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PPF dependent fixed points of generalized contractive type mappings using $C-$class functions with an application

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Abstract. In this paper, we prove the existence of PPF dependent fixed points of single-valued generalized $\alpha - \eta - \psi - \phi - F-$contraction type mappings and extend it to multi-valued $\alpha^* - \psi - \phi - F-$contraction type mappings in Banach spaces. Also, we introduce the concept of an $f - \alpha^*-$admissible mapping and prove the existence of PPF dependent coincidence points of a pair of single-valued and multi-valued mappings. A fixed point result in a Banach space endowed with a graph is obtained as an application of PPF dependent fixed point result of a single-valued mapping.

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1. Introduction

Banach contraction principle is one of the most important result in analysis and it is the main source of metric fixed point theory. The significance of the proof of the Banach fixed point theorem is that it not only provides the existence and uniqueness of fixed point, but also furnishes a method for constructing the fixed point. Several mathematicians generalized Banach's contraction condition by changing either the domain space or extending a single-valued mapping to a multi-valued mapping, for more details we refer to [1, 6, 8, 13, 14, 20, 21, 22, 25, 11, 12, 26, 27, 30]. In 2012, Samet, Vetro and Vetro [29] introduced the concept of $\alpha-$admissible self mappings and they proved the existence of fixed points by using contractive type conditions involving an $\alpha-$admissible mapping in complete metric spaces, for more details we refer to [18, 23, 28]. In 2012, Asl, Rezapour and Shahzad [5] extended these notions to multi-functions by introducing the notions of $\alpha^* - \psi-$contractive and α^*-admissible mappings and obtained some fixed points theorems, for more details we refer to [3]. In 2013, Ali and Kamran [2] extended the notion of $\alpha^* - \psi-$contractive mappings to multi-valued functions and proved some fixed

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point theorems. In 2016, Ansari, Kaewcharoen [4] introduced a new type contraction, namely the generalized $\alpha-\eta-\psi-\phi-F-$contraction type mapping and proved the existence of fixed points of such mappings.

In 1977, Bernfeld, Lakshmikantham and Reddy [10] introduced the concept of a fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point. Furthermore, they introduced a notion of Banach type contraction for non-self mapping and proved the existence of PPF dependent fixed points in the Razumikhin class for Banach type contraction mappings. The PPF dependent fixed point theorems are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data and future consideration. Several mathematicians proved the existence of a PPF dependent fixed point of single-valued and multi-valued mappings, for more details we refer to [7, 9, 16, 17, 19, 24].

In 2014, Ćirić, Alsulami, Salimi and Vetro [15] introduced the concept of triangular α_c-admissible mappings with respect to η_c non-self mappings and established the existence of PPF dependent fixed points for contraction mappings involving triangular α_c-admissible mappings with respect to η_c non-self mappings in the Razumikhin class.

In this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, and $\mathbb{N}$ is the set of all natural numbers. Let $(E, ||.||_E)$ be a Banach space and we denote it simply by E. Let $I = [a,b] \subseteq \mathbb{R}$ and $E_0 = C(I,E)$ be the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a\leq t\leq b} ||\phi(t)||_E$ for any $\phi \in E_0$. We use the following proposition in proving our results.

Proposition 1.1. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two real sequences,* $\{b_n\}$ *is bounded, then* $\liminf(a_n + b_n) \leq \liminf a_n + \limsup b_n$.

In Section 2, we present basic definitions, lemmas, and preliminaries that are needed to develop the paper. Also we extend the concept of generalized $\alpha-\eta-\psi-\phi-F-$contraction type mapping from the metric space setting to E_0 and based on this we define multi-valued $\alpha^*-\psi-\phi-F-$contraction type mapping on E_0 and also introduce the concept of a $f-\alpha^*-$admissible mapping on E_0. In Section 3, we prove the existence of PPF dependent fixed points of a single-valued generalized $\alpha-\eta-\psi-\phi-F-$contraction type mapping and draw some corollaries. In Section 4, we prove the existence of PPF dependent fixed points of multi-valued $\alpha^*-\psi-\phi-F-$contraction type mappings and PPF dependent coincidence points of a pair (f,T) where f is a single-valued function and T is a multi-valued function. In Section 5, a fixed point result in a Banach space endowed with a graph is drawn as an application of PPF dependent fixed point result of a single-valued map.

2. Preliminaries

In this section we present some basic definitions and lemmas for single and multi-valued mappings in a metric space and then we present the corresponding definitions that are related to PPF dependent fixed points.

Definition 2.1. ([29]) Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be two functions. We say that T is an $\alpha-$admissible mapping if for any $x, y \in X$ with $\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.

Definition 2.2. ([28]) Let $T : X \to X$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be three functions. We say that T is an $\alpha-$admissible mapping with respect to η if for any $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y) \implies \alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

Note that if we take $\eta(x, y) = 1$ for any $x, y \in X$, then Definition 2.2 reduces to Definition 2.1. Also, if we take $\alpha(x, y) = 1$ for any $x, y \in X$, then we say that T is an $\eta-$subadmissible mapping.

In 2013, Karapinar, Kumam and Salimi [23] introduced the notion of triangular $\alpha-$admissible mappings as follows.

Definition 2.3. ([23]) Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be two functions. Then T is said to be a triangular $\alpha-$admissible mapping if for any $x, y, z \in X$,

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1 \text{ and}$$
$$\alpha(x, z) \geq 1, \alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1.$$

Example 2.4. Let $X = \mathbb{R}$. We define $T : X \to X$ by $T(x) = x^2, x \in X$ and $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} \sqrt{x^2 + y^2} & \text{if } x \geq 1 \text{ and } y \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then T is a triangular $\alpha-$admissible mapping.

