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The Cusa-Huygens inequality revisited

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Abstract. Let $c,\ \gamma \in \mathbb{R}$, $\gamma \geq 1$, $c \geq 1$ and $T \in (0, \pi/\gamma]$ if $c = 1$, resp. $T \in (0, \pi/2\gamma]$ if $c > 1$. In this paper, we find the necessary and sufficient conditions on $a,\ b \in \mathbb{R}$ such that the inequalities

$$\frac{\sin x}{x} > a + b\cos^c(\gamma x), \quad x \in (0, T)$$

and

$$\frac{\sin x}{x} < a + b\cos^c(\gamma x), \quad x \in (0, T)$$

hold true. We also determine the best possible constants p and q such that

$$\frac{2 + \cos(px)}{3} < \frac{\sin x}{x} < \frac{2 + \cos(qx)}{3}, \quad x \in (0, \pi/2).$$

The proofs of main results contain several auxiliary results which can be of some independent interest.

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1. Introduction and preliminaries

The famous Cusa-Huygens inequality

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad x \in (0, \pi/2)$$

has been reconsidered numerous times so far (see, e.g., [2, 3, 4, 6, 8, 9, 11, 14, 15]).

The main aim of this paper is to consider the following problem. Let $a,\ b,\ c,\ \gamma \in \mathbb{R}$, $\gamma \geq 1$, $c \geq 1$ and $T \in (0, \pi/\gamma]$ if $c = 1$, resp. $T \in (0, \pi/2\gamma]$ if $c > 1$; find the necessary and sufficient conditions such that the inequalities

$$\frac{\sin x}{x} > a + b\cos^c(\gamma x), \quad x \in (0, T) \tag{1.1}$$

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and

$$\frac{\sin x}{x} < a + b\cos^c(\gamma x), \quad x \in (0, T) \tag{1.2}$$

hold true. In this way, we continue our recent research study [5], where the case $\gamma = 1$ has been analyzed.

In order to establish our main theoretical results, we need to remind ourselves of the statement which is known in the existing literature as l'Hospital's rule of monotonicity; see, e.g., [1]:

Lemma 1.1. *Let $f(x)$ and $g(x)$ be two real valued functions which are continuous on $[a, b]$ and differentiable on (a, b), where $-\infty < a < b < \infty$ and $g'(x) \neq 0$, for all $x \in (a, b)$. Let*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)}, \ x \in (a, b)$$

and

$$B(x) = \frac{f(x) - f(b)}{g(x) - g(b)}, \ x \in (a, b).$$

Then,

(i) $A(x)$ and $B(x)$ are increasing on (a, b) if $f'(x)/g'(x)$ is increasing on (a, b).

(ii) $A(x)$ and $B(x)$ are decreasing on (a, b) if $f'(x)/g'(x)$ is decreasing on (a, b).

The strictness of the monotonicity of $A(x)$ and $B(x)$ depends on the strictness of monotonicity of $f'(x)/g'(x)$.

Furthermore, we will use the following series expansions:

$$\cot x = \frac{1}{x} - \sum_{n=1}^{+\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \quad x \in (-\pi, \pi), \tag{1.3}$$

where $B_{2n} = 2(-1)^{n+1}(2n)!\zeta(2n)/(2\pi)^{2n}$ denotes the $2n$-th Bernoulli number, with $\zeta(\cdot)$ being the Riemann zeta function, and

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{+\infty} \frac{2^{2n} - 2}{(2n)!}|B_{2n}|x^{2n}, \quad x \in (-\pi, \pi). \tag{1.4}$$

They can be found in [12] and [13], respectively. From (1.3) and (1.4), the following two series expansions can be deduced:

$$x \cot x = 1 - \sum_{n=1}^{+\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n}, \quad x \in (-\pi, \pi), \tag{1.5}$$

and

(1.6)
$$\left(\frac{x}{\sin x}\right)^2 = x^2\left(-\cot x\right)' = 1 + \sum_{n=1}^{+\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)x^{2n}, \quad x \in (-\pi, \pi).$$

Finally, we will also use the following well-known result:

