Novi Sad J. Math. Vol. 50, No. 1, 2020, 67-88
https://doi.org/10.30755/NSJOM.08942

Solvability of one nonlocal Dirichlet problem for the Poisson equation

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Abstract. In this paper the solvability of a new class of nonlocal boundary value problems for the Poisson equation is studied. These problems are a generalization of the classical Dirichlet boundary value problem. Existence and uniqueness theorems for the considered problem are proved. An integral representation of the solution is established. The notion of the Green's function for the problem under consideration is introduced and an explicit form of this function is constructed. The corresponding spectral issues are also studied, namely eigenfunctions and eigenvalues of the considered problem are found. For one particular case of the problem the completeness of the system of eigenfunctions is proved.

AMS Mathematics Subject Classification (2010): 31B05; 35A09; 35C15

Key words and phrases: nonlocal boundary value problem; Poisson equation; orthogonal matrix; involution; Dirichlet problem; Green's function; eigenfunctions; eigenvalues

1. Introduction

Nonlocal boundary value problems for elliptic equations in which boundary conditions are given in the form of a connection between the values of the unknown function and its derivatives at various points of the domain's boundary are called the problems of Bitsadze-Samarskii type [1]. Numerous applications of nonlocal boundary value problems for elliptic equations to the problems of physics, engineering, and other branches of science are described in detail in [15, 16]. Solvability of nonlocal boundary value problems for elliptic equations is discussed in [2, 5, 10]. Boundary value problems for the second and fourth order elliptic equations with involution, as a special cases of the nonlocal problems, are considered in [9, 12, 13, 14, 17].

In the present paper the solvability conditions of a new class of nonlocal boundary value problems for the Poisson equation is studied.

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ $(n \geq 2)$ be the unit ball, $\partial\Omega$ be the unit sphere and S be a real orthogonal matrix $S \cdot S^T = E$. Suppose also that there exists a natural $l \in \mathbb{N}$ such that $S^l = E$. Note that if $x \in \Omega$, or $s \in \partial\Omega$, then for any $k \in \mathbb{N}$ the following containments $S^k x \in \Omega$, or $S^k s \in \partial\Omega$ hold. This is

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true because the transformation of the space $\mathbb{R}^n$ with matrix S saves the norm $|x|^2 = (x,x) = (S^TSx, x) = (Sx, Sx) = |Sx|^2$.

Let us give some simple examples of such a matrix S.

Example 1.1. Assign to any point $x \in \Omega$ the corresponding point $Sx = -x$. In this case $S = -E$. It's clear that $S \cdot S^T = -E\,(-E) = E$ and therefore $S^2 = E$, which means $l = 2$.

Example 1.2. It is obvious that the transformation performed by the matrix S can also be a rotation in space the $\mathbb{R}^n$. Indeed let $\varphi_i = 2\pi l_i/l$ and $l_i \in \mathbb{N}$. Consider the following matrix $S = C^1_{\varphi_1}C^2_{\varphi_2}\cdots C^{n-2}_{\varphi_{n-2}}$, where

$$C^i_\varphi = \begin{pmatrix} E_i & 0 & 0 & \mathbf{0} \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ \mathbf{0} & 0 & 0 & E_{n-i-2} \end{pmatrix},$$

and E_i is a $i \times i$ unit matrix, $i = \overline{1, n-2}$. Then $S^T = C^{n-2}_{-\varphi_{n-2}}\cdots C^2_{-\varphi_2}C^1_{-\varphi_1}$ and $C^i_\varphi C^i_\psi = C^i_{\varphi+\psi}$ which means

$$SS^T = C^1_{\varphi_1}C^2_{\varphi_2}\cdots C^{n-2}_{\varphi_{n-2}} \cdot C^{n-2}_{-\varphi_{n-2}}\cdots C^2_{-\varphi_2}C^1_{-\varphi_1} = E.$$

Let $a_1, a_2, \ldots, a_l$ be some real numbers, $f(x)$ and $g(x)$ be functions defined on Ω and $\partial\Omega$, respectively. Consider the following problem in Ω:

Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying the following conditions

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1.1}$$

$$\sum_{k=1}^{l} a_k u\left(S^{k-1}x\right)|_{\partial\Omega} = g(s), \quad s \in \partial\Omega. \tag{1.2}$$

When $a_1 \neq 0$, $a_k = 0$, $k = 2, 3, \ldots, l$ we have the classical Dirichlet boundary value problem for the Poisson equation. Note that in the case $n = 2$ the problem (1.1), (1.2) with matrix S, taken from Example 1.2 are studied in [11].

2. Auxiliary statements

To investigate the formulated above problem we need some auxiliary results. Consider the following matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_l \\ a_l & a_1 & \cdots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix},$$

generated by the numbers $a_1, a_2, \ldots, a_l$.

Lemma 2.1. *Let $\lambda_1 = e^{i\frac{2\pi}{l}}$ be a primitive l th root of unity. Then*

$$\det A = \prod_{k=1}^{l} \left(a_1\lambda_1^k + \ldots + a_l\lambda_l^k\right),$$

where $\lambda_k = e^{i\frac{2\pi k}{l}}$, $k = 1, \ldots, l$.

Proof. It is not difficult to see that $\lambda_k = e^{\left(i\frac{2\pi}{l}\right)k} = \lambda_1^k$ and $\lambda_l = \lambda_0 = 1$. Make sure that the number

$$\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k = \sum_{q=1}^{l} a_q\lambda_{q-1}^k, \tag{2.1}$$

where $k = 1, \ldots, l$ is an eigenvalue of the matrix A, and the vector $B_k = \left(1, \lambda_1^k, \ldots, \lambda_{l-1}^k\right)^T$ is an eigenvector corresponding to the eigenvalue μ_k. Since the indices of numbers λ_k can be calculated modulo l, then in the calculations below we can consider the indices of numbers a_k also modulo l. Then, for example, $a_0 = a_l$, $a_{-1} = a_{l-1}$ and $a_{l+1} = a_1$ and so on. We find the element of the m-th row of the following vector

$$C_k \equiv AB_k = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} \begin{pmatrix} \lambda_0^k \\ \lambda_1^k \\ \vdots \\ \lambda_{l-1}^k \end{pmatrix},$$

where $m = 1, \ldots, l$. Since the m th row of the matrix A has the form $\left(a_{2-m}, a_{3-m}, \ldots, a_{l-m+1}\right)$, then

$$\left(AB_k\right)_m = \sum_{j=1}^{l} a_{j-m+1}\lambda_{j-1}^k = \lambda_{m-1}^k \sum_{j=1}^{l} a_{j-m+1}\lambda_{j-m}^k = \mu_k\lambda_{m-1}^k,$$

where the equality $\lambda_m^k = \lambda_s^k\lambda_{m-s}^k$ was used. Therefore $AB_k = C_k = \mu_k B_k$.

