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A general fixed point theorem of Ćirić type in quasi-partial metric spaces

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Abstract. In this paper a general fixed point theorem for a mapping satisfying an implicit relation in quasi-partial metric spaces is proved.

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1. Introduction and Preliminaries

In 1994, Matthews [2] introduced the concept of partial metric space as a part of the study of denotional semantics of dataflows networks and proved the Banach principle in these spaces.

The partial metric spaces play an important role in constructing models in theory of computation.

Quite recently, Karapinar et al. [1] introduced the notion of quasi-partial metric space. Some fixed point theorems for a mapping in quasi-partial metric spaces are proved in [1].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [3], [4] and in other papers.

Common fixed point theorems for mappings satisfying implicit relations in partial metric spaces are proved in [5], [6] and in other papers.

The purpose of this paper is to prove a general fixed point theorem of Ćirić type in quasi-partial metric spaces.

Definition 1.1 ([2]). Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for all $x, y, z \in X$, the following conditions hold:

$(P_1) : p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,
$(P_2) : p(x,x) \leq p(x,y)$,
$(P_3) : p(x,y) = p(y,x)$,
$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
The pair (X, p) is called a partial metric space.

If $p(x,y) = 0$, then $x = y$, but the converse does not always hold.

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Definition 1.2 ([1]). Let X be a nonempty set. A function $q : X \times X \to \mathbb{R}_+$ is said to be a quasi-partial metric on X if for all $x, y, z \in X$, the following conditions hold:

$(Q_1) : 0 \leq q(x,x) = q(x,y) = q(y,y)$ if and only if $x = y$,

$(Q_2) : q(x,x) \leq q(y,x)$,

$(Q_3) : q(x,x) \leq q(x,y)$,

$(Q_4) : q(x,z) \leq q(x,y) + q(y,z) - q(y,y)$.

The pair (X,q) is called a quasi-partial metric space.

Lemma 1.3 ([1]). *Let (X,q) be a quasi-partial metric space. Then the following hold:*

a) If $q(x,y) = 0$, then $x = y$.

b) If $x \neq y$, then $q(x,y) > 0$ and $q(y,x) > 0$.

Lemma 1.4 ([1]). *Let (X,q) be a quasi-partial metric space. The function*

$$d_q(x,y) = q(x,y) + q(y,x) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 1.5 ([1]). Let (X,q) be a quasi-partial metric space. Then:

i) a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $q(x,x) = \lim_{n\to\infty} q(x,x_n) = \lim_{n\to\infty} q(x_n,x)$;

ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence in X if $\lim_{n,m\to\infty} q(x_n,x_m)$ and $\lim_{n,m\to\infty} q(x_m,x_n)$ exist and are finite.

iii) A quasi-partial metric space is complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_q to a point $x \in X$ such that $q(x,x) = \lim_{m,n\to\infty} q(x_m,x_n) = \lim_{m,n\to\infty} q(x_n,x_m)$.

Lemma 1.6 ([1]). *Let (X,q) be a quasi-partial metric space. The following statements are equivalent:*

a) the sequence $\{x_n\}$ is Cauchy in (X,q),

b) the sequence $\{x_n\}$ is Cauchy in (X,d_q).

Lemma 1.7 ([1]). *Let (X,q) be a quasi-partial metric space. Then, the following statements are equivalent:*

a) (X,q) is complete,

b) (X,d_q) is complete.

Moreover,

$$\lim_{n\to\infty} d_q(x,x_n) = 0 \Leftrightarrow \lim_{n\to\infty} q(x_n,x) = \lim_{n\to\infty} q(x,x_n) =$$

$$\lim_{m,n\to\infty} q(x_m,x_n) = \lim_{n,m\to\infty} q(x_n,x_m).$$

Lemma 1.8. *Let (X,q) be a quasi-partial metric space and $\{x_n\}$ a convergent sequence in X to a point $z \in X$ such that $q(z,z) = 0$ and $y \in X$. Then $\lim_{n\to\infty} q(x_n,y) = q(z,y)$ and $\lim_{n\to\infty} q(y,x_n) = q(y,z)$.*

Proof. By (Q_4), $q(z,y) \leq q(z,x_n) + q(x_n,y)$. Hence,

$$q(z,y) - q(z,x_n) \leq q(x_n,y) \leq q(x_n,z) + q(z,y).$$

Letting n tend to infinity we obtain

$$\lim_{n\to\infty} q(x_n,y) = q(z,y).$$

Similarly, $\lim_{n\to\infty} q(y,x_n) = q(y,z)$. □

2. Implicit relations

Definition 2.1. Let $\mathcal{F}_Q$ be the family of lower semi - continuous functions $F : \mathbb{R}_+^5 \to \mathbb{R}$ such that:

(F_1) : F is nonincreasing in variable t_5,

(F_2) : For all $u,v \geq 0$, there exists $h \in [0,1)$ such that $F(u,v,v,u,u+v) \leq 0$ implies $u \leq hv$,

(F_3) : For all $t,t' > 0$, there exists $k \in [0,1)$ such that $F(t,t,0,0,t+t') \leq 0$ implies $t \leq kt'$.