Definition 2.5. Let $T : X \to X$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be three functions. Then T is said to be a triangular $\alpha-$admissible mapping with respect to η if for any $x, y, z \in X$,

$$\alpha(x, y) \geq \eta(x, y) \implies \alpha(Tx, Ty) \geq \eta(Tx, Ty) \text{ and}$$
$$\alpha(x, z) \geq \eta(x, z), \quad \alpha(z, y) \geq \eta(z, y) \implies \alpha(x, y) \geq \eta(x, y).$$

Example 2.6. Let X = $\mathbb{R}$. We define $T : X \to X$ by $T(x) = x^2, x \in X$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} x - y + 2 & \text{if } x \geq y \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and

$$\eta(x, y) = \begin{cases} x - y + 1 & \text{if } x \geq y \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then T is a triangular $\alpha-$admissible mapping with respect to η.

In 2014, Ansari [3] introduced the concept of $C-$class functions as follows.

Definition 2.7. ([3]) A mapping $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a $C-$class function if it is continuous and for any $s, t \in \mathbb{R}^+$ the function F satisfies the following conditions :
i) $F(s,t) \leq s$ and
ii) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.
The family of all $C-$class functions is denoted by ζ.

Example 2.8. ([3]) The following functions belong to ζ.
i) $F(s,t) = s - t$ for all $s, t \in \mathbb{R}^+$.
ii) $F(s,t) = ks$ for all $s, t \in \mathbb{R}^+$ where $0 < k < 1$.
iii) $F(s,t) = \frac{s}{(1+t)^r}$ for all $s, t \in \mathbb{R}^+$ where $r \in (0, \infty)$.
iv) $F(s,t) = s\beta(s)$ for all $s, t \in \mathbb{R}^+$ where $\beta : \mathbb{R}^+ \to [0,1)$ is continuous.
v) $F(s,t) = s - \phi(s)$ for all $s, t \in \mathbb{R}^+$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\phi(t) = 0 \iff t = 0$.
vi) $F(s,t) = sh(s,t)$ for all $s, t \in \mathbb{R}^+$ where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $h(s,t) < 1$ for all $s, t \in \mathbb{R}^+$.

Definition 2.9. ([4]) Let (X, d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. A mapping $T : X \to X$ is said to be a generalized $\alpha - \eta - \psi - \varphi - F-$contraction type mapping if for any $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \implies \psi(d(Tx,Ty)) \leq F(\psi(M(x,y)), \varphi(M(x,y))),$$

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}, F \in \zeta,\ \psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are both continuous such that $\psi(t) = 0 \iff t = 0,\ \psi$ is a nondecreasing function and $\varphi(t) > 0$ for $t \in (0, \infty)$.

Definition 2.10. ([20]) Let (X, d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. Then X is said to be an $\alpha - \eta-$complete metric space if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for any $n \in \mathbb{N}$ converges in X.

Definition 2.11. ([20]) Let (X, d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. A mapping $T : X \to X$ is said to be an $\alpha - \eta-$continuous mapping if each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \implies Tx_n \to Tx$ as $n \to \infty$.

Theorem 2.12. *([4]) Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to \mathbb{R}^+$ and $T : X \to X$. Suppose that the following conditions are satisfied:*
i) (X, d) is an $\alpha - \eta-$complete metric space,
ii) T is a generalized $\alpha - \eta - \psi - \phi - F-$contraction type mapping,
iii) T is a triangular $\alpha-$orbital admissible mapping with respect to η,
iv) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$ and
v) T is an $\alpha - \eta-$continuous mapping.
Then $\{T^n x_1\}$ converges to x^* in X and x^* is a fixed point of T.

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \left\{\phi \in E_0 \mid ||\phi||_{E_0} = ||\phi(c)||_E\right\}$. Clearly, every constant function from I to E belongs to R_c and thus R_c is a non-empty subset of E_0 .

Definition 2.13. Let R_c be the Razumikhin class of continuous functions in E_0. Then we say that

i) the class R_c is algebraically closed with respect to the difference if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.

ii) the class R_c is topologically closed if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.

The Razumikhin class of functions R_c has the following properties.

Theorem 2.14. *([7]) Let R_c be the Razumikhin class of functions in E_0. Then*

i) for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.

ii) the Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.

iii) $\underset{c\in[a,b]}{\cap} R_c = \{\phi \in E_0 \mid \phi : I \to E \text{ is constant}\}$.

Definition 2.15. ([10]) Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $T(\phi) = \phi(c)$ for some $c \in I$.

Definition 2.16. ([10]) Let $T : E_0 \to E$ be a mapping. Then T is called a Banach type contraction if there exists $k \in [0, 1)$ such that
$||T\phi - T\psi||_E \le k\,||\phi - \psi||_{E_0}$ for any $\phi, \psi \in E_0$.