Further taking advantage of the equality

$$\det A = \prod_{k=1}^{l} \mu_k = \prod_{k=1}^{l} \left(a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k\right)$$

we obtain the necessary statement. The lemma is proved. □

Example 2.2. Let $l = 3$, then $\lambda_1 = e^{i\frac{2\pi}{3}}$ and therefore $\lambda_k = e^{i\frac{2\pi k}{3}}$. In this case we have

$$\det A = \det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix}$$
$$= \left(a_0 + a_1 e^{i\frac{2\pi}{3}} + a_2 e^{i\frac{4\pi}{3}}\right)\left(a_0 + a_1 e^{i2\frac{2\pi}{3}} + a_2 e^{i2\frac{4\pi}{3}}\right)\left(a_0 + a_1 + a_2\right)$$
$$= \left(a_0 + a_1 e^{i\frac{2\pi}{3}} + a_2 e^{i\frac{4\pi}{3}}\right)\left(a_0 + a_1 e^{i\frac{4\pi}{3}} + a_2 e^{i\frac{2\pi}{3}}\right)\left(a_0 + a_1 + a_2\right)$$
$$= \left(a_0 + a_1 + a_2\right)\left(a_0^2 + a_1^2 + a_2^2 - a_1a_2 - a_0a_1 - a_0a_2\right) = a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2.$$

Lemma 2.3. *Let the numbers μ_k be taken from (2.1) and be such that $\mu_k \neq 0$ for $k = 1, \ldots, l$, where $\lambda_k = e^{i\frac{2\pi k}{l}}$. Then there exists an inverse matrix to the matrix A*

$$A^{-1} \equiv \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \ldots & a_1 \end{pmatrix}^{-1} = \frac{1}{l} M_+ \mathrm{diag}^{-1}\left(\mu_1, \ldots, \mu_l\right) M_-^T,$$

where

$$M_+ = \begin{pmatrix} \lambda_0 & \lambda_0^2 & \ldots & \lambda_0^l \\ \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^l \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{l-1} & \lambda_{l-1}^2 & \ldots & \lambda_{l-1}^l \end{pmatrix}, \quad M_- = \begin{pmatrix} \lambda_0^{-1} & \lambda_0^{-2} & \ldots & \lambda_0^{-l} \\ \lambda_1^{-1} & \lambda_1^{-2} & \ldots & \lambda_1^{-l} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{l-1}^{-1} & \lambda_{l-1}^{-2} & \ldots & \lambda_{l-1}^{-l} \end{pmatrix}.$$

Proof. It is not difficult to see that $M_+ = (B_1, \ldots, B_l)$, where

$$B_k = \left(\lambda_0^k, \lambda_1^k, \ldots, \lambda_{l-1}^k\right)^T$$

is an eigenvector of the matrix A corresponding to the eigenvalue μ_k (see Lemma 2.1). Then $AM_+ = (\mu_1 B_1, \ldots, \mu_l B_l)$ and therefore

$$AM_+ \mathrm{diag}^{-1}\left(\mu_1, \ldots, \mu_l\right) = \left(\mu_1 B_1, \ldots, \mu_l B_l\right) \mathrm{diag}\left(\mu_1^{-1}, \ldots, \mu_l^{-1}\right)$$

$$= (B_1, \ldots, B_l) = M_+,$$

whence

$$AM_+ \mathrm{diag}^{-1}\left(\mu_1, \ldots, \mu_l\right) M_-{}^T = M_+ M_-^T.$$

Calculate the product of the resulting matrices

$$M_+ M_-^T \equiv \left(m_{i,j}\right)_{i,j=\overline{1,l}}.$$

It is not hard to see that

$$m_{i,j} = \sum_{k=1}^{l} \lambda_{i-1}^k \lambda_{j-1}^{-k} = \sum_{k=1}^{l} \left(\frac{\lambda_{i-1}}{\lambda_{j-1}}\right)^k = \sum_{k=1}^{l} \lambda_{i-j}^k, \tag{2.2}$$

where it was taken into account that $\lambda_k / \lambda_j = \lambda_{k-j}$ and $\lambda_0 = e^0 = 1$. It's obvious that $\lambda_{i-j} = \lambda_1^{i-j}$ is l th root of unity for any i and j.

Let us make sure that if λ is a l th root of unity, then

$$\sum_{k=1}^{l} \lambda^k = \begin{cases} l, & \lambda = 1 \\ 0, & \lambda \neq 1 \end{cases}. \tag{2.3}$$

Indeed, for $\lambda = 1$ the equality is obvious, and for $\lambda \neq 1$ we have

$$\lambda + \lambda^2 + \ldots + \lambda^{l-1} + \lambda^l = \frac{1}{1-\lambda}\left(\lambda + \lambda^2 + \ldots + \lambda^{l-1} + \lambda^l\right)(1-\lambda)$$
$$= \frac{1}{1-\lambda}\left(\lambda + \lambda^2 + \ldots + \lambda^{l-1} + \lambda^l - \lambda^2 - \lambda^3 - \ldots - \lambda^l - \lambda^{l+1}\right)$$
$$= \frac{1}{1-\lambda}(\lambda - \lambda) = 0.$$

Thus using (2.2) we obtain

$$m_{i,j} = \begin{cases} l, & i = j \\ 0, & i \neq j \end{cases}$$

and therefore

$$AM_+ \operatorname{diag}\left(\mu_1^{-1}, \ldots, \mu_l^{-1}\right) M_-^T = lE.$$

This proves the lemma. □

Theorem 2.4. *Let*

$$\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0, \quad k = 1, \ldots, l,$$

where $\{\lambda_j : j = 0, \ldots, l-1\}$ are l th roots of unity, then solution of the system of algebraic equations $Ab = g$ can be written as

$$b = (b_i)_{i=\overline{1,l}} = \frac{1}{l}\left(\sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{i-j} g_j\right)_{i=\overline{1,l}}.$$

Proof. Find elements of the inverse matrix, which by Lemma 2.3 exists. Similar to formula (2.2), we can write

$$(2.4) \quad \left(A^{-1}\right)_{i,j=\overline{1,l}} = \frac{1}{l} M_+ \operatorname{diag}^{-1}(\mu_1, \ldots, \mu_l) M_-^T = \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_{i-1}^k}{\mu_k}\lambda_{j-1}^{-k}$$
$$= \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_{i-j}^k}{\mu_k} = \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_k^{i-j}}{\mu_k}.$$

This implies

$$b_i = \left(A^{-1}g\right)_i = \frac{1}{l}\sum_{j=1}^{l} g_j \sum_{k=1}^{l} \frac{\lambda_k^{i-j}}{\mu_k} = \frac{1}{l}\sum_{k=1}^{l} \frac{1}{\mu_k}\sum_{j=1}^{l} \lambda_k^{i-j} g_j.$$

The theorem is proved. □

3. Uniqueness of the problem's solution

To study uniqueness of the solution of the problem (1.1), (1.2) we first give the following statement.

Lemma 3.1. *The operator $I_S u(x) = u(Sx)$ and the Laplace operator Δ commute, i.e., $\Delta I_S u(x) = I_S \Delta u(x)$. The operators*

$$\Lambda = \sum_{i=1}^{n} x_i u_{x_i}(x)$$

and I_S also commute, i.e., $\Lambda I_S u(x) = I_S \Lambda u(x)$ and the equality $\nabla I_S = I_S S^T \nabla$ holds.

