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Beale-Kato-Majda's criterion for magneto-hydrodynamic equations with zero viscosity

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Abstract. This paper is concerned with studying the blow-up criterion of smooth solutions to the three dimensional magneto-hydrodynamic equations with zero viscosity. We prove that the smooth solution may be extended by standard energy method, provided the norm of the gradient of velocity in a space much bigger than $\dot{B}^{0}_{\infty,\infty}$. The result obtained in this manuscript improves the former corresponding result.

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1. Introduction

This paper deals with the well-known problem of the breakdown of classical solutions to the incompressible magneto-hydrodynamic equations with zero viscosity in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla\pi - b\cdot\nabla b = 0, \\ \partial_t b - \Delta b + u\cdot\nabla b - b\cdot\nabla u = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is the velocity of the flows, $b = b(x,t)$ is the magnetic field, $\pi = \pi(x,t)$ is the scalar pressure, while u_0 and b_0 are given initial velocity and initial magnetic field with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$ in the sense of distribution.

The system (1.1) describes the macroscopic behavior of electrically conducting incompressible fluids (see [10]). In the turbulent flow regime which occurs when the Reynolds number is very big, we ignore the viscosity of fluids to

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obtain our system (1.1) (see e.g. [9]). In the extremely high electrical conductivity cases, which occur frequently in the cosmical and geophysical problems, we ignore the resistivity term to obtain our system (1.1) (see e.g. [4]).

The local well-posedness of the Cauchy problem of the partially viscous magneto-hydrodynamic systems (1.1) is rather standard and similar to the case of fully viscous magnetohydrodynamic system which is done in [13]. At present, there is no global-in-time existence theory for strong solutions to systems (1.1). In the absence of a well-posedness theory, the development of blowup / non-blowup theory is of major importance for both theoretical and practical purposes (see e.g. [9] and references therein). This system with zero magnetic field b leads to the Euler equations, for which the Beale-Kato-Majda blow up condition

$$\int_0^T \|\nabla \times u(\cdot,\tau)\|_{L^\infty} \, d\tau < \infty \tag{1.2}$$

is well-known (see [1]). A similar condition is known for the MHD equations. For example, Caflisch, Klapper and Steel [3] extended the well-known result of Beale, Kato and Majda on the 3D Euler equation to the 3D ideal MHD equations (i.e. without the resistivity term, Δb, in the left-hand side of $(1.1)_2$) and obtained the endpoint type continuation criterion for smooth solutions (u, b), i.e.

$$\int_0^T \|\nabla \times u(\cdot,\tau)\|_{L^\infty} \, d\tau < \infty \quad \text{and} \quad \int_0^T \|\nabla \times b(\cdot,\tau)\|_{L^\infty} \, d\tau < \infty, \tag{1.3}$$

which implies the smooth solution (u, b) can be extended beyond $t = T$. Yuan [16, 17], Zhang and Liu in [18] studied the continuation or blow-up criterion of the smooth solutions to the MHD system and the ideal MHD system, respectively. They proved that smooth solutions (u, b) can be extended beyond $t = T$ if

$$\int_0^T \|\nabla \times u(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}} \, d\tau < \infty, \tag{1.4}$$

and

$$\int_0^T \|\nabla \times b(\cdot,\tau)\|_{\dot{B}^0_{\infty,\infty}} \, d\tau < \infty, \tag{1.5}$$

for the ideal MHD system or the MHD system, respectively, where $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space.

Motivated by numerical experiments [7, 12] which seem to indicate that the velocity field plays a more important role than the magnetic field in the regularity theory of solutions to the MHD equations, in a lot of work the focus is on the regularity problem of magnetohydrodynamic equations under assumptions only on the velocity field, but not on the magnetic field (see [2, 6, 20, 19] and the references cited therein).

In their paper [5], Gala and Chen established the Beale-Kato-Majda type criterion for the system (1.1) as: the solution (u, b) is smooth up to time T provided that (1.4) holds (see also [9, 18]).

The purpose of this paper is to improve (1.2) in the homogeneous Besov type space $\dot{V}_{\Theta}$ (see the definition in the next section) in order to establish a new blow-up criterion.