Example 2.2. $F(t_1,...,t_5) = t_1 - \alpha \max\left\{t_2,t_3,t_4,\dfrac{t_5}{2}\right\}$, where $\alpha \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ such that

$$F(u,v,v,u,u+v) = u - \alpha \max\left\{u,v,\frac{u+v}{2}\right\} \leq 0.$$

If $u > v$, then $u(1-\alpha) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $0 \leq h < 1$.

(F_3) : Let $t,t' > 0$ such that

$$F(t,t,0,0,t+t') = t - \alpha \max\left\{t,\frac{t+t'}{2}\right\} \leq 0.$$

If $t > t'$, then $t(1-\alpha) \leq 0$, a contradiction. Hence, $t \leq t'$ which implies $t \leq kt'$, where $0 \leq k = \alpha < 1$.

Similarly, it is proved that the following functions satisfy properties (F_1), (F_2), (F_3).

Example 2.3. $F(t_1,...,t_5) = t_1 - \alpha \max\{t_2,t_3,t_4,t_5\}$, where $\alpha \in (0,1)$.

Example 2.4. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, where $a,b,c,d \geq 0$ and $a+b+c+d < 1$.

Example 2.5. $F(t_1,...,t_5) = t_1 - a\max\{t_2,t_3,t_4\} - bt_5$, where $a,b \geq 0$ and $a+2b < 1$.

Example 2.6. $F(t_1,...,t_5) = t_1^2 - at_2t_3 - bt_4^2 - ct_5t_6$, where $a,b \geq 0$ and $a+4b < 1$.

Example 2.7. $F(t_1,...,t_5) = t_1^2 - at_2^2 - bt_3t_4 - ct_5^2$, where $a,b,c \geq 0$ and $a+b+4c<1$.

Example 2.8. $F(t_1,...,t_5) = t_1^3 - at_1^2t_2 - bt_1t_2^2 - ct_2t_3t_4 - dt_5^2$, where $a,b,c,d \geq 0$ and $a+b+c+8d<1$.

Example 2.9. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5\}$, where $a,b,c \geq 0$ and $a+b+2c<1$.

3. Main results

Theorem 3.1. *Let (X,q) be a complete quasi-partial metric space and $f: X \to X$ such that for all $x,y \in X$*

$$F\left(\begin{array}{c} q(fx,fy), q(x,y), q(x,fx), \\ q(y,fy), q(x,fy)+q(y,fx) \end{array}\right) \leq 0 \tag{3.1}$$

for some $F \in \mathcal{F}_Q$. Then f has a unique fixed point z with $q(z,z)=0$.

Proof. Let x_0 be an arbitrary point in X. Define $\{x_n\}$ in X by $x_n = fx_{n-1}$, $n = 1,2,...$. If there exists $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of f. Suppose that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (3.1) we obtain

$$F\left(\begin{array}{c} q(fx_{n-1},fx_n), q(x_{n-1},x_n), q(x_{n-1},fx_{n-1}), \\ q(x_n,fx_n), q(x_{n-1},fx_n)+q(x_n,fx_{n-1}) \end{array}\right) \leq 0,$$

$$F\left(\begin{array}{c} q(x_n,x_{n+1}), q(x_{n-1},x_n), q(x_{n-1},x_n), \\ q(x_n,x_{n+1}), q(x_{n-1},x_{n+1})+q(x_n,x_n) \end{array}\right) \leq 0, \tag{3.2}$$

By (Q_4),

$$q(x_{n-1},x_{n+1}) \leq q(x_{n-1},x_n) + q(x_n,x_{n+1}) - q(x_n,x_n). \tag{3.3}$$

By (3.3) and (F_1) we obtain

$$F\left(\begin{array}{c} q(x_n,x_{n+1}), q(x_{n-1},x_n), q(x_{n-1},x_n), \\ q(x_n,x_{n+1}), q(x_{n-1},x_n)+q(x_n,x_{n+1}) \end{array}\right) \leq 0,$$

By (F_2), there exists $h \in [0,1)$ such that

$$q(x_n,x_{n+1}) \leq hq(x_n,x_{n-1})$$

which implies

$$q(x_n,x_{n+1}) \leq hq(x_{n-1},x_n) \leq ... \leq h^n q(x_0,x_1). \tag{3.4}$$

Let $n,m \in \mathbb{N}$ with $n>m$. By (Q_4) and (3.4) we obtain

$$\begin{aligned} q(x_m,x_n) &\leq q(x_m,x_{m+1}) + q(x_{m+1},x_{m+2}) + ... + q(x_{n-1},x_n) \\ &\leq (h^m + h^{m+1} + ... + h^n) q(x_0,x_1) \\ &\leq \frac{h^n}{1-h} q(x_0,x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Similarly,

$$q(x_n, x_m) = \frac{h^n}{1-h} q(x_1, x_0) \to 0 \text{ as } n \to \infty.$$

On the other hand,

$$\begin{aligned} d_q(x_m, x_n) &\leq q(x_n, x_m) + q(x_m, x_n) - q(x_n, x_n) - q(x_m, x_m) \\ &\leq q(x_n, x_m) + q(x_m, x_n) \to 0 \text{ for } n, m \to \infty. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, d_q).