Theorem 2.17. *([10]) Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be algebraically closed with respect to the difference and topologically closed. Then T has a unique PPF dependent fixed point in R_c.*

Definition 2.18. Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be an α_c−admissible mapping if for any $f, g \in E_0$

$$\alpha(f(c), g(c)) \ge 1 \implies \alpha(Tf, Tg) \ge 1. \tag{2.1}$$

Definition 2.19. Let $c \in I$. Let $T : E_0 \to E$, $\alpha, \eta : E \times E \to \mathbb{R}^+$ be three functions. Then T is said to be an α_c−admissible mapping with respect to η_c if for any $f, g \in E_0$,

$$\alpha(f(c), g(c)) \ge \eta(f(c), g(c)) \implies \alpha(Tf, Tg) \ge \eta(Tf, Tg). \tag{2.2}$$

Definition 2.20. ([15]) Let $c \in I$. Let $T : E_0 \to E$ and $\alpha, \eta : E \times E \to \mathbb{R}^+$ be three functions. Then T is said to be a triangular α_c−admissible mapping with respect to η_c if for any $f, g, h \in E_0$

$$\begin{aligned} &(i)\ \alpha(f(c), g(c)) \ge \eta(f(c), g(c)) \implies \alpha(Tf, Tg) \ge \eta(Tf, Tg) \text{ and} \\ &(ii)\ \alpha(f(c), g(c)) \ge \eta(f(c), g(c)),\ \ \alpha(g(c), h(c)) \ge \eta(g(c), h(c)) \\ &\qquad \implies \alpha(f(c), h(c)) \ge \eta(f(c), h(c)). \end{aligned} \tag{2.3}$$

Note that if $\eta(x, y) = 1$ for any $x, y \in E$ then we say that T is a triangular α_c−admissible mapping and if $\alpha(x, y) = 1$ for any $x, y \in E$ then we say that T is a triangular η_c−subadmissible mapping.

We use the following lemma in our main results.

Lemma 2.21. *([15]) Let T be a triangular α_c-admissible mapping with respect to η_c. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for any $n \in \mathbb{N}\cup\{0\}$, where $\phi_0 \in R_c$ is such that $\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0)$. Then $\alpha(\phi_m(c), \phi_n(c)) \geq \eta(\phi_m(c), \phi_n(c))$ for any $m, n \in \mathbb{N}$ with $m < n$.*

We denote $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \psi \text{ is continuous and } \psi(t) = 0 \iff t = 0\}$.

Now, motivated by the results of Ansari and Kaewcharoen [4] we introduce the following.

Definition 2.22. Let $c \in I$. Let $T : E_0 \to E,\ \alpha, \eta : E \times E \to \mathbb{R}^+$ be three functions. If there exist $\psi, \phi \in \Psi$, with ψ strictly monotonically increasing, functions such that

$$\begin{aligned}\alpha(f(c), g(c)) \geq \eta(f(c), g(c)) \implies \\ \psi(||Tf - Tg||_E) \leq F(\psi(M(f, g)), \phi(M(f, g))),\end{aligned} \tag{2.4}$$

where $M(f, g) = max\{||f - g||_{E_0}, ||f(c) - Tf||_E, ||g(c) - Tg||_E, \frac{1}{2}[||f(c) - Tg||_E + ||g(c) - Tf||_E]\}$
for any $f, g \in E_0$, then we say that T is a generalized $\alpha - \eta - \psi - \phi - F-$contraction type mapping.

If we take $\eta(x, y) = 1$ for any $x, y \in E$, then T is said to be a generalized $\alpha - \psi - \phi - F-$contraction type mapping.

Let $K(E)$ be the collection of all non-empty compact subsets of E. Then the Hausdorff metric induced by the norm $||.||_E$ is defined by

$$H_E(A, B) = \max\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(A, b)\},$$

where $d(a, B) = \inf_{b\in B} ||a-b||_E$ and $d(A, b) = \inf_{a\in A} ||a-b||_E$ for any $A, B \in K(E)$.

Nadler[25] proved the following lemma in metric spaces.

Lemma 2.23. *([25]) Let A and B be two non-empty compact subsets of a metric space X. If $a \in A$ then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

In 2016, Farajzadeh, Kaewcharoen and Plubtieng [17] introduced the concept of a PPF dependent fixed point and PPF dependent coincidence point of multi-valued mappings as follows.

Definition 2.24. ([17]) Let $T : E_0 \to K(E)$ be a multi-valued mapping. A point $f \in E_0$ is said to be a PPF dependent fixed point of T if $f(c) \in Tf$ for some $c \in I$.

Definition 2.25. ([17]) Let $f : E_0 \to E_0$ be a single-valued mapping and $T : E_0 \to K(E)$ be a multi-valued mapping. A point $g \in E_0$ is said to be a PPF dependent coincidence point of f and T if $fg(c) \in Tg$ for some $c \in I$.

Notice that if f is the identity mapping then clearly g is a PPF dependent fixed point of the multi-valued mapping T.

Definition 2.26. Let $c \in I$. Let $T : E_0 \to K(E),\ \alpha : E \times E \to \mathbb{R}^+$ and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ be three mappings. Then T is said to be an α^*-admissible mapping if for any $f, g \in E_0$

$$\alpha(f(c), g(c)) \geq 1 \implies \alpha^*(Tf, Tg) \geq 1,$$

where $\alpha^*(Tf, Tg) = \inf\{\alpha(a,b) \mid a \in Tf,\ b \in Tg\}$.

Based on the generalized $\alpha-\psi-\phi-F-$contraction type mapping of single-valued functions, we define the generalized $\alpha^*-\psi-\phi-F-$contraction type mapping for multi-valued functions as follows.