Proof. Let us write the orthogonal matrix S in the form $S = (s_{i,j})_{i,j=\overline{1,l}}$. Since

$$\begin{aligned}\frac{\partial}{\partial x_i} I_S u(x) &= \frac{\partial}{\partial x_i} u(Sx) = \frac{\partial}{\partial x_i} u\left((S^1_{row}, x), \dots, (S^n_{row}, x)\right) \\ &= \sum_{j=1}^{n} s_{j,i} I_S u_{x_j}(x) = \left(S^i_{col}, I_S \nabla u(x)\right) = I_S (S^i_{col}, \nabla) u(x),\end{aligned}$$

then

$$\begin{aligned}\Lambda I_S u(x) &= \Lambda u(Sx) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} u(Sx) = \sum_{i=1}^{n} x_i \left(S^i_{col}, I_S \nabla u(x)\right) \\ &= \left(\sum_{i=1}^{n} x_i S^i_{col}, I_S \nabla u(x)\right) = (Sx, I_S \nabla u(x)) = I_S (x, \nabla u(x)) = I_S \Lambda u(x).\end{aligned}$$

Further

$$\frac{\partial^2}{\partial x_i^2} I_S u(x) = \frac{\partial}{\partial x_i} I_S (S^i_{col}, \nabla) u(x) = I_S (S^i_{col}, \nabla)^2 u(x),$$

and therefore

$$\begin{aligned}\Delta I_S u(x) &= \sum_{i=1}^{n} I_S (S^i_{col}, \nabla)^2 u(x) = I_S \left| \left((S^1_{col}, \nabla), \dots, (S^n_{col}, \nabla)\right) \right|^2 u(x) \\ &= I_S \left| S^T \nabla \right|^2 u(x) = I_S (S^T \nabla, S^T \nabla) u(x) = I_S (S S^T \nabla, \nabla) u(x) = I_S \Delta u(x).\end{aligned}$$

Finally,

$$\nabla I_S u(x) = I_S \left((S^1_{col}, \nabla), \dots, (S^n_{col}, \nabla)\right) u(x) = I_S (S^T \nabla) u(x).$$

Lemma is proved. □

Corollary 3.2. *If the function $u(x)$ is harmonic in Ω, then the function $u(Sx) = I_S u(x)$ is also harmonic in Ω.*

Indeed, by the virtue of Lemma 3.1 $\Delta u(x) = 0 \Rightarrow \Delta I_S u(x) = I_S \Delta u(x) = 0$.

Corollary 3.3. *If the function $u(x)$ is harmonic in Ω, then it satisfies in Ω the homogeneous equation*

$$\sum_{k=1}^{l} a_k \Delta u(S^{k-1}x) = 0, \quad x \in \Omega. \tag{3.1}$$

Indeed, by the virtue of Lemma 3.1, for $x \in \Omega$ we have

$$\sum_{k=1}^{l} a_k \Delta u\left(S^{k-1}x\right) = \sum_{k=1}^{l} a_k \Delta I_{S^{k-1}} u\left(x\right) = \sum_{k=1}^{l} a_k I_{S^{k-1}} \Delta u\left(x\right) = 0.$$

The converse assertion is also true.

Lemma 3.4. *Let the function $u(x) \in C^2(\Omega)$ satisfy the homogeneous equation* (3.1)*. Then under the condition* $\det A \neq 0$ *the function $u(x)$ is harmonic in Ω.*

Proof. Let the function $u(x) \in C^2(\Omega)$ satisfy the homogeneous equation (3.1). Denote

$$v(x) = \sum_{k=1}^{l} a_k u(S^{k-1}x). \tag{3.2}$$

It's obvious that $v(x) \in C^2(\Omega)$ and $\Delta v(x) = 0$, $x \in \Omega$, i.e., the function $v(x)$ is harmonic in Ω. By the virtue of Corollary 3.2, the functions $v(S^k x)$ are also harmonic in Ω. On the other hand, from (3.2), due to the condition $S^l = E$, the following equalities hold

$$\begin{aligned} v(Sx) &= a_l u(x) + a_1 u(Sx) + \ldots + a_{l-1} u(S^{l-1}x) \\ v(S^2x) &= a_{l-1} u(x) + a_l u(Sx) + \ldots + a_{l-2} u(S^{l-1}x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v(S^{l-1}x) &= a_2 u(x) + a_3 u(Sx) + \ldots + a_1 u(S^{l-1}x). \end{aligned} \tag{3.3}$$

So for the functions $u(x), u(Sx), \ldots, u(S^{l-1}x)$ we obtain a system of algebraic equations (3.2), (3.3) with the matrix A

$$\begin{pmatrix} v(x) \\ v(Sx) \\ \vdots \\ v(S^{l-1}x) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u(x) \\ u(Sx) \\ \vdots \\ u(S^{l-1}x) \end{pmatrix}.$$

By the lemma's conditions the determinant of this system does not vanish. We make use of Theorem 2.4 for $b = \left(u(x), u(Sx), \ldots, u(S^{l-1}x)\right)^T$ and $g = \left(v(x), v(Sx), \ldots, v(S^{l-1}x)\right)^T$. From Theorem 2.4 when $i = 1$ it follows that

$$u(x) = b_1 = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{1-j} g_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{1-j} v(S^{j-1}x)$$

$$= \sum_{j=1}^{l} v(S^{j-1}x) \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1}\mu_k},$$

where according to (2.1) $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k$ and $\lambda_k = e^{\left(i\frac{2\pi}{l}\right)k} = \lambda_1^k$. If we denote

$$b_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1}\mu_k}, \quad j = 1, 2, \ldots, l,$$

then we obtain

$$u(x) = \sum_{j=1}^{l} b_j v(S^{j-1}x) = b_1 v(x) + b_2 v(Sx) + \ldots + b_l v(S^{l-1}x). \tag{3.4}$$

As noted above, the functions $v(S^k x)$, where $k = 0, 1, \ldots, l-1$ are harmonic in Ω, and hence the function $u(x)$ from (3.4) is also harmonic in Ω. The lemma is proved. □

According to Lemma 3.4, the following statement holds.

Theorem 3.5. *Let for all $k = 1, 2, \ldots, l$ the conditions $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ hold. If a solution of the problem* (1.1), (1.2) *exists, then it is unique.*

Proof. Let us prove that the homogeneous problem (1.1), (1.2) has only the zero solution. In this case the solution of the inhomogeneous problem (1.1), (1.2) is unique. Let $u(x)$ be a solution of the homogeneous problem (1.1), (1.2). As we already noted, if the function $u(x)$ is harmonic, then the functions $u(S^{k-1}x)$, $k = 2, 3, \ldots, l$ are also harmonic. Then the function $u(x)$ satisfies the equation (3.1). Consider the function

$$v(x) = \sum_{k=1}^{l} a_k u(S^{k-1}x), \quad x \in \Omega.$$

It's obvious that $v(x) \in C^2(\Omega) \cap C(\bar{\Omega})$. If $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$, where $k = 1, 2, \ldots, l$, then by Lemma 2.1 $\det A \neq 0$. Then, by Lemma 3.4, the function $v(x)$ is harmonic in the domain Ω and therefore it is a solution to the following Dirichlet problem

$$\Delta v(x) = 0,\ x \in \Omega; \quad v(x)|_{\partial\Omega} = 0.$$

By the virtue of the uniqueness of the Dirichlet problem, we have $v(x) \equiv 0$, $x \in \bar{\Omega}$. Then the function $u(x)$ which is determined by (3.4) is identically equal to zero, i.e. $u(x) \equiv 0$, $x \in \bar{\Omega}$. The theorem is proved. □

Remark 3.6. If $\mu_k = 0$ for some $k = 1, 2, \ldots, l$, then the homogeneous problem can have infinitely many solutions.