Lemma 1.2. *([7, 10]) Suppose that the two power series $A(x) = \sum_{n=1}^{+\infty} a_n x^n$ and $B(x) = \sum_{n=1}^{+\infty} b_n x^n$ are convergent on the interval $(-R, R)$, with $R \in (0, +\infty]$. If the sequence (a_n/b_n) is increasing (decreasing) and $b_n > 0$ for all $n \in \mathbb{N}$, then the function $A(x)/B(x)$ is also increasing (decreasing) on $(0, R)$.*

2. Formulation and proof of main results

We start this section by stating the following useful result:

Proposition 2.1. *Let $\gamma \geq 1$. Then the function*
$$F_\gamma(x) := \frac{\sin x - x\cos x}{x^2 \sin \gamma x}, \quad x \in (0, \pi/\gamma)$$
is positive and strictly increasing.

Proof. Let $\gamma > 1$. Then, it is clear that we have
$$F_\gamma(x) = \frac{\sin x - x\cos x}{x^2 \sin x}\frac{\sin x}{\sin \gamma x}, \quad x \in (0, \pi/\gamma).$$

The function
$$F(x) := \frac{\sin x - x\cos x}{x^2 \sin x}, \quad x \in (0, \pi)$$
is positive. Strictly speaking, this holds for $x = \pi/2$, while for other values of the parameter $x \in (0, \pi)$ it follows from the facts that the mapping $t \mapsto \tan t - t$, $t \in (0, \pi/2)$ ($t \in (\pi/2, \pi)$) is strictly increasing, $\cos x > 0$ if $x \in (0, \pi/2)$, $\lim_{t \to \pi/2+}(\tan t - t) = -\infty$ and $(\tan t - t)_{t=\pi} = -\pi < 0$. On the other hand, the function
$$g(x) := \frac{\sin x}{\sin \gamma x}, \quad x \in (0, \pi/\gamma)$$
is positive and strictly increasing because its first derivative is given by (see also (1.3))
$$\begin{aligned} g'(x) &= \frac{\cos x \sin(\gamma x) - \gamma \sin x \cos(\gamma x)}{\sin^2(\gamma x)} \\ &= \sin x \frac{\cot x - \gamma \cot(\gamma x)}{\sin(\gamma x)} \\ &= \sin x \frac{\sum_{n=1}^{+\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}(\gamma^{2n} - 1)}{\sin(\gamma x)}, \quad x \in (0, \pi/\gamma), \end{aligned}$$

which simply implies that $g'(x) > 0$, $x \in (0, \pi/\gamma)$. Since

$$F_\gamma(x) = \frac{\sin x - x\cos x}{x^2 \sin x} g(x), \quad x \in (0, \pi/\gamma),$$

it follows from the foregoing that it sufficies to prove of proposition for $\gamma = 1$. For $x \in (0, \zeta)$, where $2.2 \approx \zeta \in (\pi/2, \pi)$ denotes the unique solution of the equation $x \cot x = -2$, this follows from our consideration given in [5, Remark 1(i)]. Otherwise, for $x \in (\zeta, \pi)$, we have to prove that the first derivative of the function $x \mapsto x^{-2}(\sin x - x\cos x)\sin^{-1} x$, $x \in [\zeta, \pi)$ is positive, which follows from direct calculus and quite elemetary inequalities taking into account the concrete value of number ζ. ☐

Since

$$F'_\gamma(x) = \frac{x^3 \sin x \sin\gamma x - \left[2x\sin\gamma x + \gamma x^2 \cos\gamma x\right]\left[\sin x - x\cos x\right]}{x^4 \sin^2 \gamma x}, \; x \in (0, \pi/\gamma),$$

Proposition 2.1 yields the following

Corollary 2.2. *Let $\gamma \geq 1$. Then, for every $x \in (0, \pi/\gamma)$, we have*

(2.1)
$$\left(x^2 - 2\right)\sin x \sin\gamma x + \gamma x^2 \cos x \cos\gamma x + 2x\cos x \sin\gamma x \geq \gamma x \cos\gamma x \sin x.$$

The question whether there exists a number $x \in (0, \pi/\gamma)$ such that we have the equality in (2.1) is interesting but will not be considered here. Numerical calculations at symbolab.com show that a quite different result holds if $\gamma \in (0, 1)$, which is a much more complicated case to analyse. Since

$$\lim_{x \to 0+} F_\gamma(x) = \frac{1}{3\gamma} \quad \text{and} \quad \lim_{x \to T-} F_\gamma(x) = +\infty,$$

a similar line of reasoning as in the proofs of [5, Theorem 1-Theorem 2] shows that the following holds true:

