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On 2-irreducible submodules of a module

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Abstract. Let R be a commutative ring with identity and let M be an R-module. A proper submodule N of M is said to be *2-irreducible submodule* if whenever $N = H_1 \cap H_2 \cap H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 \cap H_2$ or $N = H_2 \cap H_3$ or $N = H_1 \cap H_3$. In this paper, we investigate the concept of 2-irreducible submodules of M and obtain some properties of this class of modules.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

An ideal I of R is said to be *irreducible* if $I = J_1 \cap J_2$, for ideals J_1 and J_2 of R, implies that either $I = J_1$ or $I = J_2$ [9]. An ideal I of R is said to be *2-irreducible* if whenever $I = J_1 \cap J_2 \cap J_3$ for ideals J_1, J_1 and J_3 of R, then either $I = J_1 \cap J_2$ or $I = J_1 \cap J_3$ or $I = J_2 \cap J_3$. Clearly, any irreducible ideal is a 2-irreducible ideal [10].

A proper submodule N of an R-module M is said to be *irreducible* if for submodules H_1 and H_2 of M, $N = H_1 \cap H_2$ implies that $N = H_1$ or $N = H_2$.

A submodule N of an R-module M is said to be a *2-irreducible submodule* if whenever $N = H_1 \cap H_2 \cap H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 \cap H_2$ or $N = H_2 \cap H_3$ or $N = H_1 \cap H_3$ [7].

The main purpose of this paper is to obtain some results concerning the concept of 2-irreducible submodules of an R-module M.

2. Main results

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [3].

Theorem 2.1. *Let M be a finitely generated multiplication R-module. Then we have the following.*

(a) If N is a 2-irreducible submodule of M, then $(N :_R M)$ is a 2-irreducible ideal of R.

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(b) *If N is a submodule of M such that $(N :_R M)$ is a 2-irreducible ideal of R, then N is a 2-irreducible submodule of M.*

Proof. (a) Let N be a 2-irreducible submodule of M and let $J_1 \cap J_2 \cap J_3 = (N :_R M)$ for some ideals J_1, J_2, and J_3 of R. Then

$$J_1M \cap J_2M \cap J_3M = (N :_R M)M = N$$

by [5, Corollary 1.7]. Thus by assumption, either $J_1M \cap J_2M = N$ or $J_1M \cap J_3M = N$ or $J_2M \cap J_3M = N$. Hence, either $(J_1 \cap J_2)M = (N :_R M)M$ or $(J_1 \cap J_3)M = (N :_R M)M$ or $(J_2 \cap J_3)M = (N :_R M)M$. Therefore, either $J_1 \cap J_2 = (N :_R M)$ or $J_1 \cap J_3 = (N :_R M)$ or $J_2 \cap J_3 = (N :_R M)$ by [12, Corollary of Theorem 9].

(b) Let N be a submodule of M such that $(N :_R M)$ is a 2-irreducible ideal of R and let $H_1 \cap H_2 \cap H_3 = N$ for some submodules H_1, H_2 and H_3 of M. Then we have

$$(H_1 \cap H_2 \cap H_3 :_R M)M = ((H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M))M = (N :_R M)M.$$

Thus $(H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M) = (N :_R M)$ by [12, Corollary of Theorem 9]. Hence, either $(H_1 :_R M) \cap (H_2 :_R M) = (N :_R M)$ or $(H_1 :_R M) \cap (H_3 :_R M)) = (N :_R M)$ or $(H_2 :_R M) \cap (H_3 :_R M) = (N :_R M)$, since $(N :_R M)$ is a 2-irreducible ideal of R. Therefore, either $H_1 \cap H_2 = N$ or $H_1 \cap H_3 = N$ or $H_2 \cap H_3 = N$ by [5, Corollary 1.7]. □

The following example shows that the concepts of irreducible submodules and 2-irreducible submodules are different in general.