4. Existence of the problem's solution

In this section we investigate the existence of a solution to the main problem (1.1), (1.2). Let

$$P(x,y)=\frac{1}{\omega_n}\frac{1-|x|^2}{|x-y|^n}$$

be the Poisson kernel, ω_n be the surface area of the unit sphere, $G(x,y)$ be the Green's function of the Dirichlet problem in Ω, which can be represented as (see, for example, [3])

$$G(x,y)=\frac{1}{\omega_n}\left[E(x,y)-E\left(x|y|,\frac{y}{|y|}\right)\right], \tag{4.1}$$

where $E(x,y)$ is the elementary solution of the Laplace equation

$$E(x,y)=\begin{cases}-\ln|x-y|, & n=2\\ \frac{1}{n-2}|x-y|^{2-n}, & n\geq 3\end{cases}.$$

Let us prove some auxiliary assertions.

Lemma 4.1. *Let the function $g(x)$ be continuous on $\partial\Omega$. Then for any $k\in\mathbb{N}$ the following equalities are true*

$$\int_{\partial\Omega} g(S^k y)\,ds_y=\int_{\partial\Omega} g(y)\,ds_y,\quad \int_{\Omega} g(S^k y)\,dy=\int_{\Omega} g(y)\,dy.$$

Proof. Let's prove the first equality. Let the function $w(x)$ be a solution of the Dirichlet problem for the Laplace equation in Ω with the boundary condition $w(x)=g(x)$ on $\partial\Omega$. Then the function $w(S^k x)$ is a solution of the Dirichlet problem for the Laplace equation in Ω (Corollary 3.2) with the boundary condition $w(S^k x)=g(S^k x)$ on $\partial\Omega$. It is known that the solutions of these problems are represented by Poisson integrals

$$w(x)=\int_{\partial\Omega} P(x,y)g(y)\,ds_y,\quad w(S^k x)=\int_{\partial\Omega} P(x,y)g(S^k y)\,ds_y.$$

Since

$$P(0,y)=\frac{1}{\omega_n}\frac{1}{|y|^n}=\frac{1}{\omega_n},$$

where $y\in\partial\Omega$, then

$$\frac{1}{\omega_n}\int_{\partial\Omega} g(y)\,ds_y=w(0)=\frac{1}{\omega_n}\int_{\partial\Omega} g(S^k y)\,ds_y.$$

This implies the required equality. The second equality follows from the rule of changing variables in a multiple integral

$$\int_{\Omega} g(Sy)\,dy=\int_{\Omega} g(z)|\det S^T|\,dz=\int_{\Omega} g(y)\,dy.$$

The lemma is proved. □

Lemma 4.2. *Let $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$, where $k = 1, \ldots, l$, then the matrix A^{-1} has a structure of the matrix A*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}^{-1} = \begin{pmatrix} b_1 & b_2 & \ldots & b_l \\ b_l & b_1 & \ldots & b_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_2 & b_3 & \cdots & b_1 \end{pmatrix},$$

where, similar to formula (3.4),

$$b_j = \frac{1}{l}\sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1}\mu_k} \tag{4.2}$$

for $j = 1, 2, \ldots, l$, and μ_k are defined from (2.1)*. In addition, if $k = 1, 2, \ldots, l$ the equality*

$$\mu_k(b) = 1/\mu_k(a)$$

holds, where $\mu_k(a) = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k$ and $\mu_k(b) = b_1\lambda_0^k + \ldots + b_l\lambda_{l-1}^k$.

Proof. It is clear that the indices of numbers a_k can be considered modulo l. Then, as it is easy to see, the matrix A can be written as $A = (a_{j-i+1})_{i,j=\overline{1,l}}$. By the formula (2.4) from Theorem 2.4 we can find

$$\left(A^{-1}\right)_{i,j} = \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_k^{i-j}}{\mu_k} = \frac{1}{l}\sum_{k=1}^{l} \frac{1}{\mu_k\lambda_k^{j-i}}.$$

Since the indices i and j of the elements $\left(A^{-1}\right)_{i,j}$ are the powers of numbers λ_k, then they can be calculated modulo l and the following equality is true

$$\left(A^{-1}\right)_{i,j} = \frac{1}{l}\sum_{k=1}^{l} \frac{1}{\mu_k\lambda_k^{j-i+1-1}} = b_{j-i+1},$$

where

$$b_j = \frac{1}{l}\sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1}\mu_k}.$$

Therefore we have $A^{-1} = (b_{j-i+1})_{i,j=\overline{1,l}}$.

Let us calculate $\mu_k(b)$. Bearing in mind (2.3) we can find

$$\mu_k(b) = \sum_{j=1}^{l} b_j\lambda_{j-1}^k = \frac{1}{l}\sum_{j=1}^{l}\lambda_{j-1}^k \sum_{p=1}^{l}\frac{1}{\lambda_p^{j-1}\mu_p(a)} = \frac{1}{l}\sum_{p=1}^{l}\frac{1}{\mu_p(a)}\sum_{j=1}^{l}\frac{\lambda_{j-1}^k}{\lambda_p^{j-1}}$$
$$= \sum_{p=1}^{l}\frac{1}{\mu_p(a)}\frac{1}{l}\sum_{j=1}^{l}\frac{\lambda_{j-1}^k}{\lambda_{j-1}^p} = \sum_{p=1}^{l}\frac{1}{\mu_p(a)}\frac{1}{l}\sum_{j=1}^{l}\lambda_{j-1}^{k-p} = \frac{1}{\mu_k(a)}.$$

The lemma is proved. ∎

Remark 4.3. Since $\lambda_k^l = 1$, where $k = 1, \dots, l$, then the equalities

$$\mu_l(a) = a_1\lambda_0^l + \ldots + a_l\lambda_{l-1}^l = \sum_{j=1}^{l} a_j, \quad \mu_l(b) = b_1\lambda_0^l + \ldots + b_l\lambda_{l-1}^l = \sum_{i=1}^{l} b_i,$$

are true, so by Lemma 4.2

$$\sum_{j=1}^{l} a_j \sum_{i=1}^{l} b_i = 1.$$

The following statement concerning the problem (1.1), (1.2) is true.