Theorem 2.3. *Let $a,\ b \in \mathbb{R}$, $\gamma > 1$ and $T \in (0, \pi/\gamma]$. Then, the inequality*

$$\frac{\sin x}{x} > a + b\cos(\gamma x), \quad x \in (0, T)$$

holds iff

1. *$b\gamma \leq 1/3$ and $a \leq F(T)$, or*
2. *$b\gamma \geq F_\gamma(T)$, $a \leq 1 - b$ and $T < \pi/\gamma$, or*
3. *$b\gamma \in (1/3, F_\gamma(T))$, $a \leq \min(1 - b, F(T))$ and $T < \pi/\gamma$, or $b\gamma > 1/3$, $a \leq \min(1 - b, F(T))$ and $T = \pi/\gamma$.*

Theorem 2.4. *Let* $a,\ b \in \mathbb{R}$, $\gamma > 1$ *and* $T \in (0, \pi/\gamma]$. *Then the inequality*

$$\frac{\sin x}{x} < a + b\cos(\gamma x), \quad x \in (0, T)$$

holds iff

1. $b\gamma \le 1/3$ *and* $a \ge 1 - b$, *or*
2. $b\gamma \ge F_\gamma(T)$, $a \ge F(T)$ *and* $T < \pi/\gamma$, *or*
3. $b\gamma > 1/3$, $T = \pi/\gamma$ *and* $a > F(\zeta_{b,\gamma})$, *where* $\zeta_{b,\gamma}$ *denotes the unique solution of the equation* $F_\gamma(x) = b\gamma$ *on the interval* $(0, T)$.

Corollary 2.5. *For* $x \in (0, \pi/2)$, *the smallest positive constant* p *and the greatest positive constant* q *such that*

$$\frac{2 + \cos(px)}{3} < \frac{\sin x}{x} < \frac{2 + \cos(qx)}{3}, \quad x \in (0, \pi/2)$$

are $(2/\pi)\arccos(6/\pi - 2) \approx 1.05746$ *and* 1, *respectively.*

Proof. For any $\gamma \in (0, 1]$, we have

$$\frac{2 + \cos(\gamma x)}{3} \ge \frac{2 + \cos x}{3} > \frac{\sin x}{x}, \quad x \in (0, \pi/2).$$

Further, we will prove that the inequality

$$\frac{\sin x}{x} < \frac{2 + \cos(\gamma x)}{3}, \quad x \in (0, \pi/2) \tag{2.2}$$

cannot be satisfied if $\gamma > 1$. Strictly speaking, the function $y = \gamma\sin(\pi\gamma/2) - (12/\pi^2)$, $\gamma \in (1, 2)$ is negative and has the maximal value ≈ -0.057394 at the point ≈ 1.2915. By Theorem 2.4, if $\gamma \in (1, 2]$, then the inequality (2.2) can be satisfied only if $\gamma = 2$. But, this is not the case, which can be simply inspected by considering the behaviour of both sides of this inequality around the point $x = \pi/2-$. Similarly, by considering the behaviour of both sides of this inequality around the point $x = \pi/\gamma-$, we can simply show that (2.2) does not hold for $\gamma > 2$. For remainder of proof, it suffices to show that the inequality

$$\frac{\sin x}{x} > \frac{2 + \cos(\gamma x)}{3}, \quad x \in (0, \pi/2) \tag{2.3}$$

cannot be satisfied if $1 < \gamma < (2/\pi)\arccos(6/\pi - 2)$. For this value of γ, the inequality (2.3) holds by Theorem 2.3 (the first case in part 3.). For $\gamma \in (1, (2/\pi)\arccos(6/\pi - 2))$, the conditions from part 1. and 2. of the above-mentioned theorem cannot be satisfied, as easily seen. Part 3. cannot be satisfied, likewise, because we then must have $a \le \min(1 - b, F(T))$; in our concrete situation, this reads as $\cos(\gamma\pi/2) \le 3((2/\pi) - 2/3)$, which simply implies $\gamma \ge (2/\pi)\arccos(6/\pi - 2)$. □