Definition 2.27. Let $c \in I$. Let $T : E_0 \to K(E),\ \alpha : E \times E \to \mathbb{R}^+$ and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ be three functions. If there exist functions $\psi, \phi \in \Psi$, with ψ strictly monotonically increasing, such that

$$(2.5) \qquad \alpha^*(Tf, Tg) \geq 1 \implies \psi(H_E(Tf, Tg)) \leq F(\psi(M(f,g)), \phi(M(f,g))),$$

where $M(f,g) = max\{||f-g||_{E_0}, d(f(c), Tf), d(g(c), Tg), \frac{1}{2}[d(f(c), Tg) + d(g(c), Tf)]\}$
for any $f, g \in E_0$, then we say that T is a generalized $\alpha^*-\psi-\phi-F-$contraction type mapping.

Based on the concept of α^*-admissible mappings, we define an $f-\alpha^*-$admissible mapping as follows.

Definition 2.28. Let $c \in I$. Let $T : E_0 \to K(E),\ \alpha : E \times E \to \mathbb{R}^+$, $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ and $f : E_0 \to E_0$ be four mappings. Then T is said to be an $f-\alpha^*-$admissible mapping if for any $\phi, \psi \in E_0$

$$(2.6) \qquad \alpha(f\phi(c), f\psi(c)) \geq 1 \implies \alpha^*(T\phi, T\psi) \geq 1.$$

We observe that T is an α^*-admissible mapping if f is the identity mapping.

Example 2.29. Let $E_0 = \mathbb{R}$ and $c \in [a,b] \subseteq \mathbb{R}$. Let $E_0 = C(I, E)$.
We define $T : E_0 \to K(E)$ by

$$T\phi = \begin{cases} [||\phi(c)||_E + 1,\ 3] & \text{if } ||\phi(c)||_E \leq 1 \\ [1,\ ||\phi(c)||_E] & \text{if } ||\phi(c)||_E > 1, \end{cases}$$

$f : E_0 \to E_0$ by $f(\phi) = k\phi,\ k \geq 1$ and $\phi \in E_0$,
$\alpha : E \times E \to \mathbb{R}^+$ by

$$\alpha(x,y) = \begin{cases} y - x + 2 & \text{if } x \leq y, \text{both } x \text{ and } y \text{ non-negative, or} \\ & \quad \text{both } x \text{ and } y \text{ negative, or} \\ & \quad x \text{ is negative and } y \text{ is positive,} \\ 2 & \text{if } x \geq y, \text{ both } x \text{ and } y \text{ non-negative ,} \\ 0 & \text{otherwise,} \end{cases}$$

and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ by
$\alpha^*(A,B) = \inf\{\alpha(a,b)/a \in A \text{ and } b \in B\}$ for any $A, B \in K(E)$.
Let $\phi, \psi \in E_0$ be such that $\alpha(f\phi(c), f\psi(c)) \geq 1$.
Case (i): Suppose that both $f\phi(c), f\psi(c)$ are non-negative and $f\phi(c) \leq f\psi(c)$. Since $k \geq 1$, we have both $\phi(c), \psi(c)$ are non-negative and $\phi(c) \leq \psi(c)$ and hence $||\phi(c)||_E \leq ||\psi(c)||_E$.
Subcase (i): Suppose that $||\phi(c)||_E, ||\psi(c)||_E \in [0,1]$.

We have $\alpha^*(T\phi, T\psi) = \inf\{\alpha(a,b)/a \in T\phi \text{ and } b \in T\psi\}$
$= \inf\{\alpha(a,b)/a \in [||\phi(c)||_E + 1,\ 3] \text{ and } b \in [||\psi(c)||_E + 1,\ 3]\}$.
Therefore $\alpha^*(T\phi, T\psi) = 2 > 1$.
Subcase (ii): Suppose that $||\phi(c)||_E, ||\psi(c)||_E \in (1, \infty)$.
In this case, $\alpha^*(T\phi, T\psi) = \inf\{\alpha(a,b)/a \in [1,\ ||\phi(c)||_E] \text{ and } b \in [1,\ ||\psi(c)||_E]\}$.
Therefore $\alpha^*(T\phi, T\psi) = 2 > 1$.
Subcase (iii): Suppose that $||\phi(c)||_E \in (1, \infty)$ and $||\psi(c)||_E \in [0, 1]$.
Here, $\alpha^*(T\phi, T\psi) = \inf\{\alpha(a,b)/a \in [1,\ ||\phi(c)||_E] \text{ and } b \in [||\psi(c)||_E + 1,\ 3]\}$.
Therefore $\alpha^*(T\phi, T\psi) = 2 > 1$.
Subcase (iv): Suppose that $||\psi(c)||_E \in (1, \infty)$ and $||\phi(c)||_E \in [0, 1]$.
Here, $\alpha^*(T\phi, T\psi) = \inf\{\alpha(a,b)/a \in [||\phi(c)||_E + 1,\ 3] \text{ and } b \in [1,\ ||\psi(c)||_E]\}$.
Therefore $\alpha^*(T\phi, T\psi) = 2 > 1$.
Case (ii): Suppose that both $f\phi(c), f\psi(c)$ are negative and $f\phi(c) \leq f\psi(c)$.
Since $k \geq 1$, we have both $\phi(c), \psi(c)$ are negative and $\phi(c) \leq \psi(c)$ and hence $||\phi(c)||_E \geq ||\psi(c)||_E$.
As in Case (i), we get $\alpha^*(T\phi, T\psi) = 2 > 1$.
Case (iii): Suppose that both $f\phi(c)$ is negative and $f\psi(c)$ is positive and $f\phi(c) \leq f\psi(c)$.
Since $k \geq 1$, we have $\phi(c)$ is negative and $\psi(c)$ is positive and $\phi(c) \leq \psi(c)$.
As in Case (i), we get $\alpha^*(T\phi, T\psi) = 2 > 1$.
Subcase (i): Suppose that $||\phi(c)||_E \geq ||\psi(c)||_E$.
As in Case (i), we get $\alpha^*(T\phi, T\psi) = 2 > 1$.
Subcase (ii): Suppose that $||\phi(c)||_E \leq ||\psi(c)||_E$.
As in Case (i), we get $\alpha^*(T\phi, T\psi) = 2 > 1$.
Case (iv): Suppose that both $f\phi(c), f\psi(c)$ are non-negative and $f\phi(c) \geq f\psi(c)$.
Since $k \geq 1$, we have both $\phi(c), \psi(c)$ are non-negative and $\phi(c) \geq \psi(c)$ and hence $||\phi(c)||_E \geq ||\psi(c)||_E$.
As in Case (i), we get $\alpha^*(T\phi, T\psi) = 2 > 1$.
Hence from all the above cases, we get T is $f - \alpha^*-$admissible mapping.