Example 2.2. Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$. Then $0 = \bar{3}\mathbb{Z}_6 \cap \bar{2}\mathbb{Z}_6$ implies that the 0 submodule of $\mathbb{Z}_6$ is not irreducible. But $(0 :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$ is a 2-irreducible ideal of $\mathbb{Z}$ by [10, Example 1]. Since the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a finitely generated multiplication $\mathbb{Z}$-module, 0 is a 2-irreducible submodule of $\mathbb{Z}_6$ by Theorem 2.1 (b).

Proposition 2.3. *Let N be a 2-irreducible submodule of an R-module M. Then N is a 2-irreducible submodule of T and N/K is a 2-irreducible submodule of M/K for any $K \subseteq N \subseteq T$.*

Proof. This is straightforward. □

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

An R-module M satisfies the *double annihilator conditions* (DAC for short) if for each ideal I of R we have $I = Ann_R(0 :_M I)$ [6].

An R-module M is said to be a *strong comultiplication module* if M is a comultiplication R-module and satisfies the DAC conditions [1].

Remark 2.4. [11] Let M be a strong comultiplication R-module. Consider the mapping $\phi : l(R) \to l(M)$, where $l(M)$ denotes the lattice of submodules of M, defined by $\phi(I) = (0 :_M I)$. Clearly ϕ is one-to-one, onto and order reversing with the order reversing inverse $\phi^{-1}(N) = Ann_R(N)$ for each submodule N of M. That is, ϕ is a lattice anti-isomorphism.

A non-zero submodule N of an R-module M is said to be a *sum 2-irreducible submodule* if whenever $N = H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 + H_2$ or $N = H_2 + H_3$ or $N = H_1 + H_3$. Also, M is said to be a *sum 2-irreducible module* if M is a sum 2-irreducible submodule of itself [8].

Corollary 2.5. Let M be a strong comultiplication R-module. Then every non-zero proper ideal of R is a sum 2-irreducible ideal if and only if every non-zero proper submodule of M is a 2-irreducible submodule of M.

Proof. By Remark 2.4, obviously the 2-irreducibility of submodules (which is in its essence a lattice-theoretic property) is equivalent to the dual notion in the ideal lattice. The supremum in the ideal lattice is a sum of two ideals and it corresponds to the infimum in the submodule lattice (which is the intersection, of course), so the "sum-2-irreducible" property is the dual of "2-irreducibility". □

A proper submodule P of an R-module M is said to be *prime* if, for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [4].

Proposition 2.6. Let M be a multiplication R-module and let N_1, N_2, and N_3 be prime submodules of M such that $N_1 + N_2 = N_1 + N_3 = N_2 + N_3 = M$. Then $N_1 \cap N_2 \cap N_3$ is not a 2-irreducible submodule of M.

Proof. Assume on the contrary that $N_1 \cap N_2 \cap N_3$ is a 2-irreducible submodule of M. Then $N_1 \cap N_2 \cap N_3 = N_1 \cap N_2 \cap N_3$ implies that either $N_1 \cap N_2 = N_1 \cap N_2 \cap N_3$ or $N_1 \cap N_3 = N_1 \cap N_2 \cap N_3$ or $N_2 \cap N_3 = N_1 \cap N_2 \cap N_3$. We can assume without loss of generality that $N_1 \cap N_2 = N_1 \cap N_2 \cap N_3$. Then $N_1 \cap N_2 \subseteq N_3$. It follows that $(N_1 :_R M)N_2 \subseteq N_3$. As N_3 is a prime submodule of M, we have $N_2 \subseteq N_3$ or $(N_1 :_R M) \subseteq (N_3 :_R M)$. Thus $N_2 \subseteq N_3$ or $N_1 \subseteq N_3$ since M is a multiplication R-module. Therefore, $N_3 = M$, which is a contradiction. □

Corollary 2.7. Let M be a multiplication R-module such that every proper submodule of M is 2-irreducible. Then M has at most two maximal submodules.