Definition 1.1 ([14])**.** Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity that satisfies $\widehat{\varphi} \in C_0^{\infty}\left(B_2 \backslash B_{\frac{1}{2}}\right)$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}\left(2^{-j}\xi\right)$ and $\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j(\xi) = 1$ for any $\xi \neq 0$. The homogeneous Besov space

$$\dot{B}^{s}_{p,q} = \left\{ f \in \mathcal{S}' : \|f\|_{\dot{B}^{s}_{p,q}} < \infty \right\}$$

is introduced by the norm

$$\|f\|_{\dot{B}^{s}_{p,q}} = \left(\sum_{j\in\mathbb{Z}} \left\|2^{js}\varphi_j * f\right\|^q_{L^p} \right)^{\frac{1}{q}}$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

Next we introduce the Banach space of Besov type introduced by Vishik [15], which is wider than $\dot{B}^{0}_{\infty,\infty}$.

Definition 1.2 (homogeneous Vishik's space)**.** Let $\Theta(\alpha) \geq 1$ be a nondecreasing function on $[1, +\infty[$. $\dot{V}_{\Theta} := \left\{ f \in \mathcal{S}' : \|f\|_{\dot{V}_{\Theta}} < \infty \right\}$ is introduced by the norm

$$\|f\|_{\dot{V}_{\Theta}} = \sup_{N=1,2,\ldots} \frac{\left\| \sum\limits_{j=-N}^{N} \varphi_j * f \right\|_{L^{\infty}}}{\Theta(N)}.$$

We note that the space $\dot{V}_{\Theta}$ is a homogeneous version of spaces introduced by Vishik [15]. We also note that

$$L^{\infty}(\mathbb{R}^3) \subset BMO(\mathbb{R}^3) \subset \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3) \subset \dot{V}_{\Theta}(\mathbb{R}^3) \quad \text{if} \quad \Theta(N) \geq N.$$

In order to prove our main result, we need the following logarithmic Sobolev inequality. Ogawa and Taniuchi [11] proved the same inequality for the inhomogeneous space $\dot{V}_{\Theta}$.

Lemma 1.3. *For any $s > \frac{3}{2}$ and $\Theta(\alpha) \geq 1$, there exists a constant $C(s, \Theta) > 0$ such that*

$$\|f\|_{L^{\infty}} \leq C(1 + \|f\|_{\dot{V}_{\Theta}} \Theta(\ln(e + \|f\|_{H^s})), \tag{1.6}$$

for all $f \in H^s(\mathbb{R}^3) \cap \dot{V}_{\Theta}(\mathbb{R}^3)$.

Remark 1.4. In this paper, we shall take $\Theta(\alpha) = \alpha \ln(\alpha + e)$. Then Lemma 1.3 will be

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{\dot{V}_\Theta} \ln(e + \|f\|_{H^s}) \ln(e + \ln(e + \|f\|_{H^s})).$$

Now our result reads as follows.

Theorem 1.5. *Let $T > 0$ and let $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $s \geq 3$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Suppose that (u, b) is a smooth solution to equations (1.1). If (u, b) satisfies the condition*

$$\nabla u \in L^1\left(0, T; \dot{V}_\Theta\right), \tag{1.7}$$

then (u, b) can be extended smoothly beyond $t = T$.

Theorem 1.5 implies that if T is the maximal existence time, then

$$\int_0^T \|\nabla u(\cdot, t)\|_{\dot{V}_\Theta} dt = \infty.$$

Remark 1.6. Since $\dot{V}_\Theta$ is much wider than the Besov space $\dot{B}^0_{\infty,\infty}$, hence Theorem 1.5 improves a regularity result of [5, 9, 18]. Therefore, it is possible to verify that the velocity field plays a more important role than the magnetic field in the regularity theory of solutions of the partially viscous MHD equations.

In this paper, the letter C denotes an absolute constant which may vary at different places.

2. Proof of Theorem 1.5

This section is devoted to the proof of the main Theorem.

Proof. The proof is based on the establishment of a priori estimates under condition (1.7). We will divide the proof of Theorem 1.5 into two steps. One is to establish an estimate for H^1-norm, while the second one is to do the same for H^3-norm.