We use the following lemma in our main results.

Lemma 2.30. *Let $\{\phi_n\}$ be a sequence in E_0 such that $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ as $n \to \infty$. If $\{\phi_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and two subsequences $\{\phi_{m(k)}\}$ and $\{\phi_{n(k)}\}$ of $\{\phi_n\}$ with $m(k) > n(k) > k$ such that $\left|\left|\phi_{n(k)} - \phi_{m(k)}\right|\right|_{E_0} \geq \epsilon$, $\left|\left|\phi_{n(k)} - \phi_{m(k)-1}\right|\right|_{E_0} < \epsilon$ and*

i) $\lim_{k\to\infty} \left|\left|\phi_{n(k)} - \phi_{m(k)}\right|\right|_{E_0} = \epsilon$, *ii)* $\lim_{k\to\infty} \left|\left|\phi_{n(k)} - \phi_{m(k)-1}\right|\right|_{E_0} = \epsilon$,
iii) $\lim_{k\to\infty} \left|\left|\phi_{n(k)-1} - \phi_{m(k)}\right|\right|_{E_0} = \epsilon$, *iv)* $\lim_{k\to\infty} \left|\left|\phi_{n(k)-1} - \phi_{m(k)-1}\right|\right|_{E_0} = \epsilon$.

Proof. Similar to the proof of Lemma 1.4 of [6]. □

3. PPF dependent fixed points of a single-valued mappings

Theorem 3.1. *Let $c \in I$. Let $T : E_0 \to E$ and $\alpha, \eta : E \times E \to \mathbb{R}^+$ be three functions satisfying the following conditions*:

i) T is a generalized $\alpha - \eta - \psi - \phi - F-$contraction type mapping,
ii) T is a triangular α_c-admissible mapping with respect to η_c,
iii) R_c is algebraically closed with respect to the difference,
iv) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq \eta(\phi_n(c), \phi_{n+1}(c))$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi(c)) \geq \eta(\phi_n(c), \phi(c))$ for any $n \in \mathbb{N} \cup \{0\}$, and
v) there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0)$.
Then T has a PPF dependent fixed point in R_c.

Proof. Let $\phi_0 \in R_c$ be such that $\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0)$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. We choose $\phi_1 \in R_c$ such that $x_1 = \phi_1(c)$. Then $T\phi_0 = \phi_1(c)$. Since $T\phi_1 \in E$, there exists $x_2 \in E$ such that $T\phi_1 = x_2$. We choose $\phi_2 \in R_c$ such that $x_2 = \phi_2(c)$. Then $T\phi_1 = \phi_2(c)$.
Continuing this process, we can define a sequence $\{\phi_n\}$ in R_c inductively by $T\phi_n = \phi_{n+1}(c)$ and $||\phi_{n+1} - \phi_n||_{E_0} = ||\phi_{n+1}(c) - \phi_n(c)||_E$ for any $n \in \mathbb{N} \cup \{0\}$.
If $\phi_{n+1} = \phi_n$ for some $n \in \mathbb{N} \cup \{0\}$, then $T\phi_n = \phi_{n+1}(c) = \phi_n(c)$ so that ϕ_n is a PPF dependent fixed point of T in R_c.
Suppose that $\phi_{n+1} \neq \phi_n$ for any $n \in \mathbb{N} \cup \{0\}$.
Since $T\phi_n = \phi_{n+1}(c)$ for any $n \in \mathbb{N} \cup \{0\}$ and $\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0)$, from Lemma 2.21, we have $\alpha(\phi_m(c), \phi_n(c)) \geq \eta(\phi_m(c), \phi_n(c))$ for any $m, n \in \mathbb{N}$ with $m < n$.
We consider