Theorem 4.4. *Let the numbers* $\{a_k : k = 1, \dots, l\}$ *be such that* $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ *for* $k = 1, \dots, l$, *where* $\{\lambda_k\}$ *are* l *th roots of unity and* $f \in C^\lambda(\bar{\Omega})$, $g \in C^{\lambda+2}(\partial\Omega)$, $0 < \lambda < 1$. *Then the solution to the problem* (1.1), (1.2) *exists, is unique, belongs to the class* $C^{\lambda+2}(\bar{\Omega})$ *and can be represented in the form*

$$u(x) = \int_\Omega G_S(x, y) f(y)\, dy + \int_{\partial\Omega} P_S(x, y) g(y)\, ds_y, \tag{4.3}$$

where

$$G_S(x, y) = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G\left(S^{q-1}x, \left(S^{k-1}\right)^T y\right), \tag{4.4}$$

$$P_S(x, y) = \sum_{q=1}^{l} b_q P(S^{q-1}x, y),$$

the function $G(x, y)$ *is defined in* (4.1), *and*

$$b_q = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{q-1}\mu_k}, \quad q = 1, \dots, l,$$

is defined in (4.2).

Proof. For the function $v(x)$ in the domain Ω consider the following Dirichlet boundary value problem

$$-\Delta v(x) = F(x),\ x \in \Omega; \quad v(x)\Big|_{\partial\Omega} = g(s),\ s \in \partial\Omega, \tag{4.5}$$

where

$$F(x) \equiv \sum_{k=1}^{l} a_k f(S^{k-1}x).$$

It is clear that if $f(x) \in C^\lambda(\bar{\Omega})$, then $F(x) \in C^\lambda(\bar{\Omega})$ and hence when $g(x) \in C^{\lambda+2}(\partial\Omega)$ the solution of the Dirichlet problem (4.5) exists, is unique, and belongs to the class $C^{\lambda+2}(\bar{\Omega})$ [4]. It is also known (see, for example, [3, p. 35]), that with the given functions $g(x)$ and

$$F(x) = \sum_{k=1}^{l} a_k f(S^{k-1}x)$$

the solution of the problem (4.5) can be represented in the form

$$v(x) = \sum_{k=1}^{l} a_k \int_{\Omega} G(x,y) f(S^{k-1}y)\,dy + \int_{\partial\Omega} P(x,y) g(y)\,ds_y. \tag{4.6}$$

Consider the vector $V = \big(v(x), v(Sx), \ldots, v(S^{l-1}x)\big)^T$. By Lemma 4.2, the matrix A^{-1} has a structure of the matrix A. Therefore, from the vector equality $U = A^{-1}V$, we can define the vector $U = \big(u(x), u(Sx), \ldots, u(S^{l-1}x)\big)^T$. Since $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$, then by Lemma 2.1 $\det A \neq 0$ and therefore $\det A^{-1} \neq 0$. Because $AU = V$, the function $u(x)$ is uniquely determined through the function $v(x)$ from (4.6) by the formula

$$u(x) = \sum_{j=1}^{l} b_j v(S^{j-1}x), \tag{4.7}$$

where b_j are obtained from (4.2). Let us verify that the function $u(x)$, determined from (4.7), is a solution of the problem (1.1), (1.2). Indeed $f \in C^{\lambda}(\bar{\Omega})$, $g \in C^{\lambda+2}(\partial\Omega) \Rightarrow v \in C^{\lambda+2}(\bar{\Omega}) \Rightarrow u \in C^{\lambda+2}(\bar{\Omega})$. Therefore, according to Lemma 3.1 and equality (4.5), in the domain Ω we have

$$\begin{aligned}
-\Delta u(x) &= -\sum_{j=1}^{l} b_j \Delta v(S^{j-1}x) = -\sum_{j=1}^{l} b_j I_{S^{j-1}} \Delta v(x) \\
&= \sum_{j=1}^{l} b_j I_{S^{j-1}} (-\Delta) v(x) = \sum_{j=1}^{l} b_j I_{S^{j-1}} \sum_{k=1}^{l} a_q f(S^{q-1}x).
\end{aligned}$$

We investigate the functions

$$u_j(x) = I_{S^{j-1}} \left(\sum_{q=1}^{l} a_q f(S^{q-1}x) \right), \quad j = 1, 2, \ldots, l.$$

If $j = 1$ we get

$$u_1(x) = \sum_{q=1}^{l} a_q f(S^{q-1}x).$$

Further, when $j = 2$ we have

$$\begin{aligned}
u_2(x) &= I_S \big(a_1 f(x) + a_2 f(Sx) + \ldots + a_l f(S^{l-1}x)\big) \\
&= a_1 f(Sx) + a_2 f(S^2x) + \ldots + a_{l-1} f(S^{l-1}x) + a_l f(x).
\end{aligned}$$

If we assume here $a_l = a_0$, then we get

$$u_2(x) = a_0 f(x) + \sum_{q=1}^{l-1} a_q f(S^q x) = \sum_{q=1}^{l} a_{q-1} f(S^{q-1}x).$$

Continuing in this way and assuming $a_{-q} = a_{l-q}$, by the induction, we get

$$u_j(x) = \sum_{q=1}^{l} a_{q-j+1} f(S^{q-1}x), j = 1, 2, \ldots, l.$$

Thus

$$-\Delta u(x) = \sum_{j=1}^{l} b_j \sum_{q=1}^{l} a_{q-j+1} f(S^{q-1}x) = \sum_{q=1}^{l} f(S^{q-1}x) \sum_{j=1}^{l} a_{q-j+1} b_j.$$

Calculate the inner sum in the obtained equality. To do this, we change the indices $p = q - j + 1 \Rightarrow j = q - p + 1$ and, remembering the meanings of b_j from (4.2) and μ_k from (2.1), we calculate

$$\sum_{j=1}^{l} a_{q-j+1} b_j = \sum_{p=1}^{l} a_p b_{q-p+1} = \frac{1}{l} \sum_{p=1}^{l} a_p \sum_{k=1}^{l} \frac{1}{\lambda_k^{q-p} \mu_k}$$

$$= \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{q-1} \mu_k} \sum_{p=1}^{l} \lambda_{p-1}^{k} a_p = \frac{1}{l} \sum_{k=1}^{l} \frac{\mu_k}{\lambda_k^{q-1} \mu_k} = \frac{1}{l} \sum_{k=1}^{l} \lambda_k^{1-q} = \frac{1}{l} \sum_{k=1}^{l} \lambda_{1-q}^{k}.$$

Taking into account (2.3), we get

$$\sum_{p=1}^{l} a_p b_{q-p+1} = \begin{cases} 1, & q = 1 \\ 0, & q \neq 1 \end{cases} \tag{4.8}$$

and then equation (1.1) is satisfied

$$-\Delta u(x) = \sum_{q=1}^{l} f(S^{q-1}x) \sum_{j=1}^{l} a_{q-j+1} b_j = f(x).$$

