard sequence $\{x_n\}$ by $x_{n+1} = f x_n = f^n x_0$, for all $n \in \mathbb{N} \cup \{0\}$.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that there exist $x_0, x_1 \in M$ such that $\alpha(x_0) \geq 1 \implies \beta(f x_0) = \beta(x_1) \geq 1$ and $\beta(x_0) \geq 1 \implies \alpha(f x_0) = \alpha(x_1) \geq 1$. By continuing the above process, we have $\alpha(x_n) \geq 1 \implies \beta(f x_n) = \beta(x_{n+1}) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(f x_n) = \alpha(x_{n+1}) \geq 1$.

Since $\alpha(x_m) \geq 1 \implies \beta(f x_m) = \beta(x_{m+1}) \geq 1$ and $\beta(x_m) \geq 1 \implies \alpha(f x_m) = \alpha(x_{m+1}) \geq 1$, for all $m, n \in \mathbb{N}$ with $n < m$. Moreover, since $\alpha(x_m) \geq 1 \implies \beta(x_{m+2}) \geq 1$ and $\beta(x_m) \geq 1 \implies \alpha(x_{m+2}) \geq 1$, for all $m, n \in \mathbb{N}$ with $n < m$.

By continuing this process, we have $\alpha(x_n) \geq 1 \implies \beta(x_m) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(x_m) \geq 1$, for all $m, n \in \mathbb{N}$. Hence the result. □

Lemma 2.3. *Let (M, d) be a metric space, $f : M \to M$ be a self-mapping and f be a generalized $(\alpha, \beta, \mathcal{Z}_{\mathcal{G}})-$ contraction. Suppose that there exists a Picard sequence $\{x_n\} \subseteq M$ defined by $x_{n+1} = f^n x_0 = f x_n$ such that $x_n \neq x_{n+1}$. Then the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.*

Proof. Suppose that there is a Picard sequence $\{x_n\}$ such that $x_{n+1} = f^n x_0 = f x_n$, where $n \in \mathbb{N} \cup \{0\}$. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 2.2, we have $\alpha(x_n) \geq 1 \implies \beta(x_m) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(x_m) \geq 1$, for all $m, n \in \mathbb{N}$. Thus $\alpha(x_n)\beta(x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Substituting $u = x_n$, $v = x_{n+1}$ in (2.1) we obtain that

$$\begin{aligned} C_G &\leq \zeta\left(d\left(f x_n, f x_{n+1}\right), d\left(x_n, x_{n+1}\right)\right) = \zeta\left(d\left(x_{n+1}, x_{n+2}\right), d\left(x_n, x_{n+1}\right)\right) \\ &< G\left(d\left(x_n, x_{n+1}\right), d\left(x_{n+1}, x_{n+2}\right)\right). \end{aligned}$$

Using $(C_G 1)$ of Definition 1.2, we have $d\left(x_n, x_{n+1}\right) > d\left(x_{n+1}, x_{n+2}\right)$. Hence, for all $n \in \mathbb{N} \cup \{0\}$ we get that $d\left(x_{n+1}, x_{n+2}\right) < d\left(x_n, x_{n+1}\right)$.

Further we have to prove that $x_n \neq x_m$ for $n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n > m$. Then we choose $x_{n+1} = x_{m+1}$ (which is obviously possible by the definition of the Picard sequence $\{x_n\}$). Then following the previous arguments, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) < \cdots < d(x_m, x_{m+1}) = d(x_n, x_{n+1}),$$

which is a contradiction. Hence $x_n \neq x_m$.