$$\psi(||\phi_{n+1} - \phi_{n+2}||_{E_0}) = \psi(||T\phi_n - T\phi_{n+1}||_E)$$

$$\leq F(\psi(M(\phi_n, \phi_{n+1})), \phi(M(\phi_n, \phi_{n+1}))) \tag{3.1}$$

$$\leq \psi(M(\phi_n, \phi_{n+1})). \tag{3.2}$$

We consider

$$\begin{aligned} M(\phi_n, \phi_{n+1}) &= max\{||\phi_n - \phi_{n+1}||_{E_0}, ||\phi_n(c) - T\phi_n||_E, ||\phi_{n+1}(c) - T\phi_{n+1}||_E, \\ &\qquad \tfrac{1}{2}[||\phi_n(c) - T\phi_{n+1}||_E + ||\phi_{n+1}(c) - T\phi_n||_E]\} \\ &= max\{||\phi_n - \phi_{n+1}||_{E_0}, ||\phi_{n+1} - \phi_{n+2}||_{E_0}, \\ &\qquad \tfrac{1}{2}[||\phi_n - \phi_{n+2}||_{E_0}]\} \\ &\leq max\{||\phi_n - \phi_{n+1}||_{E_0}, ||\phi_{n+1} - \phi_{n+2}||_{E_0}, \\ &\qquad \tfrac{1}{2}[||\phi_n - \phi_{n+1}||_{E_0} + ||\phi_{n+1} - \phi_{n+2}||_{E_0}]\} \\ &= max\{||\phi_n - \phi_{n+1}||_{E_0}, ||\phi_{n+1} - \phi_{n+2}||_{E_0}\} \\ &\leq M(\phi_n, \phi_{n+1}). \end{aligned}$$

Hence $M(\phi_n, \phi_{n+1}) = max\{||\phi_n - \phi_{n+1}||_{E_0}, ||\phi_{n+1} - \phi_{n+2}||_{E_0}\}$.
Suppose that $max\{||\phi_n - \phi_{n+1}||_{E_0}, ||\phi_{n+1} - \phi_{n+2}||_{E_0}\} = ||\phi_{n+1} - \phi_{n+2}||_{E_0}$.
Then $M(\phi_n, \phi_{n+1}) = ||\phi_{n+1} - \phi_{n+2}||_{E_0}$.
From (3.1),

$$\begin{aligned} \psi(||\phi_{n+1}-\phi_{n+2}||_{E_0}) &\leq F(\psi(||\phi_{n+1}-\phi_{n+2}||_{E_0}), \phi(||\phi_{n+1}-\phi_{n+2}||_{E_0})) \\ &\leq \psi(||\phi_{n+1} - \phi_{n+2}||_{E_0}) \end{aligned}$$

and hence $F(\psi(||\phi_{n+1}-\phi_{n+2}||_{E_0}), \phi(||\phi_{n+1}-\phi_{n+2}||_{E_0})) = \psi(||\phi_{n+1}-\phi_{n+2}||_{E_0})$.
Therefore either $\psi(||\phi_{n+1} - \phi_{n+2}||_{E_0}) = 0$ or $\phi(||\phi_{n+1} - \phi_{n+2}||_{E_0}) = 0$ and hence $\phi_{n+1} = \phi_{n+2}$, a contradiction. Therefore $M(\phi_n, \phi_{n+1}) = ||\phi_n - \phi_{n+1}||_{E_0}$.
From (3.2), $\psi(||\phi_{n+1} - \phi_{n+2}||_{E_0}) \leq \psi(||\phi_n - \phi_{n+1}||_{E_0})$.
Since ψ is strictly monotonically increasing, we have
$||\phi_{n+1} - \phi_{n+2}||_{E_0} \leq ||\phi_n - \phi_{n+1}||_{E_0}$.
Therefore the sequence $\{||\phi_n - \phi_{n+1}||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$ and hence it is convergent.
Let $\lim_{n\to\infty} ||\phi_n - \phi_{n+1}||_{E_0} = r$. We now show that $r = 0$.
From (3.1), $\psi(||\phi_{n+1} - \phi_{n+2}||_{E_0}) \leq F(\psi(||\phi_n - \phi_{n+1}||_{E_0}), \phi(||\phi_n - \phi_{n+1}||_{E_0}))$.
On applying limits as $n \to \infty$, we get $\psi(r) \leq F(\psi(r), \phi(r))) \leq \psi(r)$ and hence $F(\psi(r), \phi(r)) = \psi(r)$. Therefore either $\psi(r) = 0$ or $\phi(r) = 0$ and hence $r = 0$. Therefore

$$\lim_{n\to\infty} ||\phi_n - \phi_{n+1}||_{E_0} = 0. \tag{3.3}$$

We now show that the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.
Suppose that the sequence $\{\phi_n\}$ is not a Cauchy sequence.
By Lemma 2.30, there exist an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_k-1}||_{E_0} < \epsilon$ and

$$\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon. \tag{3.4}$$

By the triangular inequality, we have
$||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} \leq ||\phi_{n_k+1} - \phi_{n_k}||_{E_0} + ||\phi_{n_k} - \phi_{m_k}||_{E_0} + ||\phi_{m_k} - \phi_{m_k+1}||_{E_0}$.
On applying limit superior as $k \to \infty$ on both sides we get

$$\limsup_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} \leq \epsilon. \tag{3.5}$$

By the triangular inequality, we have
$||\phi_{n_k} - \phi_{m_k}||_{E_0} \leq ||\phi_{n_k} - \phi_{n_k+1}||_{E_0} + ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} + ||\phi_{m_k+1} - \phi_{m_k}||_{E_0}$.
Now by applying Proposition 1.1 with $a_k = ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0}$ and $b_k = ||\phi_{n_k} - \phi_{n_k+1}||_{E_0} + ||\phi_{m_k+1} - \phi_{m_k}||_{E_0}$ we have

$$\epsilon \leq \liminf_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0}. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\lim_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon. \tag{3.7}$$

From (3.4) and (3.7) we have

$$\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon = \lim_{k\to\infty} ||\phi_{m_k} - \phi_{n_k+1}||_{E_0}. \tag{3.8}$$