First of all, for classical solutions to (1.1), one has the following basic energy law

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\nabla b\|_{L^2}^2 = 0.$$

Step 1. H^1 estimates. Multiplying the first equation of (1.1) by Δu, after integration by parts and taking the divergence free property into account, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k u_j \cdot \partial_i u_j dx + \int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k b_j \cdot \partial_i u_j dx \\ &\quad - \int_{\mathbb{R}^3} b_k \cdot \partial_i \partial_k u_j \cdot \partial_i b_j dx. \end{aligned} \tag{2.1}$$

Similarly, multiplying the second equation of (1.1) by Δb, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k b_j \cdot \partial_i b_j dx + \int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k u_j \cdot \partial_i b_j dx \\
&\qquad + \int_{\mathbb{R}^3} b_k \cdot \partial_k \partial_i u_j \cdot \partial_i b_j dx. \qquad (2.2)
\end{aligned}$$

Combining (2.1) and (2.2) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \|\Delta b\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k u_j \cdot \partial_i u_j dx + \int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k b_j \cdot \partial_i u_j dx \\
&\qquad -\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k b_j \cdot \partial_i b_j dx + \int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k u_j \cdot \partial_i b_j dx \\
&\quad \leq C\,\|\nabla u\|_{L^\infty} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2), \qquad (2.3)
\end{aligned}$$

Under (1.7), one concludes that for any small $\epsilon > 0$, there exists $T_0 < T$ such that

$$\int_{T_0}^{T} \|\nabla u(\cdot,\tau)\|_{\dot{V}_\Theta} d\tau < \epsilon. \qquad (2.4)$$

Now, let

$$y(t) = \sup_{T_0 \leq \tau \leq t} \left[\|u(\cdot,\tau)\|_{H^3}^2 + \|b(\cdot,\tau)\|_{H^3}^2\right], \quad \text{for all } T_0 \leq t < T. \qquad (2.5)$$

Step 2. H^3 estimates. We will show how to deduce H^α estimates from H^1. Let $\alpha \geq 1$ be an integer. Taking the operation ∇^α on both sides of (1.1), then multiplying them by $\nabla^\alpha u$ and $\nabla^\alpha b$ respectively, after integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^\alpha u(\cdot,t)\|_{L^2}^2 + \|\nabla^\alpha b(\cdot,t)\|_{L^2}^2\right) + \|\nabla^\alpha \nabla b(\cdot,t)\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \nabla^\alpha (u\cdot\nabla u)\,\nabla^\alpha u dx + \int_{\mathbb{R}^3} \nabla^\alpha (b\cdot\nabla b)\,\nabla^\alpha u dx \\
&\qquad -\int_{\mathbb{R}^3} \nabla^\alpha (u\cdot\nabla b)\,\nabla^\alpha b dx + \int_{\mathbb{R}^3} \nabla^\alpha (b\cdot\nabla u)\,\nabla^\alpha b dx.
\end{aligned}$$

Noting that $\nabla \cdot u = \nabla \cdot b = 0$ and integrating by parts, we write (2.6) as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^{\alpha}u(\cdot,t)\|_{L^2}^2 + \|\nabla^{\alpha}b(\cdot,t)\|_{L^2}^2\right) + \|\nabla^{\alpha}\nabla b(\cdot,t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3}\left[\nabla^{\alpha}(u\cdot\nabla u) - u\cdot\nabla^{\alpha}\nabla u\right]\nabla^{\alpha}u dx \\
&\quad -\int_{\mathbb{R}^3}\left[\nabla^{\alpha}(u\cdot\nabla b) - u\cdot\nabla^{\alpha}\nabla b\right]\nabla^{\alpha}b dx \\
&\quad +\int_{\mathbb{R}^3}\left[\nabla^{\alpha}(b\cdot\nabla b) - b\cdot\nabla^{\alpha}\nabla b\right]\nabla^{\alpha}u dx \\
&\quad +\int_{\mathbb{R}^3}\left[\nabla^{\alpha}(b\cdot\nabla u) - b\cdot\nabla^{\alpha}\nabla u\right]\nabla^{\alpha}b dx \\
&= \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 \qquad (2.6)
\end{aligned}$$

Let $\alpha = 3$ and we will show the estimate of the right hand side of (2.6). Now, we recall the commutator estimate given by Kato and Ponce [8] :

$$\|\Lambda^{\alpha}(fg) - f\Lambda^{\alpha}g\|_{L^2} \leq C\left(\|g\|_{L^\infty}\|f\|_{H^\alpha} + \|\nabla f\|_{L^\infty}\|g\|_{H^{\alpha-1}}\right).$$

The above inequality yields

$$\|\nabla^{\alpha}(u\cdot\nabla u) - u\cdot\nabla^{\alpha}\nabla u\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|u\|_{H^\alpha}, \quad \alpha \geq 1. \qquad (2.7)$$