Therefore there exists $t \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = t \geq 0$. Suppose that $t > 0$. Since $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ and both $d(x_{n+1}, x_{n+2})$ and $d(x_n, x_{n+1})$ tend to t, using $(\mathcal{Z}_G2)$ of Definition 1.3, we get

$$C_G \leq \limsup_{n\to\infty} \zeta(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < C_G,$$

which is a contradiction. Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = t = 0$. □

Lemma 2.4. *Let (M, d) be a metric space, $f : M \to M$ be a self-mapping and f be a generalized $(\alpha, \beta, \mathcal{Z}_\mathcal{G})-$ contraction. Suppose that there exists a Picard sequence $\{x_n\} \subseteq M$ defined by $x_{n+1} = f^n x_0 = f x_n$ such that $x_n \neq x_{n+1}$. Then the Picard sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Suppose that there is a Picard sequence $\{x_n\}$ such that $x_{n+1} = f^n x_0 = f x_n$ where $n \in \mathbb{N} \cup \{0\}$. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemmas 2.2 and 2.3, we have that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Now, we have to show that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that it is not. Putting $x = x_{m(k)}$, $y = x_{n(k)}$ in (2.1), we obtain

$$\begin{aligned} C_G &\leq \zeta\left(d\left(f x_{m(k)}, f x_{n(k)}\right), d\left(x_{m(k)}, x_{n(k)}\right)\right) \\ &< G\left(d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right). \end{aligned} \tag{2.2}$$

Using (C_G1) of Definition 1.2, it follows that

$$d\left(x_{m(k)}, x_{n(k)}\right) > d\left(x_{m(k)+1}, x_{n(k)+1}\right).$$

Now, since the sequence $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.5, we have $d\left(x_{m(k)}, x_{n(k)}\right)$, $d\left(x_{m(k)+1}, x_{n(k)+1}\right)$, $d\left(x_{m(k)}, x_{n(k)+1}\right)$ and $d\left(x_{n(k)}, x_{m(k)+1}\right)$ tend to $\varepsilon > 0$, as $k \to \infty$. Therefore, using (2.1), we have

$$C_G \leq \limsup_{n\to\infty} \zeta\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) < C_G,$$

which is a contradiction. Therefore, the Picard sequence $\{x_n\}$ is a Cauchy sequence.

□

Theorem 2.5. *Let (M, d) be a complete metric space, $f : M \to M$ be a mapping and $\alpha, \beta : M \to [0, 1)$ be two functions. Suppose that the following conditions hold.*

(1) *f is a generalized $(\alpha, \beta, \mathcal{Z}_\mathcal{G})-$ contraction mapping;*

(2) *There exists an element* $x_0 \in M$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$;

(3) f *is sequentially continuous;*

or

If the sequence $\{x_n\}$ *in* M *converges to* $x \in M$ *with the property* $\alpha(x_n) \geq 1$ (*or* $\beta(x_n) \geq 1$) *for all* $n \in \mathbb{N}$, *then* $\alpha(x) \geq 1$ (*or* $\beta(x) \geq 1$).

Then f *is a weakly Picard operator.*

Proof. Assume that there exist $x_0 \in M$ such that $\alpha(x_0) \geq 1$. Define a Picard sequence $\{x_n\}$ by $x_{n+1} = fx_n = f^n x_0$, for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $u_{n_0} = fu_{n_0}$, then we are done. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that there exist $x_0, x_1 \in M$ such that $\alpha(x_0) \geq 1 \implies \beta(fx_0) = \beta(x_1) \geq 1$ and $\beta(x_0) \geq 1 \implies \alpha(fx_0) = \alpha(x_1) \geq 1$. Using Lemma 2.2, we have $\alpha(x_n) \geq 1 \implies \beta(x_m) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(x_m) \geq 1$, for all $m, n \in \mathbb{N}$. Thus $\alpha(x_n)\beta(x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$.

Using Lemma 2.3, we have that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Using Lemma 1.5, we obtain that the Picard sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a Cauchy sequence.

Now as (M, d) is a complete metric space, there exists $x \in M$ such that $\{x_n\}$ converges to x.

The continuity of f and uniqueness of the limit implies $fx = x$, thus we get a fixed point.

