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Fixed points of mappings over a locally convex topological vector space and Ulam-Hyers stability of fixed point problems

Kushal Roy[1 2] **and Mantu Saha**[3]

Abstract. This paper deals with the Theory of fixed points of mappings which are analogous to contraction mappings and Kannan mappings over a locally convex topological vector space. Some common fixed point theorems for a pair of mappings involving their iterates are proved. The purpose of this paper is to examine the validity of established results on fixed points of contraction mappings and Kannan mappings over a locally convex topological vector space. It is revealed that a suitable local base in locally convex topological vector space plays an important role in finding fixed points of above mappings over that space. Also an application related to stability of fixed point equation for Kannan-type contractive mappings is obtained here.

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1. Introduction

Theory of fixed points over a metric space finds applications in areas like differential equations, integral equations, implicit function theorem etc. Historically, Schauder fixed point theorem [27], Brouwer fixed point theorem [5], Tychonoff [30] and Morales [19] as early as Banach contraction principle [2], everywhere fixed point theorems as found in literature depend heavily on continuity of the operators involved over underlying spaces. Early twentieth century had witnessed researchers in fixed point theory dealing with operators that are not necessarily continuous, and consequently we had seen a surge in development of fixed point theory with enormous speed and volume, and researchers have seen that Kannan operators [16], Ćirić operators [7] and similar operators (see [3],[4],[6],[8],[9],[12],[13],[22],[23],[21],[25]) had occupied a stable position in fixed point theory demanding further relaxation in operators and in underlying spaces or in both. Thus one can find representative fixed point theorems

[1]Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India. e-mail: kushal.roy93@gmail.com
[2]Corresponding author
[3]Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India. e-mail: mantusaha.bu@gmail.com

in various topological structured metric spaces (see [1],[11],[17],[18]) in recent times of the new millennium.

However, though researchers have been trying to involve TVS (Topological Vector Space) as a ground space to develop fixed point theory, but efforts are yet to gain the desired momentum. Our investigations in this paper rest on this platform, that is we investigate non-linear mappings over a TVS X. Very recently Tang et. al. [29] had proved fixed point theorem for (ψ, ϕ)-contractive mapping in a locally convex TVS using Minkowski functional, while our results do not involve such functionals. Therefore our paper provides a new direction for researchers in proving fixed point theorems over linear topological spaces without using Minkowski functional. Details follow in Sections 2 and 3. Moreover, an application related to Ulam-Hyers stability (see [14]) of fixed point of mapping is given here.

2. Preliminaries

In the following, we give some basic definitions and properties corresponding to a topological vector space (see [10],[20],[24] and [26]).

Definition 2.1. Let X be a vector space and C a subset of X. Then C is said to be convex if for any two elements $x, y \in C$ and for any scalar $0 \leq \alpha \leq 1$, $\alpha x + (1-\alpha)y \in C$, that is the line segment joining two points x, y, must lie in the set C. Equivalently, $\alpha C + (1-\alpha)C \subset C$ for all scalars α satisfying $0 \leq \alpha \leq 1$.

Lemma 2.2. *A subset C of a vector space X is convex iff for all positive scalars s and t, $(s+t)C = sC + tC$.*

Definition 2.3. A subset S of a vector space X is said to be symmetric if $-S \subset S$, equivalently $S = -S$.

Definition 2.4. A subset B of a vector space X is said to be balanced if $\alpha B \subset B$ for all scalars α, whenever $|\alpha| \leq 1$.

Definition 2.5. A set A in a vector space X is said to be absorbing if for each $x \in X$ there exists an $\epsilon > 0$ such that $\alpha x \in A$, whenever $|\alpha| \leq \epsilon$.

Lemma 2.6. *A convex set C of a vector space X is balanced iff it is symmetric.*

Definition 2.7. A balanced set B of a vector space X is absorbing iff for each $x \in X$, there corresponds a scalar $\beta \neq 0$ such that $\beta x \in B$.

Definition 2.8. A vector space X over a field F ($\mathbb{R}$ or $\mathbb{C}$) equipped with a T_1 topology τ is said to be a topological vector space (TVS) if the following conditions are satisfied.

(i) The mapping from $X \times X$ to X defined by $(x, y) \to x + y$, $x, y \in X$, is continuous, that is, for every neighborhood W of $x + y$ we can find neighborhoods V_1 of x and V_2 of y such that $V_1 + V_2 \subset W$ and also

(ii) The mapping from $F \times X \to X$ defined by $(\alpha, x) \to \alpha x$, $\alpha \in F, x \in X$, is continuous, that is, for any neighborhood W of αx we can find a neighborhood of α say $(\alpha-\delta,\alpha+\delta),\delta>0$ and a neighborhood V of x such that $\gamma V\subset W$ whenever $\gamma\in(\alpha-\delta,\alpha+\delta)$.

We now quote the following useful definitions and known results (see [9]).

Definition 2.9. (Local base) By local base of a TVS (X,τ) we mean a neighborhood base $\mathbb{B}$ of $\theta\in X$ that is for every neighborhood V of θ there exists a member $B\in\mathbb{B}$ such that $\theta\in B\subset V$.

Definition 2.10. A TVS X is said to be locally convex if X has a local base whose members are all convex sets.

Lemma 2.11. *A TVS X has a balanced local base.*

Lemma 2.12. *Every neighborhood of θ in a TVS X contains an absorbing neighborhood of $\theta\in X$.*

Lemma 2.13. *In a locally convex TVS X every neighborhood of θ contains a absorbing, balanced and convex neighborhood of θ.*

Lemma 2.14. *Every TVS is regular.*

Lemma 2.15. *Let X be a TVS. Then the following hold.*

(i) If $A\subset X$ then $\overline{A}=\cap(A+V)$, where $V\in\mathbb{N}(\theta)$, $\mathbb{N}(\theta)$ is the collection of all neighborhoods of $\theta\in X$.