Hence, it is easy to see that

$$|\Pi_1| \leq C\|\nabla u\|_{L^\infty}\|u\|_{H^3}^2. \qquad (2.8)$$

After integrating by parts, we obtain

$$\begin{aligned}
|\Pi_2| + |\Pi_4| &\leq 4\|\nabla u\|_{L^\infty}\|b\|_{H^3}^2 \\
&\quad +3\left|\int_{\mathbb{R}^3}\nabla^3 b\left[\nabla^2 u\cdot\nabla^2 b\right]dx\right| + 3\left|\int_{\mathbb{R}^3}\nabla^3 b\left(\nabla^3 u\cdot\nabla b\right)dx\right| \\
&\quad +3\left|\int_{\mathbb{R}^3}\nabla^3 b\nabla^2 b.\nabla^2 u dx\right| + 3\left|\int_{\mathbb{R}^3}\nabla^3 b\nabla b.\nabla^3 u dx\right| \qquad (2.9)\\
&\leq 14\|\nabla u\|_{L^\infty}\|b\|_{H^3}^2 + 10\|\nabla u\|_{L^\infty}\left\|\nabla^2 b\right\|_{L^2}\left\|\nabla^4 b\right\|_{L^2} \\
&\quad +4\left\|\nabla^2 u\right\|_{L^4}\|\nabla b\|_{L^4}\left\|\nabla^4 b\right\|_{L^2}.
\end{aligned}$$

By the following interpolation inequalities

$$\begin{aligned}
\|f\|_{L^\infty} &\leq C\left\|\nabla^2 f\right\|_{L^2}^{\frac{3}{4}}\|f\|_{L^2}^{\frac{1}{4}}, \\
\|f\|_{L^4} &\leq C\left\|\nabla^2 f\right\|_{L^2}^{\frac{3}{8}}\|f\|_{L^2}^{\frac{5}{8}}, \\
\|\nabla f\|_{L^2} &\leq C\left\|\nabla^2 f\right\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}, \\
\left\|\nabla^k f\right\|_{L^{\frac{2\alpha}{k}}} &\leq C\|f\|_{L^\infty}^{1-\frac{k}{\alpha}}\|\nabla^\alpha f\|_{L^2}^{\frac{k}{\alpha}}, \quad 0 \leq k \leq \alpha,
\end{aligned}$$

and (2.5), we do the following estimate

$$
\begin{aligned}
& 10\left\|\nabla u\right\|_{L^\infty}\left\|\nabla^2 b\right\|_{L^2}\left\|\nabla^4 b\right\|_{L^2} \\
\leq\ & \frac{1}{8}\left\|\nabla^4 b\right\|_{L^2}^2 + C\left\|\nabla u\right\|_{L^\infty}^2\left\|\nabla^2 b\right\|_{L^2}^2 \\
\leq\ & \frac{1}{8}\left\|\nabla^4 b\right\|_{L^2}^2 + C\left\|\nabla u\right\|_{L^\infty}\left\|\nabla^3 u\right\|_{L^2}^{\frac{3}{4}}\left\|\nabla u\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla^3 b\right\|_{L^2}\left\|\nabla b\right\|_{L^2} \\
\leq\ & \frac{1}{8}\left\|\nabla^4 b\right\|_{L^2}^2 + C\left\|\nabla u\right\|_{L^\infty}\left(\left\|u\right\|_{H^3}^2 + \left\|b\right\|_{H^3}^2\right)^{\frac{7}{8}}\left\|\nabla b\right\|_{L^2}\left\|\nabla u\right\|_{L^2}^{\frac{1}{4}} \\
\leq\ & \frac{1}{8}\left\|\nabla^4 b\right\|_{L^2}^2 + C_0\left\|\nabla u\right\|_{L^\infty}[y(t)]^{\frac{7}{8}}\left(1+y(t)\right)^{\frac{3C\epsilon}{4}}. \qquad (2.10)
\end{aligned}
$$

Here we made use of the Young's inequality

$$ab \leq \delta a^q + C(\delta) b^{q'}$$

for any $a, b, \delta > 0$ and any $q, q' > 1$ $\frac{1}{q} + \frac{1}{q'} = 1$, where $C(\delta) = (\delta q)^{-\frac{q'}{q}}(q')^{-1}$.