Corollary 2.5 can be deduced by using l'Hospital's rule of monotonicity, directly. Since the proof is complicated and contains many interesting points, we would like to present it, as well:

Proof with l'Hospital's rule of monotonicity. Let us consider the function

$$f(x) := \frac{\arccos(3\sin x/x - 2)}{x} := \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) := \arccos(3\sin x/x - 2)$ and $f_2(x) := x$ $(x \in (0, \pi/2))$. Then, we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{3(\sin x - x\cos x)}{x\sqrt{12x\sin x - 9\sin^2 x - 3x^2}}, \quad x \in (0, \pi/2).$$

Moreover, for every $\in (0, \pi/2)$, we have

$$\begin{aligned}\left[\frac{f_1'(x)}{f_2'(x)}\right]^2 &= \frac{3(\sin x - x\cos x)^2}{x^2(4x\sin x - 3\sin^2 x - x^2)} \\ &= \frac{3(\sin x - x\cos x)}{x^2 \sin x} \, \frac{\sin x(\sin x - x\cos x)}{4x\sin x - 3\sin^2 x - x^2} = f_3(x) f_4(x),\end{aligned}$$

where

$$f_3(x) := 3(\sin x - x\cos x)/(x^2 \sin x), \quad x \in (0, \pi/2)$$

and

$$f_4(x) := \sin x(\sin x - x\cos x)/(4x\sin x - 3\sin^2 x - x^2), \quad x \in (0, \pi/2).$$

By Proposition 2.1, the function $f_3(\cdot)$ is positive and strictly increasing on $(0, \pi/2)$. Now let us show that the function

$$G(x) := \frac{\sin x(\sin x - x\cos x)}{4x\sin x - 3\sin^2 x - x^2}, \quad x \in (0, \pi/2)$$

is positive and strictly increasing on $(0, \pi/2)$. The fact that $G(x)$ is positive on $(0, \pi/2)$ follows from the facts that $\sin x - x\cos x > 0$ and $3x\sin x > x^2 + 2\sin^2 x$ for $x \in (0, \pi/2)$, which can be easily seen by using a simple inequality

$$2\sqrt{2} \le 2\frac{\sin x}{x} + \frac{x}{\sin x} < 3, \quad x \in (0, \pi/2), \tag{2.4}$$

which can be proved as follows. The function $y = \sin x/x$, $x \in (0, \pi/2)$ is strictly decreasing so that $\sin x/x \in (2/\pi, 1)$ for $x \in (0, \pi/2)$. Consider the function $q : [2/\pi, 1] \to \mathbb{R}$ given by $q(t) := 2t + t^{-1} - 3$, $t \in [2/\pi, 1]$. Since $q'(t) = t^{-2}(2t^2 - 1)$, $t \in [2/\pi, 1]$, we have that the function $q(\cdot)$ is strictly decreasing on the interval $[2/\pi, 1/\sqrt{2}]$ and the function $q(\cdot)$ is strictly increasing on the interval $[1/\sqrt{2}, 1]$, which simply implies (2.4) because $q(2/\pi) < 0$ and $q(1) = 0$. Further on, let us rewrite the function $G(x)$ as $G(x) = A(x)/B(x)$, $x \in (0, \pi/2)$, where $A(x) := 1 - x\cot x$ and $B(x) := 4x/\sin x - 3 - (x/\sin x)^2$

$(x \in (0, \pi/2))$. We now aim to apply Lemma 1.2. By using the series expansions (1.4)-(1.6), we get

$$A(x) = \sum_{n=1}^{+\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n} = \sum_{n=1}^{+\infty} a_n \left(x^2\right)^n, \quad x \in (0, \pi/2),$$

where $a_n = 2^{2n}|B_{2n}|/(2n)!$, and

$$\begin{aligned} B(x) &= 4 + \sum_{n=1}^{+\infty} \frac{4(2^{2n}-2)}{(2n)!}|B_{2n}|x^{2n} - 3 - 1 - \sum_{n=1}^{+\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|(2n-1)x^{2n} \\ &= \sum_{n=1}^{+\infty} \frac{|B_{2n}|}{(2n)!}\left[4(2^{2n}-2) - 2^{2n}(2n-1)\right]x^{2n} = \sum_{n=1}^{+\infty} b_n \left(x^2\right)^n, \quad x \in (0, \pi/2), \end{aligned}$$

where $b_n = (2^{2n}(5-2n) - 8)|B_{2n}|/(2n)!$ for any $n \in \mathbb{N}$.