Proof. This follows from Proposition 2.6 □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.8. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a proper submodule of M. If N is a 2-irreducible submodule of M, then either $N_1 = M_1$ and N_2 is 2-irreducible submodule of M_2 or $N_2 = M_2$ and N_1 is a 2-irreducible submodule of M_1 or N_1, N_2 are irreducible submodules of M_1, M_2, respectively.*

Proof. Let $N = N_1 \times N_2$ be a 2-irreducible submodule of M such that $N_2 = M_2$. From our hypothesis, N is proper, so $N_1 \neq M_1$. Set $\acute{M} = M/(0 \times M_2)$. One can see that $\acute{N} = N/(0 \times M_2)$ is a 2-irreducible submodule of $\acute{M}$. Also, observe that $\acute{M} \cong M_1$ and $\acute{N} \cong N_1$. Thus N_1 is a 2-irreducible submodule of M_1. By a similar argument as in the previous case, if $N_1 = M_1$, then N_2 is a 2-irreducible submodule of M_2. Now suppose that $N_1 \neq M_1$ and $N_2 \neq M_2$. We show that N_1 is an irreducible submodule of M_1. Suppose that $H_1 \cap K_1 = N_1$ for some submodules H_1 and K_1 of M_1. Then

$$(H_1 \times M_2) \cap (M_1 \times N_2) \cap (K_1 \times M_2) = (H_1 \cap K_1) \times N_2 = N_1 \times N_2.$$

Thus by assumption, either $(H_1 \times M_2) \cap (M_1 \times N_2) = N_1 \times N_2$ or $(H_1 \times M_2) \cap (K_1 \times M_2) = N_1 \times N_2$ or $(M_1 \times N_2) \cap (K_1 \times M_2) = N_1 \times N_2$. Therefore, $H_1 = N_1$ or $K_1 = N_1$ since $N_2 \neq M_2$. Thus N_1 is an irreducible submodule of M_1. Similarly, we can show that N_2 is an irreducible submodule of M_2. □

Theorem 2.9. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. Then for a proper submodule N of M, if N is a 2-irreducible submodule of M, then either $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a 2-irreducible submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$ or $N = \times_{i=1}^{n} N_i$ such that for some $k, m \in \{1, 2, ..., n\}$, N_k is an irreducible submodule of M_k, N_m is an irreducible submodule of M_m, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k, m\}$.*

Proof. We use induction on n. For $n = 2$ the result holds by Theorem 2.8. Now let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \cdots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 2.8, N is a 2-irreducible submodule of M if and only if either $N = L \times M_n$ for some 2-irreducible submodule L of K or $N = K \times L_n$ for some 2-irreducible submodule L_n of M_n or $N = L \times L_n$ for some irreducible submodule L of K and some irreducible submodule L_n of M_n. Note that a proper submodule L of K is an irreducible submodule of K if and only if $L = \times_{i=1}^{n-1} N_i$ such that for some $k \in \{1, 2, ..., n-1\}$, N_k is an irreducible submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, ..., n-1\} \setminus \{k\}$. Consequently the claim is now verified. □

Definition 2.10. We say that an element a of a lattice $(L; \wedge, \vee)$ is a *2-irreducible lattice element* if for all $b, c, d \in L$, if $b \wedge c \wedge d = a$ then either $b \wedge c = a$ or $b \wedge d = a$ or $c \wedge d = a$.

Proposition 2.11. Let $f : M \to \acute{M}$ be an epimorphism of R-modules. Then we have the following.

(a) If N is a 2-irreducible submodule of M such that $Ker(f) \subseteq N$, then $f(N)$ is a 2-irreducible submodule of $\acute{M}$.

(b) If $\acute{N}$ is a 2-irreducible submodule of $\acute{M}$, then $f^{-1}(\acute{N})$ is a 2-irreducible submodule of M.

Proof. Any submodule N of M which contains $\ker f$ is 2-irreducible iff N is a 2-irreducible lattice element in the interval lattice $[\ker f, M]$ of the submodule lattice iff $f(N)$ is a 2-irreducible submodule of $\acute{M}$. The first iff holds because 2-irreducibility of N depends only on submodules which contain N, and those are all in the interval $[\ker f, M]$. The second iff is by the Correspondence Theorem. □

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