(ii) If $A\subset X$ and $B\subset X$ then $\overline{A}+\overline{B}\subset\overline{A+B}$.

(iii) If Y is a subspace of X then $\overline{Y}$ is also a subspace of X.

(iv) If C is a convex set in X then $\overline{C}$ and $Int(C)$ are also convex.

(v) If $E\subset X$ is balanced then $\overline{E}$ is also balanced, moreover if $\theta\in Int(E)$ then $Int(E)$ is also balanced.

(vi) If A is an absorbing subset of X then $\overline{A}$ is also absorbing.

Lemma 2.16. *The following conditions are equivalent in a TVS X.*

(i) X is T_0.

(ii) X is T_2.

(iii) $\cap_{V\in\mathbb{N}(\theta)}=\{\theta\}$, where $\mathbb{N}(\theta)$ is the collection of all neighborhoods of $\theta\in X$.

Lemma 2.17. *In a locally convex TVS X, the balanced, closed, convex neighborhood of θ forms a neighborhood base of $\theta\in X$.*

Definition 2.18. Let X be a TVS. Fix a base $\mathbb{B}$ for X. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if to every $V\in\mathbb{B}$ there corresponds a $N\in\mathbb{N}$ such that $x_n-x_m\in V$ whenever $m>n\geq N$.

Definition 2.19. A sequence $\{x_n\}\subset X$ is said to be convergent to an element $x\in X$ if for any basic neighborhood V, there exists a positive integer $N\in\mathbb{N}$ such that $x_n-x\in V$ whenever $m\geq N$. We write $x_n\to x$ as $n\to\infty$ and we say that x is the limit of $\{x_n\}$.

Definition 2.20. A TVS X is said to be complete if every Cauchy sequence in X is convergent to an element in X.

Lemma 2.21. *A TVS X is Hausdorff iff every sequence in X has at most one limit.*

Lemma 2.22. *A complete subset of a Hausdorff TVS is closed.*

Lemma 2.23. *Let $A \subset X$ be complete. Then every closed subset of A is complete.*

Definition 2.24. A TVS X is said to be an $F-$space if its topology τ is induced by a complete invariant metric. A TVS X is a Frechet space if it is a locally convex $F-$space.

Definition 2.25. Let X and Y be two TVSs. Also let $T : X \to Y$ be a mapping. Then T is said to be continuous at $x_0 \in X$ if for every sequence $\{x_n\}$ in X such that $x_n \to x_0$ as $n \to \infty$ implies $Tx_n \to Tx_0$ as $n \to \infty$.

In the following we give the definition of $\mathcal{U}-$contraction mapping and state a fixed point theorem related to it.

Definition 2.26. [28] Let E be a separated locally convex topological vector space and $\mathcal{U}$ be a neighborhood basis of the origin consisting of absolutely convex open subsets of E. Also let S be a nonempty subset of E. A mapping $g : S \to E$ is a $U-$contraction ($U \in \mathcal{U}$) iff for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon, U) > 0$ such that if $x, y \in S$ and if

$$x - y \in (\epsilon + \delta)U, \text{ then } g(x) - g(y) \in \epsilon U. \tag{2.1}$$

If g is a $U-$contraction for each $U \in \mathcal{U}$, then g is a $\mathcal{U}-$contraction.

Theorem 2.27. *[28] Let S be a sequentially complete subset of E and $g : S \to E$ be a $\mathcal{U}-$contraction. If g satisfies the condition:*

for each $x \in S$ with $g(x) \notin S$, there is a $z \in (x, g(x)) \cap S$ such that $g(z) \in S$ then g has a unique fixed point in S. In the above condition, $(x, y) = \{z \in E : z = \mu x + (1-\mu)y, 0 < \mu < 1\}$ for any $x, y \in E$.

3. Main results

In this section following the references [2], [3], [15] and [16] we have defined contraction mapping, Kannan-type contractive mapping, $T-$contraction mapping and $T-$Kannan-type contractive mapping over a locally convex TVS and we have been able to prove some fixed point theorems and common fixed point theorems over it.

Definition 3.1. Let (X, τ) be a locally convex TVS. A mapping $T : X \to X$ is said to be a contraction mapping if for any neighborhood U of $\theta \in X$ there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, whenever $x - y \in U$, then $Tx - Ty \in \alpha U$.

Definition 3.2. Let (X, τ) be a locally convex TVS. A mapping $T : X \to X$ is said to be a Kannan-type contractive mapping if for every neighborhood U of $\theta \in X$ there exists $0 < \alpha < \frac{1}{2}$ such that for all $x, y \in X$, whenever $x - Tx \in U$, then $(Tx - Ty) - \alpha(y - Ty) \in \alpha U$.

Definition 3.3. Let (X, τ) be a locally convex TVS and $T : X \to X$ be a mapping. Then a mapping $S : X \to X$ is said to be a $T-$contraction if for any neighborhood U of $\theta \in X$ there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, whenever $Tx - Ty \in U$, then $TSx - TSy \in \alpha U$.

Definition 3.4. Let (X, τ) be a locally convex TVS and $T : X \to X$ be a mapping. Then a mapping $S : X \to X$ is said to be a $T-$Kannan-type contractive mapping if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ and any neighborhood U of $\theta \in X$, whenever $Tx - TSx \in U$, then $(TSx - TSy) - \alpha(Ty - TSy) \in \alpha U$.