Now, let us set $c_n := b_n/a_n = 5 - 2n - 2^{3-2n}$, $n \geq 1$. Then, $c_n - c_{n+1} = 2 - 3 \times 2^{1-2n} > 1/2 > 0$ for $n \geq 1$, implying that (c_n) is decreasing and, a fortiori, (a_n/b_n) is increasing. It follows from Lemma 1.2 that $G(x)$ is also increasing on $(0, \pi/2)$. Moreover, the zeros of $G'(x)$ cannot form an interval so that the function $G(\cdot)$ is, in fact, strictly increasing on $(0, \pi/2)$. Therefore, $f_1'(\cdot)/f_2'(\cdot)$ is strictly increasing and, by l'Hospital's rule of monotonicity (see Lemma 1.1), $f(\cdot)$ is strictly increasing on $(0, \pi/2)$ with $f(0+) < f(x) < f(\pi/2)$. Then we find $f(0+) = 1$ by l'Hospital's rule and $f(\pi/2) = 2\arccos(6/\pi - 2)/\pi$. We end the proof by noticing that $q = f(0+) < f(x) < f(\pi/2) = p$ is equivalent to the desired inequalities, i.e., $(2+\cos(px))/3 < \sin x/x < (2+\cos(qx))/3$. □

Remark 2.6. It is worth noting that we have not used above any inequality and estimate with Bernoulli numbers since we had used them only for getting the precise series expansion of functions $A(\cdot)$ and $B(\cdot)$; after that, the Bernoulli numbers disappered in the concrete formula for the sequence (c_n), which is crucial for applying Lemma 1.2. For more details about the Bernoulli numbers and related inequalities, the reader may consult the paper [16] by F. Qi and references cited therein.

The result in Corollary 2.5 can be visualized in Figure 1.

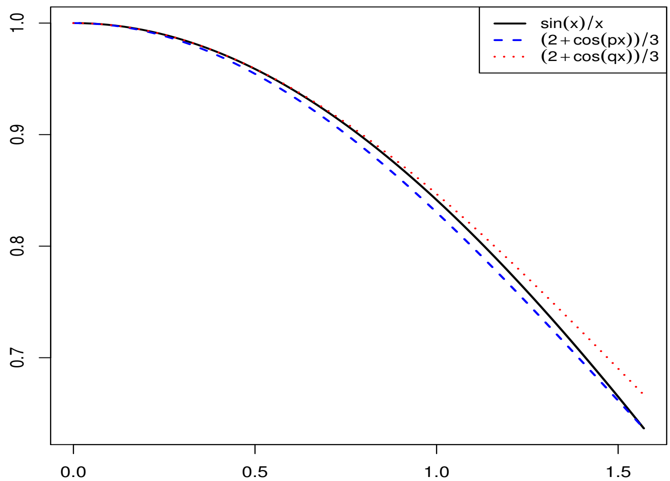

Figure 1: Illustration of the inequalities in Theorem 2.5.

Remark 2.7. Observe also that the function

$$G_0(x) = \frac{\sin x - x\cos x}{4x\sin x - 3\sin^2 x - x^2}, \quad x \in (0, \pi/2)$$

is not strictly monotone on $(0, \pi/2)$. In actual fact, we have $\lim_{x\to 0+} G_0(x) = +\infty$ as well as there exists a unique point $\zeta \approx 0.9161$ such that $G_0'(x) < 0$ for $x \in (0, \zeta)$ and $G_0'(x) > 0$ for $x \in (\zeta, \pi/2)$; this can be verified by using, e.g., a simple computation and the graphing calculator at www.symbolab.com. Hence, the mapping $G_0(x)$ is strictly decreasing on $(0, \zeta)$ and strictly increasing on $(\zeta, \pi/2)$. Similar conclusions hold for the function

$$G_\theta(x) = \frac{\sin^\theta x \cdot (\sin x - x\cos x)}{4x\sin x - 3\sin^2 x - x^2}, \quad x \in (0, \pi/2),$$

where $\theta \in (0, 1)$.