Next, we check the boundary conditions of the problem (1.1), (1.2). For $s \in \partial\Omega$ from the equality (4.7) we obtain

$$u(x)\Big|_{\partial\Omega} = \sum_{q=1}^{l} b_q v(S^{q-1}x)\Big|_{\partial\Omega} = \sum_{q=1}^{l} b_q g(S^{q-1}s),$$

$$u(Sx)\Big|_{\partial\Omega} = I_S u(x)\Big|_{\partial\Omega} = I_S \left(\sum_{q=1}^{l} b_q g(S^{q-1}s) \right) = \sum_{q=1}^{l} b_q g(S^q s)$$

$$= b_l g(s) + \sum_{q=1}^{l-1} b_q g(S^q s) = b_0 g(s) + \sum_{q=2}^{l} b_{q-1} g(S^{q-1}s) = \sum_{q=1}^{l} b_{q-1} g(S^{q-1}s).$$

Then by the induction

$$u(S^{k-1}x)\Big|_{\partial\Omega} = \sum_{q=1}^{l} b_{q-k+1} g(S^{q-1}s), \quad k = 1, 2, \ldots, l.$$

Hence,

$$\sum_{k=1}^{l} a_k u(S^{k-1}x)\Big|_{\partial\Omega} = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_{q-k+1} g(S^{q-1}s) = \sum_{q=1}^{l} g(S^{q-1}s) \sum_{k=1}^{l} a_k b_{q-k+1}.$$

Using (4.8) we finally obtain

$$\sum_{k=1}^{l} a_k u(S^{k-1}x)|_{\partial\Omega} = g(s),$$

i.e. the boundary condition (1.2) is also satisfied.

Further, substituting the representation of the function $v(x)$ given by (4.6) to the equality (4.7) and taking into account formulas (4.4) we obtain

$$u(x) = \sum_{q=1}^{l} b_q v(S^{q-1}x) = \sum_{q=1}^{l} b_q \left(\sum_{k=1}^{l} a_k \int_{\partial\Omega} G(S^{q-1}x, y) f(S^{k-1}y)\, dy \right)$$
$$+ \sum_{q=1}^{l} b_q \int_{\partial\Omega} P(S^{q-1}x, y) g(y)\, ds_y = \int_{\Omega} \left[\sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G(S^{q-1}x, \left(S^{k-1}\right)^T y) \right] f(y)\, dy$$
$$+ \int_{\partial\Omega} \left[\sum_{q=1}^{l} b_q P(S^{q-1}x, y) \right] g(y)\, ds_y = \int_{\Omega} G_S(x, y) f(y)\, dy + \int_{\partial\Omega} P_S(x, y) g(y)\, ds_y.$$

Thus, representation (4.3) for the function $u(x)$ holds. The theorem is proved. $\square$

Remark 4.5. Since $S^T = S^{l-1} = S^{-1}$, then

$$\left(S^{k-1}\right)^T = \left(S^T\right)^{k-1} = \left(S^{-1}\right)^{k-1} = S^{-k+1} = S^{l-k+1}$$

and therefore, under the conditions of Theorem 4.4, the Green's function of the problem (1.1), (1.2) can be represented in the form

$$G_S(x, y) = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G(S^{q-1}x, S^{l-k+1}y).$$

Example 4.6. Consider a particular case of the problem (1.1), (1.2), when $f(x) = -x_i$ and $g(s) = s_j^2$, $i, j = 1, \ldots, n$. The auxiliary problem (4.5) has the form

$$\Delta v(x) = (a_1 - a_2)x_i,\ x \in \Omega;\ v\big|_{\partial\Omega} = s_j^2.$$

In this case it is better to use the results of [7]. It is not hard to find

$$v(x) = x_j^2 + \left(1 - |x|^2\right)\left(\frac{1}{n} - \frac{a_1 - a_2}{2(n+2)} x_i\right)$$

and therefore the problem's solution according to (6.5) is

$$\begin{aligned} u(x) = {} & \frac{a_1}{a_1^2 - a_2^2}\left(x_j^2 + \left(1 - |x|^2\right)\left(\frac{1}{n} - \frac{a_1 - a_2}{2(n+2)} x_i\right)\right) \\ & - \frac{a_2}{a_1^2 - a_2^2}\left(x_j^2 + \left(1 - |x|^2\right)\left(\frac{1}{n} + \frac{a_1 - a_2}{2(n+2)} x_i\right)\right) \\ & = \frac{x_j^2}{a_1 + a_2} + \left(1 - |x|^2\right)\left(\frac{1}{n(a_1 + a_2)} - \frac{x_i}{2(n+2)}\right). \end{aligned}$$

Let us check this solution. Obviously the boundary condition (1.2) is fulfilled

$$a_1 u(x) + a_2 u(-x)\big|_{\partial\Omega} = s_j^2,$$

as well as the equation (1.1)

$$\Delta u(x) = \frac{2}{a_1 + a_2} - \frac{\Delta |x|^2}{n(a_1 + a_2)} + \frac{\Delta x_i |x|^2}{2(n+2)} = x_i.$$

Here is used the equality $\Delta\left(|x|^{2m} P_s(x)\right) = 2m(2m+2s+n-2)|x|^{2m-2} P_s(x)$, where $P_s(x)$ is a homogeneous harmonic polynomial of degree s (see [7]).

5. The Green's function of the nonlocal Dirichlet problem

As in the classical case for the problem (1.1), (1.2) we can introduce the concept of Green's function.

Definition 5.1. Green's function of the problem (1.1), (1.2) is the function $G_S(x, y)$ that satisfies the conditions:

1) the function $G_S(x, y)$ is harmonic in $x \in \Omega$ and $y \in \Omega$ if S-orbits of the points x and y do not intersect

$$\left\{S^k x : k = 1, \ldots, l\right\} \cap \left\{S^k y : k = 1, \ldots, l\right\} = \varnothing \tag{5.1}$$

2) for the function $G_S(x, y)$ the equalities

$$\sum_{k=1}^{l} a_k G_S(S^{k-1} x, y) = 0, \quad \sum_{k=1}^{l} a_k G_S(x, \left(S^{k-1}\right)^T y) = 0, \quad x \in \partial\Omega,\ y \in \Omega$$

hold.

From Theorem 4.4 or rather from the representation (4.4) the following assertion follows.

Theorem 5.2. *Let the numbers $\{a_k : k = 1, \dots, l\}$ be such that $\mu_k = a_1\lambda_0^k + \dots + a_l\lambda_{l-1}^k \neq 0$ for $k = 1, \dots, l$, where $\{\lambda_k\}$ are l th roots of unity. Then Green's function of the problem* (1.1), (1.2) *exists, is unique and can be represented in the form*

$$G_S(x,y) = \sum_{q=1}^{l} b_q \sum_{k=1}^{l} a_k G\left(S^{q-1}x, \left(S^{k-1}\right)^T y\right), \tag{5.2}$$

where the numbers b_q, $q = 1, \dots, l$ are taken from (4.2), *and $G(x,y)$ is Green's function of the Dirichlet problem* (4.1). *The following symmetry of Green's function takes place $G_S(x,y) = G_{S^T}(y,x)$.*

Proof. We show that the function $G_S(x,y)$ satisfies conditions 1) and 2) from Definition 5.1. It is known that Green's function of the Dirichlet problem $G(x,y)$ from (4.1) is harmonic in $x \in \Omega$ and $y \in \Omega$ if $x \neq y$. It is not difficult to see that if

$$\left\{S^k x : k = 1, \dots, l\right\} \cap \left\{S^k y : k = 1, \dots, l\right\} = \varnothing,$$

then the inequalities $S^{q-1}x \neq S^{l+1-p}y$ take place for any $q, p = 1, 2, \dots, l$, which means

$$S^{q-1}x \neq \left(S^T\right)^{p-1} y = \left(S^{p-1}\right)^T y.$$

Therefore the function $G\left(S^{q-1}x, \left(S^{p-1}\right)^T y\right)$ for $q, p = 1, 2, \dots, l$ is also harmonic in $x \in \Omega$ and $y \in \Omega$, satisfying the condition (5.1). So the function $G_S(x,y)$ is also harmonic in $x \in \Omega$ and $y \in \Omega$ subject to the condition (5.1).