By Lemma 1.6, $\{x_n\}$ is a Cauchy sequence in (X, q). Since (X, q) is complete, $\{x_n\}$ is convergent in (X, q) to a point z with $q(z, z) = 0$. Since $\lim_{n\to\infty} d_q(z, x_n) = 0$, then by Lemma 1.7,

$$\lim_{n\to\infty} q(x_n, z) = \lim_{n\to\infty} q(z, x_n) = \lim_{n,m\to\infty} q(x_m, x_n) = \lim_{n\to\infty} q(x_n, x_m). \tag{3.5}$$

By (3.1), for $x = x_n$ and $y = z$ we obtain

$$F\begin{pmatrix} q(fx_n, fz), q(x_n, z), q(x_n, fx_n), \\ q(z, fz), q(x_n, fz) + q(z, fx_n) \end{pmatrix} \leq 0,$$

$$F\begin{pmatrix} q(x_{n+1}, fz), q(x_n, z), q(x_n, x_{n+1}), \\ q(z, fz), q(x_n, fz) + q(z, x_{n+1}) \end{pmatrix} \leq 0. \tag{3.6}$$

Letting n tend to infinity, by (3.6), (3.5) and Lemma 1.8, we obtain

$$F(q(z, fz), 0, 0, q(z, fz), q(z, fz)) \leq 0.$$

By (F_2), $q(z, fz) = 0$, which implies $z = fz$. Hence, z is a fixed point of f with $q(z, z) = 0$.

Suppose that there exists another fixed point $z' \neq z$ of f such that $q(z', z') = 0$. By Lemma 1.3, $q(z, z') > 0$. By (3.1) we obtain

$$F(q(fz, fz'), q(z, z'), q(z, fz), q(z', fz'), q(z, fz') + q(z', fz)) \leq 0,$$

$$F(q(z, z'), q(z, z'), 0, 0, q(z, z') + q(z', z)) \leq 0.$$

By (F_3), there exists $k \in [0, 1)$ such that

$$q(z, z') \leq kq(z', z).$$

Similarly we obtain

$$q(z', z) \leq kq(z', z) \leq k^2 q(z, z'),$$

which implies

$$q(z, z')(1-k) \leq 0.$$

Hence, $q(z, z') = 0$. By Lemma 1.3, $z = z'$. □

By Theorem 3.1 and Example 2.2 we obtain a theorem of Ćirić type in complete quasi-partial metric spaces.

Theorem 3.2. *Let* (X, q) *be a complete quasi-partial metric space and* $f : X \to X$ *such that for all* $x, y \in X$

$$q(fx, fy) \leq \alpha \max\{q(x,y), q(x,fx), q(y,fy), q(x,fy) + q(y,fx)\},$$

where $\alpha \in (0,1)$. *Then* f *has a unique fixed point* z *with* $q(z,z) = 0$.

Remark 3.3. By Theorem 3.1 and Examples 2.3 - 2.9 we obtain new results.

References

[1] Karapinar, E., Erhan, I. M., and Öztürk, A. Fixed point theorems on quasi-partial metric spaces. *Math. Comput. Modelling 57*, 9-10 (2013), 2442–2448.

[2] Matthews, S. G. Partial metric topology. In *Papers on general topology and applications (Flushing, NY, 1992)*, vol. 728 of *Ann. New York Acad. Sci.* New York Acad. Sci., New York, 1994, pp. 183–197.

[3] Popa, V. Fixed point theorems for implicit contractive mappings. *Stud. Cercet. Ştiinţ. Ser. Mat. Univ. Bacău*, 7 (1997), 127–133 (1999).

[4] Popa, V. Some fixed point theorems for compatible mappings satisfying an implicit relation. *Demonstratio Math. 32*, 1 (1999), 157–163.

[5] Popa, V., and Patriciu, A.-M. A general fixed point theorem for a pair of mappings in partial metric spaces. *Acta Univ. Apulensis Math. Inform.*, 43 (2015), 93–103.

[6] Vetro, C., and Vetro, F. Common fixed points of mappings satisfying implicit relations in partial metric spaces. *J. Nonlinear Sci. Appl. 6*, 3 (2013), 152–161.

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