We consider
$M(\phi_{n_k}, \phi_{m_k}) = max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\phi_{n_k}(c) - T\phi_{n_k}||_E, ||\phi_{m_k}(c) - T\phi_{m_k}||_E,$

$$\frac{1}{2}[||\phi_{n_k}(c) - T\phi_{m_k}||_E + ||\phi_{m_k}(c) - T\phi_{n_k}||_E]\}$$
$$= max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\phi_{n_k} - \phi_{n_k+1}||_{E_0}, ||\phi_{m_k} - \phi_{m_k+1}||_{E_0}, \frac{1}{2}[||\phi_{n_k} - \phi_{m_k+1}||_{E_0} + ||\phi_{m_k} - \phi_{n_k+1}||_{E_0}]\}.$$

On applying limits as $k \to \infty$, we get

$$\lim_{k\to\infty} M(\phi_{n_k}, \phi_{m_k}) = max\{\epsilon, 0, 0, \frac{1}{2}[\epsilon + \epsilon]\} = \epsilon. \tag{3.9}$$

We consider

$$\psi(||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0}) = \psi(||T\phi_{n_k} - T\phi_{m_k}||_E),$$
$$\leq F(\psi(M(\phi_{n_k}, \phi_{m_k})), \phi(M(\phi_{n_k}, \phi_{m_k}))).$$

On applying limits as $k \to \infty$, we get $\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon)$ and hence $F(\psi(\epsilon), \phi(\epsilon)) = \psi(\epsilon)$. Therefore either $\psi(\epsilon) = 0$ or $\phi(\epsilon) = 0$ and hence $\epsilon = 0$, a contradiction. Therefore the sequence $\{\phi_n\}$ is a Cauchy sequence in $R_c \subseteq E_0$.
Since E_0 is complete, we have $\phi_n \to \phi^*$ as $n \to \infty$ for some $\phi^* \in E_0$.
Since R_c is topologically closed, we have $\phi^* \in R_c$.
From (iv), we have $\alpha(\phi_n(c), \phi^*(c)) \geq \eta(\phi_n(c), \phi^*(c))$ for any $n \in \mathbb{N} \cup \{0\}$.
Since T is a generalized $\alpha - \eta - \psi - \phi - F-$ contraction type mapping, we have

$$\psi(||\phi_{n+1}(c) - T\phi^*||_E) = \psi(||T\phi_n - T\phi^*||_E)$$
$$\leq F(\psi(M(\phi_n, \phi^*)), \phi(M(\phi_n, \phi^*))), \tag{3.10}$$

where

$$||\phi^*(c) - T\phi^*||_E \leq M(\phi_n, \phi^*)$$
$$= max\{||\phi_n - \phi^*||_{E_0}, ||\phi_n(c) - T\phi_n||_E, ||\phi^*(c) - T\phi^*||_E, \frac{1}{2}[||\phi_n(c) - T\phi^*||_E + ||\phi^*(c) - T\phi_n||_E]\}$$
$$\leq max\{||\phi_n - \phi^*||_{E_0}, ||\phi_n - \phi_{n+1}||_{E_0}, ||\phi^*(c) - T\phi^*||_E, \frac{1}{2}[||\phi_n(c) - T\phi^*||_E + ||\phi^* - \phi_n||_{E_0} + ||\phi_n - \phi_{n+1}||_{E_0}]\}.$$

On applying limits as $n \to \infty$, we get

$$||\phi^*(c) - T\phi^*||_E \leq \lim_{n\to\infty} M(\phi_n, \phi^*) \leq max\{0, 0, ||\phi^*(c) - T\phi^*||_E, \frac{1}{2}[||\phi^*(c) - T\phi^*||_E]\}$$
$$= ||\phi^*(c) - T\phi^*||_E.$$

Hence $\lim_{n\to\infty} M(\phi_n, \phi^*) = ||\phi^*(c) - T\phi^*||_E$.
On applying limits as $n \to \infty$ to inequality (3.10), we get

$$\psi(||\phi^*(c) - T\phi^*||_E) \leq F(\psi(||\phi^*(c) - T\phi^*||_E), \phi(||\phi^*(c) - T\phi^*||_E))$$
$$\leq \psi(||\phi^*(c) - T\phi^*||_E)$$

and hence

$$F(\psi(||\phi^*(c) - T\phi^*||_E), \phi(||\phi^*(c) - T\phi^*||_E)) = \psi(||\phi^*(c) - T\phi^*||_E).$$

Therefore $\psi(||\phi^*(c) - T\phi^*||_E) = 0$ or $\phi(||\phi^*(c) - T\phi^*||_E) = 0$ and hence $T\phi^* = \phi^*(c)$. Therefore $\phi^* \in R_c$ is a PPF dependent fixed point of T. □

Corollary 3.2. *Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions satisfying the following conditions:*
i) T is a generalized $\alpha - \psi - \phi - F-$contraction type mapping,
ii) T is a triangular α_c-admissible mapping,
iii) R_c is algebraically closed with respect to the difference,

iv) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, and

v) there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$.

Then T has a PPF dependent fixed point in R_c.