Check the condition 2). Because

$$G(x,y)|_{x\in\partial\Omega \vee y\in\partial\Omega} = 0 \Rightarrow G\left(S^{q-1}x, \left(S^{k-1}\right)^T y\right)\Big|_{x\in\partial\Omega \vee y\in\partial\Omega} = 0,$$

then the following conditions hold

$$G_S(x,y)|_{x\in\partial\Omega} = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G(S^{q-1}x, \left(S^{k-1}\right)^T y)\Big|_{x\in\partial\Omega} = 0,$$

$$G_S(x,y)|_{y\in\partial\Omega} = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G(S^{q-1}x, \left(S^{k-1}\right)^T y)\Big|_{y\in\partial\Omega} = 0.$$

Further, let $x \in \partial\Omega$. Then by the virtue of (4.8)

$$\sum_{j=1}^{l} a_j G_S(S^{j-1}x, y) = \sum_{j=1}^{l} a_j I_{S^{j-1}} G_S(x, y) =$$

$$= \sum_{j=1}^{l} a_j I_{S^{j-1}} \left(\sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G\Big(S^{q-1}x, \left(S^{k-1}\right)^T y\Big) \right)$$

$$= \sum_{k=1}^{l} a_k \sum_{j=1}^{l} a_j \left(\sum_{q=1}^{l} b_{q-j+1} G\Big(S^{q-1}x, \left(S^{k-1}\right)^T y\Big) \right)$$

$$= \sum_{k=1}^{l} a_k \sum_{j=1}^{l} a_j \left(\sum_{q=1}^{l} b_{q-j+1} G\Big(S^{q-1}x, \left(S^{k-1}\right)^T y\Big) \right)$$

$$= \sum_{k=1}^{l} a_k \sum_{q=1}^{l} G\Big(S^{q-1}x, \left(S^{k-1}\right)^T y\Big) \sum_{j=1}^{l} a_j b_{q-j+1} = \sum_{k=1}^{l} a_k G\Big(x, \left(S^{k-1}\right)^T y\Big).$$

Therefore we have

$$G\left(x, \left(S^{k-1}\right)^T y\right)\Big|_{x\in\partial\Omega} = 0 \Rightarrow \sum_{j=1}^{l} a_j\, G_S\left(S^{j-1}x, y\right)\big|_{x\in\partial\Omega} = 0.$$

Similarly, if $y \in \partial\Omega$, then we can get

$$\sum_{j=1}^{l} a_j G_S(S^{j-1}x, y)\Bigg|_{y\in\partial\Omega} = \sum_{k=1}^{l} a_k G\left(x, \left(S^{k-1}\right)^T y\right)\Bigg|_{y\in\partial\Omega}$$

$$= \sum_{k=1}^{l} a_k G(S^{k-1}x, y)\Bigg|_{y\in\partial\Omega} = 0.$$

Property 2) is proved.

Finally, since the function $G(x, y)$ is symmetric and the equalities

$$\omega_n G(x, S^k y) = E(x, S^k y) - E\left(x/|x|, |x|S^k y\right) = E\left(\left(S^k\right)^T x, y\right)$$
$$- E\left(\left(S^k\right)^T x/\left|\left(S^k\right)^T x\right|, \left|\left(S^k\right)^T x\right| y\right) = \omega_n G\left(\left(S^k\right)^T x, y\right),$$

are fulfilled, then we get

$$G_S(x, y) = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G_D\left(S^{q-1}x, \left(S^{k-1}\right)^T y\right)$$

$$= \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G_D\left(S^{k-1}x, \left(S^{q-1}\right)^T y\right) = \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G_D\left(\left(S^{q-1}\right)^T y, S^{k-1}x\right)$$

$$= \sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q G_D\left(\left(S^T\right)^{q-1} y, \left(\left(S^T\right)^{k-1}\right)^T x\right) = G_{S^T}(y, x).$$

The theorem is proved. □

Remark 5.3. In the case $l=2$, because of the equality $S^T=S$ Green's function $G_S(x,y)$ is symmetric.

6. Eigenfunctions and eigenvalues of the nonlocal Dirichlet problem

Consider the following spectral problem

$$-\Delta u(x)=\lambda u(x),\ x\in\Omega, \tag{6.1}$$

$$\sum_{k=1}^{l} a_k u\left(S^{k-1}x\right)|_{\partial\Omega}=0. \tag{6.2}$$

We call a function $u(x)$ which belongs to to the class $u(x)\in C^2(\Omega)\cap C(\bar{\Omega})$ and which satisfies the conditions (6.1), (6.2) in the classical sense the solution to the problem (6.1), (6.2).

Consider the function

$$v(x)=\sum_{k=1}^{l} a_k u\left(S^{k-1}x\right).$$

Let us apply the Laplace operator to this function. We get

$$\Delta v(x)=\sum_{k=1}^{l} a_k\Delta u\left(S^{k-1}x\right)=-\lambda\sum_{k=1}^{l} a_k u\left(S^{k-1}x\right)=-\lambda v(x).$$

In addition, from (6.2) it follows

$$v(x)|_{\partial\Omega}=\sum_{k=1}^{l} a_k u\left(S^{k-1}x\right)\Big|_{\partial\Omega}=0.$$

Thus, for the function $v(x)$ we obtain the spectral problem

$$-\Delta v(x)=\lambda v(x)\,,\ x\in\Omega;\quad v(x)|_{\partial\Omega}=0. \tag{6.3}$$