Definition 3.5. Let (X, τ) be a locally convex TVS and $T : X \to X$ be a mapping. Then T is said to be sequentially convergent if, for any sequence $\{y_n\}$ in X, convergence of $\{Ty_n\}$ in X implies that $\{y_n\}$ is convergent in X.

Definition 3.6. Let (X, τ) be a locally convex TVS and $T : X \to X$ be a mapping. Then T is said to be subsequentially convergent if, for any sequence $\{y_n\}$ in X, convergence of $\{Ty_n\}$ in X implies that $\{y_n\}$ has a convergent subsequence in X.

Definition 3.7. Let (X, τ) be a locally convex TVS and $\{T_n\}$ be a sequence of self maps on X. Then $\{T_n\}$ converges uniformly to a self map T on X if for each neighborhood U of $\theta \in X$ there exists $N \in \mathbb{N}$ such that for all $x \in X$, whenever $n > N$, then $T_n x - Tx \in U$.

Lemma 3.8. *Let (X, τ) be a locally convex TVS and $\{x_n\}$ be a sequence in X. If for any neighborhood V of $\theta \in X$ there exists some $t > 0$ such that for any $n \in \mathbb{N}$, $x_n - x_{n+1} \in \alpha^n tV$ for some $\alpha \in (0, 1)$, then $\{x_n\}$ is Cauchy in X.*

Proof. Let U be an arbitrary neighborhood of $\theta \in X$. Then there exists some $k > 0$ such that $x_n - x_{n+1} \in \alpha^n kU$ for any $n \in \mathbb{N}$. Therefore, for $p \geq 1$ and for any $n \in \mathbb{N}$, we get

$$\begin{aligned} x_n - x_{n+p} &= (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \ldots + (x_{n+p-1} - x_{n+p}) \\ &\in (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+p-1})kU \\ &= \alpha^n \frac{1 - \alpha^p}{1 - \alpha} kU \subset \frac{\alpha^n}{1 - \alpha} kU = \alpha^n \frac{k}{1 - \alpha} U. \end{aligned} \tag{3.1}$$

Since $\alpha \in (0, 1)$, there exists $N \in \mathbb{N}$ such that $\alpha^N < \frac{1-\alpha}{k}$. So whenever $n \geq N$, we get $x_n - x_{n+p} \in \alpha^n \frac{k}{1-\alpha} U \subset \alpha^N \frac{k}{1-\alpha} U \subset U$.
Since U is arbitrary it follows that $\{x_n\}$ is a Cauchy sequence in X. □

Theorem 3.9. *Let (X, τ) be a complete locally convex topological vector space. Then a contraction mapping T possesses a unique fixed point in X.*

Proof. Any contraction mapping is a $\mathcal{U}-$contraction mapping and by Theorem 2.27 the proof follows immediately. □

Theorem 3.10. *Let (X, τ) be a complete locally convex topological vector space and f be a continuous mapping from X into itself. Let $g : X \to X$ be a mapping such that it commutes with f and satisfies $g(X) \subset f(X)$. If for any neighborhood U of θ in X there exists $0 < \alpha < 1$ such that $gx - gy \in \alpha U$ whenever $fx - fy \in U$ $\forall x, y \in X$ then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be fixed. Then there exists $x_1 \in X$ such that $fx_1 = gx_0$. Since $x_1 \in X$, there exists $x_2 \in X$ such that $fx_2 = gx_1$. Proceeding in this way we get $fx_n = gx_{n-1}$ $\forall n \in \mathbb{N}$. Let $y_n = fx_n = gx_{n-1}$ $\forall n \in \mathbb{N}$.

Let U be a neighbourhood of θ. Without loss of generality we can take U as convex, balanced and absorbing. So there exists a $\lambda > 0$ such that $y_1 - y_2 = fx_1 - fx_2 \in \beta U$ whenever $|\beta| \geq \lambda$. Thus we get $y_1 - y_2 = fx_1 - fx_2 \in \lambda U$, which implies $y_2 - y_3 = gx_1 - gx_2 = fx_2 - fx_3 \in \alpha\lambda U$, consequently $y_3 - y_4 = gx_2 - gx_3 \in \alpha^2 \lambda U$. Proceeding in this way we get, $y_n - y_{n+1} \in \alpha^{n-1}\lambda U$ $\forall n \in \mathbb{N}$.

Therefore by Lemma 3.8, $\{y_n\}$ is Cauchy in X and since X is complete, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Since f is continuous, we see that g is also continuous on X. So, $fy_n \to fz$ and $gy_n \to gz$ as $n \to \infty$. Now, $fy_n = fgx_{n-1} = gfx_{n-1} = gy_{n-1}$ and hence $fz = gz$.

Let V be any neighbourhood of θ. Without loss of generality we can assume that V is convex, balanced and absorbing. So there exists $\mu > 0$ such that $gz - g^2 z \in \gamma V$ whenever $|\gamma| \geq \mu$. Thus whenever $|\gamma| \geq \mu$ we get $gz - g^2 z \in \gamma V = V_\gamma$ (say), implies $fz - g(fz) \in V_\gamma$, which in turn implies that $fz - f(gz) \in V_\gamma$. Hence $gz - g^2 z \in \alpha V_\gamma$. Continuing in this way we get $gz - g^2 z \in \alpha^n \gamma V$ $\forall n \in \mathbb{N}$. So we get $gz - g^2 z \in V$. Since V is arbitrary, we have $g^2 z = gz$.
Now, $f(gz) = g(fz) = g^2 z = g(gz) = gz$ so f and g have a common fixed point $gz = a$ (say) in X. Uniqueness of a is also obvious. □