Comparing Theorem 2.3 and Theorem 2.4 to [5, Theorem 1 and Theorem 2], it is necessary to say that we have considered the case $\gamma \neq 1$ here as well as increased the range of values of parameter T from $(0, \pi/2\gamma]$ to $(0, \pi/\gamma]$. The prolongation of interval is no longer possible if we consider the inequality (1.1) with a, b and $c > 1$. To explain this in more detail, let us consider the function

$$M_1(x) := \frac{\sin x}{x} - b\cos^c(\gamma x), \quad x \in (0, \pi/\gamma),$$

whose first derivative is given by

$$M_1'(x) = \cos^{c-1}(\gamma x)\sin(\gamma x)\Big[\frac{x\cos x - \sin x}{x^2\sin(\gamma x)\cos^{c-1}(\gamma x)} + bc\gamma\Big], \quad x \in (0, \pi/\gamma).$$

Using Proposition 2.1 and the assumption $c > 1$, it follows that the function

$$x \mapsto \frac{x\cos x - \sin x}{x^2 \sin(\gamma x)\cos^{c-1}(\gamma x)}, \quad x \in (0, T)$$

is strictly decreasing for $T \leq \pi/2\gamma$; moreover, the range of this function is equal to $(-\infty, (-1)/3)$, if $T = \pi/2\gamma$, resp. $(-F(T)\cos^{1-c}(\gamma T)/\sin(\gamma T), (-1)/3)$, if $T < \pi/2\gamma$. Using the foregoing arguments, we can clarify the following result:

Theorem 2.8. *Suppose that $a,\ b \in \mathbb{R}$, $c > 1$ and $T \in (0, \pi/2\gamma]$. Then, we have the following:*

(i) The inequality (1.1) *holds iff:*

1. $bc\gamma \leq 1/3$ *and* $a \leq M_1(T)$, *or*
2. $bc\gamma \geq F_\gamma(T)\cos^{1-c}(\gamma T)$, $a \leq 1 - b$ *and* $T < \pi/2\gamma$, *or*
3. $bc\gamma \in (1/3, F_\gamma(T)\cos^{1-c}(\gamma T))$, $a \leq \min(1 - b, M_1(T))$ *and* $T < \pi/2\gamma$, *or* $bc\gamma > 1/3$, $a \leq \min(1 - b, M_1(T))$ *and* $T = \pi/2\gamma$.

(ii) The inequality (1.2) *holds iff:*

1. $bc\gamma \leq 1/3$ *and* $a \geq 1 - b$, *or*
2. $bc\gamma \geq F_\gamma(T)\cos^{1-c}(\gamma T)$, $a \geq M_1(T)$ *and* $T < \pi/2\gamma$, *or*
3. $bc\gamma > 1/3$, $T = \pi/2\gamma$ *and* $a > F(\eta_{b,\gamma})$, *where* $\eta_{b,\gamma}$ *denotes the unique solution of equation*

$$\frac{x\cos x - \sin x}{x^2 \sin(\gamma x)\cos^{c-1}(\gamma x)} + bc\gamma = 0$$

on the interval $(0, T)$.

Further on, a straightforward computation shows that, for every $x \in (0, \pi/\gamma)$, we have:

$$\begin{aligned}
&\Big(\frac{x\cos x - \sin x}{x^2 \sin(\gamma x)\cos^{c-1}(\gamma x)}\Big)' \\
&= \frac{\sin x - x\cos x}{x^2}\cos^{-c}(\gamma x)(c-1)\gamma + \frac{\sin x - x\cos x}{x^2}\cos^{-c}(\gamma x) \\
&\times \left[\cos(\gamma x)\frac{(x^2-2)\sin x \sin\gamma x + \gamma x^2 \cos x\cos\gamma x + 2x\cos x\sin\gamma x - \gamma x\cos\gamma x\sin x}{x(\sin x - x\cos x)\sin^2\gamma x}\right] \\
&=: \frac{\sin x - x\cos x}{x^2}\cos^{-c}(\gamma x)\big[(c-1)\gamma + W_\gamma(x)\big].
\end{aligned}$$