It is known (see, for example, [3]) that problem (6.3) has a complete in $L_2(\Omega)$ orthogonal system of eigenfunctions $\{v_j(x),\ j\in\mathbb{N}\}$, corresponding to the eigenvalues $\lambda^{(j)}$, $j\in\mathbb{N}$. Let the numbers $\{a_k: k=1,\dots,l\}$ be such that $\mu_k=a_1\lambda_0^k+\dots+a_l\lambda_{l-1}^k\neq 0$ for $k=1,\dots,l$, where $\{\lambda_k: k=1,\dots,l\}$ are l th roots of unity, and b_k are given by (4.2). It is easy to show that the functions

$$u_j(x)=\sum_{k=1}^{l} b_k v_j(S^{k-1}x),\ j\in\mathbb{N} \tag{6.4}$$

are the eigenfunctions of the problem (6.1), (6.2), which correspond to the eigenvalues $\lambda^{(j)}$, $j \in \mathbb{N}$. Indeed

$$\Delta u_j(x) = \sum_{k=1}^{l} b_k \Delta v_j(S^{k-1}x) = \sum_{k=1}^{l} b_k I_{S^{k-1}} \Delta v_j(x)$$
$$= -\lambda^{(j)} \sum_{k=1}^{l} b_k I_{S^{k-1}} v_j(x) = -\lambda^{(j)} \sum_{k=1}^{l} b_k v_j(S^{k-1}x) = -\lambda^{(j)} u_j(x),$$

as well as for any $q = 1, 2, \dots, l$ we can write

$$u_j(S^{q-1}x)\big|_{\partial\Omega} = \sum_{k=1}^{l} b_{k-q+1} v_j(S^{k-1}x)\Big|_{\partial\Omega} = 0,\ j \in \mathbb{N}.$$

Thus we have

$$\sum_{k=1}^{l} a_k u_j\left(S^{k-1}x\right)|_{\partial\Omega} = 0,\ j \in \mathbb{N}.$$

We have proved the following statement.

Theorem 6.1. *Let the numbers $\{a_k : k = 1, \dots, l\}$ be such that $\mu_k = a_1\lambda_0^k + \dots + a_l\lambda_{l-1}^k \neq 0$ for $k = 1, \dots, l$, where $\{\lambda_k\}$ are l th roots of unity. If $\left\{v_j(x), \lambda^{(j)} :\ j \in \mathbb{N}\right\}$ are eigenfunctions and eigenvalues, respectively, of the Dirichlet problem* (6.3) *and b_k are defined in* (4.2)*, then the system of functions* (6.4) *is a system of eigenfunctions of the problem* (6.1)*,* (6.2) *corresponding to eigenvalues $\lambda^{(j)}$.*

Consider one particular case of the problem (6.1), (6.2).

Theorem 6.2. *Let $Sx = -x$ and $a_1 \neq \pm a_2$, then the system of eigenfunctions of the problem* (6.1)*,* (6.2) *is orthogonal and complete in $L_2(\Omega)$.*

Proof. According to Theorem 6.1, the system of functions determined from (6.4) is a system of eigenfunctions of the problem (6.1), (6.2), where $\{v_j(x), \lambda^{(j)} : j \in \mathbb{N}\}$ is a complete in $L_2(\Omega)$ system of eigenfunctions and eigenvalues of the problem (6.3).

In our case $Sx = -x$, and therefore $S^2 = E$ and $l = 2$. Further, $\lambda_1 = e^{i\pi} = -1$, $\lambda_2 = e^{2i\pi} = 1$, and according to (2.1) $\mu_1 = a_1 - a_2$, $\mu_2 = a_1 + a_2$. By the formula (4.3) we find

$$b_1 = \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\lambda_k^0 \mu_k} = \frac{1}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) = \frac{1}{2}\left(\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\right) = \frac{a_1}{a_1^2 - a_2^2},$$
$$b_2 = \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\lambda_k^1 \mu_k} = \frac{1}{2}\left(\frac{1}{-\mu_1} + \frac{1}{\mu_2}\right) = \frac{1}{2}\left(-\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\right) = \frac{-a_2}{a_1^2 - a_2^2}.$$

Therefore the system of functions (6.4) has the form

$$u_j(x) = \frac{a_1}{a_1^2 - a_2^2} v_j(x) - \frac{a_2}{a_1^2 - a_2^2} v_j(-x). \tag{6.5}$$

According to [6, 8] the eigenfunctions of the Dirichlet problem $v_j(x)$ can be taken in the form

$$v_k^{(\lambda)}(x) = g_{n+2k}(\lambda|x|^2)H_k(x), \quad k \in \mathbb{N}_0,$$

where $H_k(x)$ are homogeneous harmonic polynomials of degree k,

$$g_m(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{(2,2)_k (m,2)_k},$$

and λ is a root of the function $g_{n+2k}(t)$. Using expansion of Bessel functions of the first kind $J_m(t)$ in a series on t, it is not difficult to get the following connection between $g_m(t)$ and $J_m(t)$

$$J_m(t) = \frac{t^m}{2^m \Gamma(m+1)} \, g_{2m+2}(t^2).$$

Therefore, the system of functions $\{v_j(x) : j \in \mathbb{N}\}$ can be chosen so that the condition $v_j(-x) = \pm v_j(x)$ hold.

We check that the system $\{u_j(x) : j \in \mathbb{N}\}$ defined in (6.5) is orthogonal. Indeed,

$$\int_\Omega u_j(x)u_k(x)\,dx = \frac{1}{a_1^2 - a_2^2}\int_\Omega (a_1 v_j(x) - a_2 v_j(-x))(a_1 v_k(x) - a_2 v_k(-x))\,dx$$

$$= \int_\Omega \frac{a_1^2 v_j(x)v_k(x) - a_1 a_2 (v_j(x)v_k(-x) - v_j(-x)v_k(x)) + a_2^2 v_j(-x)v_k(-x)}{a_1^2 - a_2^2}dx$$

$$= \frac{-a_1 a_2}{a_1^2 - a_2^2}\int_\Omega (v_j(x)v_k(-x) + v_j(-x)v_k(x))\,dx$$

$$= \frac{-2a_1 a_2}{a_1^2 - a_2^2}\int_\Omega v_j(x)v_k(-x)\,dx = 0.$$

Let us show that the system of functions $\{u_j(x) : j \in \mathbb{N}\}$ is also complete in $L_2(\Omega)$. Indeed, suppose that the function $f(x) \in L_2(\Omega)$ is orthogonal to all functions of the system (6.5). Then for $j \in \mathbb{N}$ we have

$$0 = (u_j, f) = \int_\Omega u_j(x) f(x)\,dx = \frac{1}{a_1^2 - a_2^2}\int_\Omega [a_1 v_j(x) - a_2 v_j(-x)]\,f(x)\,dx$$

$$= \frac{1}{a_1^2 - a_2^2}\int_\Omega v_j(x)\,[a_1 f(x) - a_2 f(-x)]\,dx$$

$$= \frac{1}{a_1^2 - a_2^2}\int_\Omega v_j(x)\,[a_1 f(x) - a_2 f(-x)]\,dx.$$

Further, since the system $\{v_j(x)\}$ is complete in $L_2(\Omega)$, then $a_1 f(x) = a_2 f(-x)$ for almost all $x \in \Omega$, which means $a_1 f(-x) = a_2 f(x)$ and therefore $a_1^2 f(x) = a_1 a_2 f(-x) = a_2^2 f(x)$, whence $(a_1^2 - a_2^2) f(x) = 0$ and therefore $f(x) = 0$ for almost all $x \in \Omega$. This proves the completeness of the system (6.5) in $L_2(\Omega)$. The theorem is proved. $\square$

Acknowledgements

The work was supported by Act 211 of the Government of the Russian Federation, contract no. 02.A03.21.0011, and by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (grant no. AP05131268).

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Received by the editors January 4, 2019
First published online August 22, 2019