Theorem 3.11. *Let (X, τ) be a complete locally convex topological vector space and $T : X \to X$ be a map such that T is injective, continuous and subsequentially convergent in X. If S is a continuous $T-$contraction map with the constant $0 < \alpha < 1$, then S has a unique fixed point in X. Also if T is sequentially convergent then for each $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point of S.*

Proof. Let $x_0 \in X$ and we construct the sequence $\{x_n\}$ in X by $x_n = Sx_{n-1} = S^n x_0$ for all $n \in \mathbb{N}$.
Let U be a neighborhood of $\theta \in X$. Without loss of generality we can assume that U is convex, balanced and absorbing. So there exists $k > 0$ such that $Tx_0 - Tx_1 \in \gamma U$ whenever $|\gamma| \geq k$. Since $Tx_0 - Tx_1 \in kU$, $TSx_0 - TSx_1 \in \alpha kU$, that is, $Tx_1 - Tx_2 \in \alpha kU$. So we have $TSx_1 - TSx_2 \in \alpha^2 kU$. Proceeding in a similar fashion, we get $Tx_{n-1} - Tx_n \in \alpha^{n-1} kU$ for all $n \in \mathbb{N}$. Then by Lemma 3.8 we see that $\{Tx_n\}$ is a Cauchy sequence in X. Since (X, τ) is complete, $\{Tx_n\}$ is convergent and let it be convergent to $a \in X$. Since T is subsequentially convergent in X, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some $b \in X$(say). Now T is continuous in X, so

$Tx_{n_k} \to Tb$ as $n \to \infty$. Thus $Tb = a$. Also S is continuous in X so $Sx_{n_k} \to Sb$, i.e., $x_{n_k+1} \to TSb$ and so $TSb = a$. Therefore $TSb = Tb$ and since T is injective, then $Sb = b$. So b is a fixed point of S. Uniqueness of b is also clear. □

Theorem 3.12. *Let (X, τ) be a complete locally convex topological vector space. Also let $\{T_n\}$ be a sequence of mappings on X such that for any neighborhood U of $\theta \in X$ there exists $0 < \alpha < 1$ such that for all $n \in \mathbb{N}$, $T_n x - T_n y \in \alpha U$ whenever $x - y \in U$, $\forall x, y \in X$. Suppose that for each $x \in X$ the sequence $\{T_n x\}$ converges to Tx, where T is a self map on X. Then T is also a contraction mapping on X with the same constant α.*

Proof. Let V be an arbitrary neighborhood of $\theta \in X$. Then there exists a neighborhood W of $\theta \in X$ such that $W + W \subset V$.
Let U be a neighborhood of θ in X and $x - y \in U$, where $x, y \in X$. By Lemma 2.17 there exists a closed, convex, balanced and absorbing neighborhood P of $\theta \in X$ such that $x - y \in P \subset U$. Since $x - y \in P$, $T_n x - T_n y \in \alpha P$ for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} Tx - Ty &= Tx - T_n x + T_n x - T_n y + T_n y - Ty \\ &= (Tx - T_n x) + (T_n x - T_n y) + (T_n y - Ty). \end{aligned} \tag{3.2}$$

Since $T_n x \to Tx$ and $T_n y \to Ty$ as $n \to \infty$, then there exists $N_1, N_2 \in \mathbb{N}$ such that $T_n x - Tx \in W$ whenever $n \geq N_1$ and also $T_n y - Ty \in W$ whenever $n \geq N_2$. If we set $N = max\{N_1, N_2\}$ then from (1) we have $Tx - Ty \in W + W + \alpha P \subset V + \alpha P$. Since V is a neighborhood of θ in X, then

$$Tx - Ty \in \cap_{\theta \in V}(V + \alpha P) = \overline{\alpha P} = \alpha \overline{P} = \alpha P \subset \alpha U. \tag{3.3}$$

Since U is an arbitrary neighborhood of $\theta \in X$, therefore T is also a contraction map with the same constant α. □

Theorem 3.13. *Let (X, τ) be a complete locally convex topological vector space and let $T : X \to X$ be a Kannan-type contractive mapping with the constant α. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ and let U be a neighborhood of $\theta \in X$. Let us define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. We may assume that U is convex, balanced and absorbing. Now $x_0 - Tx_0 = x_0 - x_1 \in X$. So there exists a scalar $\lambda > 0$ such that $x_0 - x_1 \in \eta U$ whenever $|\eta| \geq \lambda$. As $x_0 - x_1 \in \lambda U$ then $(Tx_0 - Tx_1) - \alpha(x_1 - Tx_1) \in \alpha\lambda U$, that is, $x_1 - x_2 \in \frac{\alpha}{1-\alpha}\lambda U$. Proceeding in a similar fashion we get $x_n - x_{n+1} \in (\frac{\alpha}{1-\alpha})^n \lambda U$ for all $n \in \mathbb{N}$. Then by Lemma 3.8 we see that $\{x_n\}$ is Cauchy in X. Since X is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Let V be a neighborhood of $\theta \in X$. Then there exists a balanced, convex and absorbing neighborhood W of $\theta \in X$ such that $W \subset \frac{1-\alpha}{2} V$. Now since $x_n \to z$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $x_n - z \in W$ and $x_n - x_{n+1} \in W$ for all $n \geq N$. So for all $n \geq N$,

$$\begin{aligned}
(z - Tz) - \alpha(z - Tz) &= z - x_{n+1} + x_{n+1} - Tz - \alpha(z - Tz) \\
&= (z - x_{n+1}) + [(Tx_n - Tz) - \alpha(z - Tz)] \\
&\in W + \alpha W \subset W + W \subset (1 - \alpha)V. \qquad (3.4)
\end{aligned}$$