Now, suppose that the sequence $\{x_n\}$ in X converges to $x \in X$ with the property $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in \mathbb{N}$, then $\alpha(x) \geq 1$ (or $\beta(x) \geq 1$). Hence $\alpha(x)\beta(x) \geq 1$

Further, we claim that $fx = x$. Suppose not, that is, $fx \neq x$. So $d(fx, x) > 0$ and $d(x, fx) = \lim_{n\to\infty} d(x_{n+1}, fx) = \lim_{n\to\infty} d(fx_n, fx) \neq 0$. Using (2.1) we have

$$\begin{aligned} C_G &\leq \zeta\left(d\left(fx_n, fx\right), d\left(x_n, x\right)\right) \\ &< G\left(d\left(x_n, x\right), d\left(fx_n, fx\right)\right). \end{aligned} \tag{2.3}$$

Taking $n \to \infty$ and using property $(C_G 1)$ of Definition 1.2, we have $d(x, fx) \leq 0$, which is a contradiction. We, thus, obtain that f has a fixed point $fx = x$. Hence f is a weakly Picard operator. □

Here, we have an example that if f satisfies all the hypotheses of Theorem 2.5, then the fixed point of f may not necessarily be unique.

Example. Let $X = [0, 1]$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in [0, +\infty)$, and consider the mapping $f : X \to X$ given, for all $x \in X$, by $fx = x^2$. Define $\alpha, \beta : X \to \mathbb{R}$ as

$$\alpha(x) = \beta(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases}$$

However, putting $\zeta(t,s) = \frac{s}{s+1} - t$, $G(s,t) = s - t$, $C_G = 0$, we have that f is a generalized $(\alpha, \beta, \mathcal{Z}_{\mathcal{G}})-$ contraction with respect to ζ. Hence using Theorem 2.5, we have 0 and 1 are fixed points of f. Hence f is a weakly Picard operator.

References

[1] Alizadeh, S., Moradlou, F., and Salimi, P. Some fixed point results for (α,β)-(ψ,ϕ)-contractive mappings. *Filomat 28*, 3 (2014), 635–647.

[2] Ansari, A. H., Işik, H., and Radenović, S. Coupled fixed point theorems for contractive mappings involving new function classes and applications. *Filomat 31*, 7 (2017), 1893–1907.

[3] Berinde, V. On the approximation of fixed points of weak contractive mappings. *Carpathian J. Math. 19*, 1 (2003), 7–22.

[4] Berinde, V. Approximating fixed points of weak contractions using the picard iteration. *Nonlinear Anal. Forum. 9*, 1 (2004), 43–53.

[5] Berinde, V. *Iterative Approximation of Fixed Points.* Springer-Verlag, Berlin Heidelberg, 2007.

[6] Khojasteh, F., Shukla, S., and Radenović, S. A new approach to the study of fixed point theory for simulation functions. *Filomat 29*, 6 (2015), 1189–1194.

[7] Liu, X.-L., Ansari, A. H., Chandok, S., and Radenović, S. On some results in metric spaces using auxiliary simulation functions via new functions. *J. Comput. Anal. Appl. 24*, 6 (2018), 1103–1114.

[8] Nastasi, A., and Vetro, P. Fixed point results on metric and partial metric spaces via simulation functions. *J. Nonlinear Sci. Appl. 8*, 6 (2015), 1059–1069.

[9] Olgun, M., Biçer, O., and Alyi ldiz, T. A new aspect to Picard operators with simulation functions. *Turkish J. Math. 40*, 4 (2016), 832–837.

[10] Radenović, S., and Chandok, S. Simulation type functions and coincidence points. *Filomat 32*, 1 (2018), 141–147.

[11] Radenović, S., Kadelburg, Z., Jandrlić, D., and Jandrlić, A. Some results on weakly contractive maps. *Bull. Iranian Math. Soc. 38*, 3 (2012), 625–645.

[12] Roldán-López-de Hierro, A.-F., Karapi nar, E., Roldán-López-de Hierro, C., and Martínez-Moreno, J. Coincidence point theorems on metric spaces *via* simulation functions. *J. Comput. Appl. Math. 275* (2015), 345–355.

[13] Rus, I. A. Weakly picard mappings. *Comment. Math. Univ. Caroline. 34*, 4 (1993), 769–773.

[14] Rus, I. A. *Picard operator and applications.* Babes-Bolyai Univ., Cluj-Napoca, 1996.

[15] Wang, S., Ansari, A. H., and Chandok, S. Some fixed point results for non-decreasing and mixed monotone mappings with auxiliary functions. *Fixed Point Theory Appl.* (2015), 2015:209, 16.

Received by the editors April 15, 2019
First published online June 6, 2020