The main problem in extending [5, Theorem 4] to the case in which $c < 1$ and $\gamma > 1$ is the question whether the function $W_\gamma(\cdot)$ is strictly decreasing on $(0, \pi/2\gamma)$. This is the first open problem we would like to address to our readers. In order to formulate the second problem, set, for every $x \in (0, \pi/2\gamma)$,

$$\begin{aligned}
&W_{0,\gamma}(x) \\
&:= \cos(\gamma x)\frac{(x^2-2)\sin x\sin\gamma x + \gamma x^2\cos x\cos\gamma x + 2x\cos x\sin\gamma x - \gamma x\cos\gamma x\sin x}{x^3\sin^3(\gamma x)}.
\end{aligned}$$

Due to Corollary 2.2, we have $W_{0,\gamma}(x) \geq 0$ for all $x \in (0, \pi/2\gamma)$. Since

$$W_\gamma(x) = \frac{x^2 \sin(\gamma x)}{\sin x - x \cos x} W_{0,\gamma}(x), \quad x \in (0, \pi/2\gamma),$$

Proposition 2.1 shows that the function $W_\gamma(\cdot)$ would be strictly decreasing on $(0, \pi/2\gamma)$ provided that the function $W_{0,\gamma}(\cdot)$ is strictly decreasing on $(0, \pi/2\gamma)$. By [5, Lemma 5], the function $W_{0,1}(\cdot)$ is strictly decreasing on $(0, \pi/2)$. Therefore, it is natural to ask whether the function $W_{0,\gamma}(\cdot)$ is strictly decreasing on $(0, \pi/2\gamma)$ in the case that $\gamma > 1$. We close the paper with the observation that the online graphic calculators show that this is actually true in many concrete situations (in cases $\gamma = 2, 3, 4, 5, 6$, e.g.).

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References

[1] Anderson, G. D., Vamanamurthy, M. K., and Vuorinen, M. *Conformal invarients, inequalities and quasiconformal maps.* John Wiley and Sons, 1997.

[2] Bagul, Y. J. Remark on the paper of Zheng Jie Sun and Ling Zhu. *J. Math. Ineqal. 5*, 2 (2019), 801–803.

[3] Bagul, Y. J., and Chesneau, C. Refined forms of Oppenheim and Cusa-Huygens type inequalities. *preprint* (2019).

[4] Bagul, Y. J., and Chesneau, C. Some new simple inequalities involving exponential, trigonometric and hyperbolic functions. *Cubo 21*, 1 (2019), 21–35.

[5] Bagul, Y. J., Chesneau, C., and Kostić, M. On the Cusa-Huygens inequality. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* (2019), in press.

[6] Bhayo, B. A., Klén, R., and Sándor, J. New trigonometric and hyperbolic inequalities. *Miskolc Math. Notes 18*, 1 (2017), 125–137.

[7] Biernacki, M., and Krzyż, J. On the monotonicity of certain functionals in the theory of analytic functions. *Ann. Univ. Mariae Curie-Sklodowska Sect. A 9*, 1 (1955), 135–147.

[8] Chen, C. P., and Cheung, W. S. Sharp Cusa and Becker-Stark inequalities. *J. Inequal. Appl. 2011*, 136 (2011), 6 pp.

[9] Chen, C. P., and Sándor, J. Inequality chains for Wilker, Huygens and Lazarević type inequalities. *J. Math. Inequal. 8*, 136 (2014), 55–67.

[10] Heikkala, V., Vamanamurthy, M. K., and Vuorinen, M. Generalized elliptic integrals. *Comput. Methods Funct. Theory 9*, 1 (2009), 75–109.

[11] Huygens, C. *Oeuvres completes*, vol. 1. Société Hollondaise des Sciences, 2009.

[12] Jeffrey, A. *CHandbook of Mathematical Formulas and Integrals. 3^{rd} edition.* Elsevier Academic Press, San Diego, CA, 2004.

[13] Li, J.-L. An identity related to Jordan's inequality. *Int. J. Math. Math. Sci. 2006*, Article ID 76782 (2006), pp. 1–6.

[14] Mitrinović, D. S. *Analytic Inequalities.* Springer-Verlag, New-York, Berlin, 1970.

[15] Mortici, C. The natural approach of Wilker-Cusa-Huygens inequalities. *Math. Inequal. Appl. 14*, 3 (2006), 535–541.

[16] Qi, F. A double inequality for the ratio of two consecutive Bernoulli numbers. *J. Comput. Applied Math. 351*, 3 (2019), 1–5.

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