Then $(1 - \alpha)(z - Tz) \in (1 - \alpha)V$ whenever $n \geq N$, that is, $z - Tz \in V$. Since V is arbitrary neighborhood of $\theta \in X$, then we have $Tz = z$. Clearly the fixed point of T is unique. $\square$

Theorem 3.14. *Let (X, τ) be a complete locally convex topological vector space and f a continuous self map on X. Let $g : X \to X$ be a mapping such that it commutes with f and satisfies $g(X) \subset f(X)$. If for every neighborhood U of $\theta \in X$ there exists an $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$, $fx - gx \in U$ implies $gx - gy - \alpha(fy - gy) \in \alpha U$, then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Then there exists $x_1 \in X$ such that $fx_1 = gx_0$. Since $x_1 \in X$, there exists $x_2 \in X$ such that $fx_2 = gx_1$. Continuing in this way, we get $fx_n = gx_{n-1}$ $\forall n \in \mathbb{N}$. Let us take $\{y_n\} \subset X$ defined by $y_n = fx_n = gx_{n-1}$ for all $n \in \mathbb{N}$.
Let U be a neighborhood of θ in X. Assume that U is convex, absorbing and balanced. So there exists a $t > 0$ such that $y_1 - y_2 = fx_1 - gx_1 \in \zeta U$ whenever $|\zeta| \geq t$. Therefore, $(gx_1 - gx_2) - \alpha(fx_2 - gx_2) \in \alpha t U$, that is, $y_2 - y_3 \in \frac{\alpha}{1-\alpha} t U$. Proceeding in this manner we get $y_n - y_{n+1} \in \frac{\alpha}{1-\alpha}^{n-1} t U$ for all $n \in \mathbb{N}$. So by applying Lemma 3.8 we see that $\{y_n\}$ is Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Since f is continuous we have $fy_n \to fz$ as $n \to \infty$. Now $fy_n = fgx_{n-1} = gfx_{n-1} = gy_{n-1}$ for all $n \geq 2$. Therefore, $gy_n \to fz$ as $n \to \infty$.

Let V be a neighborhood of $\theta \in X$. Let $W = \frac{1-\alpha}{2\alpha} V$. Then there exists $N_1 \in \mathbb{N}$ such that $fy_n - gy_n = fy_n - fy_{n+1} \in W$ and $fz - gy_n \in W$ whenever $n \geq N_1$. If $n \geq N_1$ then $(gy_n - gz) - \alpha(fz - gz) \in \alpha W$, implying that $(gy_n - gz) - \alpha(fz - gy_n + gy_n - gz) \in \alpha W$. Thus $(1 - \alpha)(gy_n - gz) \in 2\alpha W$, that is, $gy_n - gz \in \frac{2\alpha}{1-\alpha} W = V$ whenever $n \geq N_1$. Since V is arbitrary, therefore $gy_n \to gz$ as $n \to \infty$. Thus we get $fz = gz$. Since $fz - gz = \theta \in \frac{1}{\alpha} P$ for any neighborhood P of $\theta \in X$, we have $(gz - g^2z) - \alpha(fgz - g^2z) \in \alpha \frac{1}{\alpha} P = P$, which in turn implies that $gz - g^2z \in P$. Hence $g^2z = gz$ and so $f(gz) = g(fz) = g^2z = gz$. Therefore $gz = a$ (say) is a common fixed point of f and g in X. Uniqueness of a is evident. $\square$

Theorem 3.15. *Let (X, τ) be a complete locally convex topological vector space. Also let $T, S : X \to X$ be two mappings satisfying (i) $(Tx - Sy) - \alpha(y - Sy) \in \alpha U$ whenever $x - Tx \in U$ and (ii) $(Sx - Ty) - \alpha(y - Ty) \in \alpha U$ whenever $x - Sx \in U$, for any $x, y \in X$ and for any neighborhood U of $\theta \in X$, where $0 < \alpha < \frac{1}{2}$. Then T, S have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be fixed. The sequence $\{x_n\}$ in X is defined by

$$x_n = \begin{cases} Tx_{n-1}, & \text{when } n \text{ is odd} \\ Sx_{n-1}, & \text{when } n \text{ is even} \end{cases}$$

Now let U be any neighborhood of $\theta \in X$. We can assume that U is balanced, absorbing and convex. Now $x_0 - x_1 = x_0 - Tx_0 \in X$, so there exists some $l > 0$ such that $x_0 - x_1 \in \beta U$ whenever $|\beta| \geq l$. Thus we get $x_0 - Tx_0 \in lU$ implying that $(Tx_0 - Sx_1) - \alpha(x_1 - Sx_1) \in \alpha l U$ (using condition (i)). That is, $x_1 - Sx_1 = x_1 - x_2 \in \frac{\alpha}{1-\alpha} lU$, which implies that $(Sx_1 - Tx_2) - \alpha(x_2 - Tx_2) \in \alpha\frac{\alpha}{1-\alpha} lU$ (using condition (ii)). Thus $x_2 - x_3 \in (\frac{\alpha}{1-\alpha})^2 lU$, and proceeding in a similar way we have $x_n - x_{n+1} \in (\frac{\alpha}{1-\alpha})^n lU$ for all $n \in \mathbb{N}$. So by Lemma 3.8 $\{x_n\}$ is a Cauchy sequence in X, since X is complete, there exists $z \in X$ to which $\{x_n\}$ converges. So $\{Tx_{2n}\}_{n\geq 0}$ converges to z and also $\{Sx_{2n-1}\}_{n\in\mathbb{N}}$ converges to z.