Similarly to (2.10), we obtain

$$
\begin{aligned}
& 4\left\|\nabla^2 u\right\|_{L^4}\left\|\nabla b\right\|_{L^4}\left\|\nabla^4 b\right\|_{L^2} \\
\leq\ & 4\left\|\nabla u\right\|_{L^\infty}^{\frac{1}{2}}\left\|\nabla^3 u\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla^3 b\right\|_{L^2}^{\frac{3}{8}}\left\|\nabla b\right\|_{L^2}^{\frac{5}{8}}\left\|\nabla^4 b\right\|_{L^2} \\
\leq\ & \frac{1}{8}\left\|\nabla^4 b\right\|_{L^2}^2 + C\left\|\nabla u\right\|_{L^\infty}\left\|u\right\|_{H^3}\left\|b\right\|_{H^3}^{\frac{3}{4}}\left\|\nabla b\right\|_{L^2}^{\frac{5}{4}} \\
\leq\ & \frac{1}{8}\left\|\nabla^4 b\right\|_{L^2}^2 + C_0\left\|\nabla u\right\|_{L^\infty}[y(t)]^{\frac{7}{8}}\left(1+y(t)\right)^{\frac{5C\epsilon}{4}}.
\end{aligned}
$$

Thus, if we choose $\epsilon > 0$ be small enough such that

$$3C\epsilon \leq 1,$$

then, by (2.9), we derive

$$|\Pi_2| + |\Pi_4| \leq \frac{1}{4}\left\|\nabla^4 b\right\|_{L^2}^2 + C_0\left\|\nabla u\right\|_{L^\infty}\left(1+y(t)\right). \qquad (2.11)$$

It remains to estimate the term Π_3 on the right hand side of (2.6). Integrating by parts, we obtain

$$\left|\int_{\mathbb{R}^3}\nabla^3 u\nabla^2 b\cdot\nabla^2 b dx\right| \leq \left|\int_{\mathbb{R}^3}\nabla^2 u\nabla^3 b\cdot\nabla^2 b dx\right| + \left|\int_{\mathbb{R}^3}\nabla^2 u\nabla^2 b\cdot\nabla^3 b dx\right|.$$

Then

$$
\begin{aligned}
|\Pi_3| \leq\ & \left|\int_{\mathbb{R}^3}\nabla^3 u\nabla^3 b\cdot\nabla b dx\right| + 3\left|\int_{\mathbb{R}^3}\nabla^3 u\nabla^2 b\cdot\nabla^2 b dx\right| \\
& +3\left|\int_{\mathbb{R}^3}\nabla^3 u\nabla b\cdot\nabla^3 b dx\right| \qquad (2.12) \\
\leq\ & \frac{1}{4}\left\|\nabla^4 b\right\|_{L^2}^2 + C\left\|\nabla u\right\|_{L^\infty}\left(1+y(t)\right).
\end{aligned}
$$

Combining (2.6) with (2.8), (2.11), (2.12) and using (2.7), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^3}^2+\|b\|_{H^3}^2\right)+\|\nabla b\|_{H^3}^2\\
&\leq \; C\|\nabla u\|_{L^\infty}(e+y(t))\\
&\leq \; C(1+\|\nabla u\|_{\dot{V}_\Theta}\Theta(\ln(e+y(t)))(e+y(t))
\end{aligned} \tag{2.13}$$

for all $T_0 \leq t < T$. Integrating (2.13) on the time interval $[T_0, t)$ and using (1.7), we have

$$\begin{aligned}
&\ln(e+\|u(\cdot,t)\|_{H^3}^2+\|b(\cdot,t)\|_{H^3}^2)\\
&\leq \; \ln(e+\|u(\cdot,T_0)\|_{H^3}^2+\|b(\cdot,T_0)\|_{H^3}^2)\\
&\quad +C\int_{T_0}^{t}\|\nabla u(\cdot,\tau)\|_{\dot{V}_\Theta}\ln(e+\ln(e+y(\tau)))\ln(e+y(\tau))\,d\tau.
\end{aligned}$$

Then the Gronwall inequality yields that

$$\begin{aligned}
&e+\|u(\cdot,t)\|_{H^3}^2+\|b(\cdot,t)\|_{H^3}^2\\
&\leq \; \left(e+\|u(\cdot,T_0)\|_{H^3}^2+\|b_0(\cdot,T_0)\|_{H^3}^2\right)\\
&\quad \cdot\exp\left\{\exp\left(C\int_{T_0}^{t}\|\nabla u(\cdot,\tau)\|_{\dot{V}_\Theta}\,d\tau\right)\right\}
\end{aligned} \tag{2.14}$$

for all $T_0 \leq t < T$. Noting that the right hand side of (2.14) is independent of t, one concludes that (2.14) is also valid for $t = T$. Hence we have the H^3 regularity for the solution at $t = T$ and the solution can be continued after $t = T$. This completes the proof of Theorem 1.5.

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