Let V be any neighborhood of θ in X. It can be assumed that V is convex, balanced and absorbing. Then there exists $N \in \mathbb{N}$ such that $x_{2n} - x_{2n+1} \in \frac{1-\alpha}{2\alpha}V$ and $Tx_{2n} - z \in \frac{1-\alpha}{2\alpha}V$ whenever $n \geq N$. Therefore, we get $(Tx_{2n} - Sz) - \alpha(z - Sz) \in \alpha\frac{1-\alpha}{2\alpha}$ whenever $n \geq N$ (from condition (i)). Thus $(Tx_{2n} - Sz) - \alpha(z - Tx_{2n} + Tx_{2n} - Sz) \in \frac{1-\alpha}{2}V$, that is, $(1-\alpha)(Tx_{2n} - Sz) \in \alpha(z - Tx_{2n}) + \frac{1-\alpha}{2}V \subset \alpha\frac{1-\alpha}{2\alpha}V + \frac{1-\alpha}{2}V = (1-\alpha)V$. From this we see that $Tx_{2n} - Sz \in V$ whenever $n \geq N$. So $Tx_{2n} \to Sz$ as $n \to \infty$. Since X is Hausdorff, then $Sz = z$. In a similar fashion using condition (ii) we have $Tz = z$. So z is a common fixed point of T and S. Uniqueness of z is also clear. □

Theorem 3.16. *Let (X, τ) be a complete locally convex topological vector space. Let $\{T_n\}$ be a sequence of Kannan-type contractive mappings on X with the same constant $\alpha \in (0, \frac{1}{2})$, which is uniformly convergent to T. Then T is also a Kannan-type contractive mapping with the constant α. Also if $\{u_n\}$ is the sequence of fixed points of $\{T_n\}$ in X then it converges to the fixed point of T.*

Proof. Let V be any neighborhood of $\theta \in X$. Also let K be a neighborhood of θ in X such that $x - Tx \in K$ for some $x \in X$. Now by Lemma 2.17 there exists a closed, balanced, absorbing and convex neighborhood G of $\theta \in X$ such that $x - Tx \in G \subset K$. Now T_n converges uniformly to T. So for each $j \in \mathbb{N}$ we have $Tx - T_n x \in \frac{1}{j}G\ \forall x \in X$, whenever $n \geq N_j$, where $\{N_j\}$ is a strictly increasing sequence in $\mathbb{N}$. Then if $n \geq N_j$ $x - T_n x = (x - Tx) + (Tx - T_n x) \in G + \frac{1}{j}G = (1 + \frac{1}{j})G$ for all $j \in \mathbb{N}$. In particular, for all $j \geq 1$, $x - T_{N_j}x \in (1 + \frac{1}{j})G$. Therefore for each $j \in \mathbb{N}$ $(T_{N_j}x - T_{N_j}y) - \alpha(y - T_{N_j}y) \in \alpha(1 + \frac{1}{j})G$ for all $y \in X$. Now,

$$\begin{aligned} &(Tx - Ty) - \alpha(y - Ty) \\ &\quad= (Tx - T_{N_j}x + T_{N_j}x - T_{N_j}y + T_{N_j}y - Ty) \\ &\quad\quad -\alpha(y - T_{N_j}y + T_{N_j}y - Ty) \qquad (3.5)\\ &\quad= (T_{N_j}x - T_{N_j}y) - \alpha(y - T_{N_j}y) + (Tx - T_{N_j}x) + (1-\alpha)(T_{N_j}y - Ty). \end{aligned}$$

Now there exists $N \in \mathbb{N}$ such that for every $a \in X$ $T_{N_j}a - Ta \in \frac{1}{2-\alpha}V$ if $j \geq N$. Therefore $(Tx - Ty) - \alpha(y - Ty) \in \alpha(1 + \frac{1}{j})G + V$ for all $j \geq N$ implying that $(Tx - Ty) - \alpha(y - Ty) \in \alpha G + V$. Since V is arbitrary it follows that $(Tx - Ty) - \alpha(y - Ty) \in \alpha G \subset \alpha K$. Therefore T is a Kannan-type contractive mapping and hence it has a unique fixed point $u \in X$. Now, $u_n - u = T_n u_n - Tu = T_n u_n - Tu_n + Tu_n - Tu$. But $u - Tu = \theta \in \frac{1}{1+2\alpha}W$ for any neighborhood W of $\theta \in X$. So for all $n \in \mathbb{N}$ $(Tu - Tu_n) - \alpha(u_n - Tu_n) \in \frac{\alpha}{1+2\alpha}W$ implying that $(Tu - Tu_n) - \alpha(T_n u_n - Tu_n) \in \frac{\alpha}{1+2\alpha}W$, which again implies that $Tu - Tu_n \in \alpha(T_n u_n - Tu_n) + \frac{\alpha}{1+2\alpha}W$. Now $T_n \to T$ uniformly as $n \to \infty$ so there exists $N_0 \in \mathbb{N}$ such that $T_n u_n - Tu_n \in \frac{1}{1+2\alpha}W$ whenever $n \geq N_0$. Hence, if $n \geq N_0$ then $u_n - u \in W$. Therefore $\{u_n\}$ converges to the fixed point u of T. □

Theorem 3.17. *Let (X, τ) be a complete locally convex topological vector space. Let $\{T_n\}$ be a sequence of self mappings in X such that T_i and T_j commute for every $i, j \in \mathbb{N}$. Suppose that there exists a sequence of non-negative integers $\{m_n\}$ such that for every neighborhood U of $\theta \in X$, $(T_i^{m_i}x - T_j^{m_j}y) - \alpha(y - T_j^{m_j}y) \in \alpha U$ for all $x, y \in X$ and for every $i, j (i \neq j)$ whenever $x - T_i^{m_i}x \in U$, where $0 < \alpha < \frac{1}{2}$. Then the sequence of mappings $\{T_n\}$ has a unique common fixed point in X.*

Proof. Let us denote $F_i = T_i^{m_i}$ for all $i \in \mathbb{N}$. Then by the given condition we get for every i, j $(i \neq j)$, for all $x, y \in X$ and any neighborhood U of $\theta \in X$, whenever $x - F_i x \in U$, then $(F_i x - F - jy) - \alpha(y - F_j y) \in \alpha U$.

Now let $x_0 \in X$ be fixed. Let us construct a sequence $\{x_n\}$ in X by $x_n = F_n(x_{n-1})$ for all $n \geq 1$. Now let U be a convex, balanced and absorbing neighborhood of $\theta \in X$. So for $x_0 - x_1 \in X$ there exists a $t > 0$ such that $x_0 - x_1 \in \lambda U$ for all scalars λ satisfying $|\lambda| \geq t$. Now, in particular, $x_0 - F_1 x_0 \in tU$, implying that $(F_1 x_0 - F_2 x_1) - \alpha(x_1 - F_2 x_1) \in \alpha tU$, which in turn implies that $x_1 - x_2 = x_1 - F_2 x_1 \in \frac{\alpha}{1-\alpha} tU$. Proceeding in this way we get $x_n - x_{n+1} \in \frac{\alpha}{1-\alpha}^n tU$ for all $n \geq 1$. By applying Lemma 3.8 we get $\{x_n\}$ is Cauchy in X, and since X is complete, it is convergent in X and converges to some $z \in X$. Now for any $n \in \mathbb{N}$ we have $z - F_n z = (z - x_{m+1}) + (F_{m+1}x_m - F_n z)$ for all $m \geq 1$.

Let V be any neighborhood of $\theta \in X$. Assume that V is convex, balanced and absorbing. Let $n \in \mathbb{N}$ be fixed. Then there exists $N \in \mathbb{N}$ such that $N > n$ and for all $m \geq N$ we get $x_m - x_{m+1} \in \frac{1-\alpha}{1+\alpha}V$ and $x_m - z \in \frac{1-\alpha}{1+\alpha}V$. Then, $x_m - F_{m+1}x_m \in \frac{1-\alpha}{1+\alpha}V$ whenever $m \geq N$ implying that $(F_{m+1}x_m - F_n z) - \alpha(z - F_n z) \in \alpha\frac{1-\alpha}{1+\alpha}V$ whenever $m \geq N$. Therefore for all $m \geq N$ we have, $(z - F_n z) - \alpha(z - F_n z) \in \alpha\frac{1-\alpha}{1+\alpha}V + \frac{1-\alpha}{1+\alpha}V$, that is, $z - F_n z \in V$. Since V is an arbitrary neighborhood of $\theta \in X$, we have $F_n z = z$. So, for all $n \geq 1$ $F_n z = z$. Now let $z_0 \in X$ be such that $F_n z_0 = z_0$ $\forall$ $n \in \mathbb{N}$. Then, $z - F_1 z \in \frac{1}{\alpha}K$ for any neighborhood K of θ, implying that $(F_1 z - F_2 z_0) - \alpha(z_0 - F_2 z_0) \in \alpha\frac{1}{\alpha}K$, which implies that $z - z_0 \in K$. Since K is any neighborhood of θ it follows that $z = z_0$. Now we see that for any fixed $i \in \mathbb{N}$, $T_i z = T_i(F_n z) = T_i(T_n^{m_n} z) = T_n^{m_n}(T_i z) = F_n(T_i z)$ [as T_i and T_n commute] for all $n \geq 1$, implying that $T_i z = z$. Therefore for all $i \in \mathbb{N}$ we get $T_i z = z$. Hence z is a unique common fixed point of the sequence of mappings $\{T_n\}$. □

Theorem 3.18. *Let (X, τ) be a complete locally convex topological vector space and $T : X \to X$ be an one-one, continuous and subsequentially convergent mapping. If S is a T-Kannan-type contractive mapping then S has a unique fixed point in X. Also if T is sequentially convergent then for each $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

Proof. Let $x_0 \in X$ and let us construct the sequence $\{x_n\}$ in X by $x_n = Sx_{n-1} = S^n x_0$ for all $n \in \mathbb{N}$.
Let U be a neighborhood of $\theta \in X$. Without loss of generality we can assume that U is convex, balanced and absorbing. So there exists $h > 0$ such that $Tx_0 - Tx_1 \in \gamma U$ whenever $|\gamma| \geq h$. In particular $Tx_0 - Tx_1 \in hU$, so we get $(TSx_0 - TSx_1) - \alpha(Tx_1 - TSx_1) \in \alpha hU$, implying that $Tx_1 - Tx_2 \in \frac{\alpha}{1-\alpha} hU$. Proceeding in this way, we get $Tx_n - Tx_{n+1} \in (\frac{\alpha}{1-\alpha})^n hU$ for all $n \in \mathbb{N}$. Then by Lemma 3.8 we see that $\{Tx_n\}$ is Cauchy sequence in X and since X is complete, there exists $a \in X$ such that $\lim Tx_n = a$. Now since T is subsequentially convergent then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that it is convergent and converges to $b \in X$. Since T is continuous, so $\lim Tx_{n_k} = Tb$, implying that $Tb = a$.

Now let V be a neighborhood of $\theta \in X$. Since $\{Tx_n\}$ is convergent then there exists $N \in \mathbb{N}$ such that $Tx_{n_k} - Tx_{n_k+1} \in \frac{1-\alpha}{1+\alpha} V$ and $Tx_{n_k+1} - Tb \in \frac{1-\alpha}{1+\alpha} V$ whenever $k \geq N$, which implies that

$$\begin{aligned}(Tb - TSb) - \alpha(Tb - TSb) &= Tb - TSx_{n_k} + TSx_{n_k} - TSb - \alpha(Tb - TSb) \\ &= (Tb - Tx_{n_k+1}) \\ &\quad + [TSx_{n_k} - TSb - \alpha(Tb - TSb)] \\ &\in \frac{1-\alpha}{1+\alpha} V + \alpha \frac{1-\alpha}{1+\alpha} V = (1-\alpha)V. \qquad (3.6)\end{aligned}$$

The above equality implies that $Tb - TSb \in V$. Since V is any neighborhood of $\theta \in X$ then we get $TSb = Tb$. Since T is injective then we have $Sb = b$ and therefore b is a fixed point of S in X. Uniqueness of b is also obvious. □

We now cite the following examples in support of our theorems.
Let us consider the sequence of subsets $\{K_m\}_{m \geq 1}$ of $\mathbb{R}^n$, where $K_m = B[\theta, m]$, $m \in \mathbb{N}$. Let us take the space $C_c^\infty(K_m)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact support contained in K_m. Then $C_c^\infty(K_m)$ is a Frechet space, where the topology τ_m is built by the family of seminorms given by, for each $r \in \mathbb{N}$, $||f||_r^{(m)} = \sup_{x \in K_m} |D^r f(x)|$ for all $f \in C_c^\infty(K_m)$. Then from the family of topological spaces $\{(C_c^\infty(K_m), \tau_m) : m \in \mathbb{N}\}$ we have the natural $LF-$space structure on $C_c^\infty(\mathbb{R}^n)$. We know that $C_c^\infty(\mathbb{R}^n)$ with this structure is a complete locally convex TVS but not a Frechet space.

Example 3.19. Let us consider the $LF-$space $X = C_c^\infty(\mathbb{R}^n)$ and a mapping $T : X \to X$ is defined by $Tf = \frac{1}{3} f \ \forall f \in X$. Then clearly it is a contraction map on X and the function $g \in X$ such that $g(x) = 0$ for all $x \in \mathbb{R}^n$ is the unique fixed point of T in X.

Example 3.20. Let $X = C_c^\infty(\mathbb{R}^n)$ be the $LF-$space and $g, f : X \to X$ be two mappings defined by $gx = \frac{1}{4}x$ and $fx = \frac{1}{2}x$ for all $x \in X$. Then clearly f is continuous, g commutes with f, $g(X) \subset f(X)$ and also satisfies the contractive condition for pair of mappings due to Theorem 3.10. We see that the zero function is the unique common fixed point of f and g in X.

Example 3.21. Let us take the $LF-$space $X = C_c^\infty(\mathbb{R}^n)$ and $T : X \to X$ by $Tx = -\frac{1}{2}x$ for all $x \in X$. Then it is a Kannan-type contractive mapping in X for the constant $\alpha = \frac{1}{3}$ and we have $f \in C_c^\infty(\mathbb{R}^n)$, defined by $ft = 0\ \forall t \in \mathbb{R}^n$, is the unique fixed point of T in X.

Example 3.22. Let $X = C_c^\infty(\mathbb{R}^n)$ be the $LF-$space and $g, f : X \to X$ be two mappings defined by $gx = -\frac{1}{10}x$ and $fx = \frac{1}{5}x$ for all $x \in X$. Then f is continuous, g commutes with f, $g(X) \subset f(X)$ and clearly f and g satisfy the contractive condition for pair of mappings given in Theorem 3.14. Here the null function is the unique common fixed point of f and g in X.

4. An application to Ulam-Hyers stability

Let (X, τ) be a locally convex topological vector space and $T : X \to X$ be a given mapping. Let us consider the fixed point equation

$$Tx = x \tag{4.1}$$

and for some neighborhood U of $\theta \in X$

$$v - Tv \in U. \tag{4.2}$$

Any point $v \in X$ which satisfies the above equation (4.2) is called an $U-$solution of the mapping T. We say that the fixed point problem (4.1) is Ulam-Hyers stable in a locally convex topological vector space if there exists a $c > 0$ such that for each absolutely convex neighborhood U of $\theta \in X$ and an $U-$solution $v \in X$, there exists a solution u of the fixed point equation (4.1) such that

$$v - u \in c\,U. \tag{4.3}$$

Theorem 4.1. *Let (X, τ) be a complete locally convex topological vector space and let $T : X \to X$ be a Kannan-type contractive mapping with the constant α. Then the fixed point equation* (4.1) *of T is Ulam-Hyers stable.*

Proof. From Theorem 3.13 we see that T has a unique fixed point in X, that is the fixed point equation (4.1) of T has a unique solution say u. Let U be an arbitrary absolutely convex neighborhood of $\theta \in X$ and v be an $U-$solution that is $v - Tv \in U$.

Since T is Kannan-type contractive mapping with the constant α and $u - Tu = u - u = \theta \in U$ therefore

$$(Tu - Tv) - \alpha(v - Tv) \in \alpha U. \tag{4.4}$$

Now

$$\begin{aligned} v-u=v-Tu &= (v-Tv)+(Tv-Tu) \\ &= (v-Tv)-(Tu-Tv) \\ &= (1-\alpha)(v-Tv)-[(Tu-Tv)-\alpha(v-Tv)] \\ &\in (1-2\alpha)U. \end{aligned} \tag{4.5}$$

Here $c = 1-2\alpha > 0$ and consequently the fixed point problem of T is Ulam-Hyers stable. □

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