Proof. Follows by choosing $\eta(\phi(c), \psi(c)) = 1$ for any $\phi, \psi \in E_0$ in Theorem 3.1. □

Corollary 3.3. *Let $c \in I$. Let $T : E_0 \to E$ and $\alpha, \eta : E \times E \to \mathbb{R}^+$ be three functions satisfying the following conditions:*

i) T satisfies the inequality

$$\alpha(f(c), g(c)) \geq \eta(f(c), g(c)) \implies ||Tf - Tg||_E \leq k \cdot \max\{||f - g||_{E_0}, ||f(c) - Tf||_E, ||g(c) - Tg||_E, \frac{1}{2}[||f(c) - Tg||_E + ||g(c) - Tf||_E]\}$$

for any $f, g \in E_0$, where $0 < k < 1$,

ii) T is a triangular α_c-admissible mapping with respect to η_c,

iii) R_c is algebraically closed with respect to the difference,

iv) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq \eta(\phi_n(c), \phi_{n+1}(c))$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi^*(c)) \geq \eta(\phi_n(c), \phi^*(c))$ for any $n \in \mathbb{N} \cup \{0\}$, and

v) there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq \eta(\phi_0(c), T\phi_0)$.

Then T has a PPF dependent fixed point in R_c.

Proof. Follows by choosing $F(s, t) = ks$ where $0 < k < 1$ and $\psi(t) = t$ for any $s, t \in \mathbb{R}^+$ in Theorem 3.1. □

Corollary 3.4. *Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions satisfying the following conditions:*

i) T satisfies the inequality

$$\alpha(f(c), g(c)) \geq 1 \implies ||Tf - Tg||_E \leq k\, max\{||f - g||_{E_0}, ||f(c) - Tf||_E, ||g(c) - Tg||_E, \frac{1}{2}[||f(c) - Tg||_E + ||g(c) - Tf||_E]\}$$

for any $f, g \in E_0$, where $0 < k < 1$,

ii) T is a triangular α_c-admissible mapping,

iii) R_c is algebraically closed with respect to the difference,

iv) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, and

v) there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$.

Then T has a PPF dependent fixed point in R_c.

Proof. Follows by choosing $\eta(\phi(c), \psi(c)) = 1$ for any $\phi, \psi \in E_0$ in Corollary 3.3. □

4. PPF dependent fixed points and coincidence points of multi-valued mappings

Theorem 4.1. *Let $c \in I$. Let $T : E_0 \to K(E)$, $\alpha : E \times E \to \mathbb{R}^+$ and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ be three functions satisfying the following conditions:*
i) T is a generalized $\alpha^* - \psi - \phi - F-$contraction type mapping,
ii) T is an α^*-admissible mapping,
iii) R_c is algebraically closed with respect to the difference,
iv) $T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,
v) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, and
vi) there exist $\phi_0 \in R_c$ and $\phi_1(c) \in T\phi_0$ such that $\alpha(\phi_0(c), \phi_1(c)) \geq 1$.
Then T has a PPF dependent fixed point in R_c.

Proof. Let $\phi_0 \in R_c$ and $\phi_1(c) \in T\phi_0$ be such that $\alpha(\phi_0(c), \phi_1(c)) \geq 1$.
If $\phi_0 = \phi_1$ then ϕ_0 is a PPF dependent fixed point of T. Suppose that $\phi_0 \neq \phi_1$.
Since T is an α^*-admissible mapping, we have $\alpha^*(T\phi_0, T\phi_1) \geq 1$.
Since T is a generalized $\alpha^* - \psi - \phi - F-$ contraction type mapping, we have $\psi(H_E(T\phi_0, T\phi_1)) \leq F(\psi(M(\phi_0, \phi_1)), \phi(M(\phi_0, \phi_1)))$.
Since $x_1 \in T\phi_0$, by Lemma 2.23 there exists $x_2 \in T\phi_1$ such that $||x_1 - x_2||_E \leq H_E(T\phi_0, T\phi_1)$. Since $x_2 \in T\phi_1$ and $T\phi_1 \subseteq R_c(c)$, we choose $\phi_2 \in R_c$ such that $x_2 = \phi_2(c) \in T\phi_1$. If $\phi_1 = \phi_2$ then ϕ_1 is a PPF dependent fixed point of T. Suppose that $\phi_1 \neq \phi_2$. Clearly $\alpha(\phi_1(c), \phi_2(c)) \geq \alpha^*(T\phi_0, T\phi_1) \geq 1$ and hence $\alpha(\phi_1(c), \phi_2(c)) \geq 1$. Since T is an α^*-admissible mapping, we have $\alpha^*(T\phi_1, T\phi_2) \geq 1$. Since T is a generalized $\alpha^* - \psi - \phi - F-$contraction type mapping, we have $\psi(H_E(T\phi_1, T\phi_2)) \leq F(\psi(M(\phi_1, \phi_2)), \phi(M(\phi_1, \phi_2)))$.
Since $x_2 \in T\phi_1$, by Lemma 2.23 there exists $x_3 \in T\phi_2$ such that $||x_2 - x_3||_E \leq H_E(T\phi_1, T\phi_2)$. On continuing this process, we get a sequence $\{\phi_n\}$ in R_c satisfying the following:
for any $n \in \mathbb{N}$,

$$(4.1) \quad \begin{cases} \phi_{n-1} \neq \phi_n, \\ x_n = \phi_n(c) \in T\phi_{n-1}, \\ ||\phi_n - \phi_{n+1}||_{E_0} = ||\phi_n(c) - \phi_{n+1}(c)||_E \\ \qquad\qquad = ||x_n - x_{n+1}||_E \leq H_E(T\phi_{n-1}, T\phi_n), \\ \alpha^*(T\phi_{n-1}, T\phi_n) \geq 1 \text{ and hence} \\ \psi(H_E(T\phi_{n-1}, T\phi_n)) \leq F(\psi(M(\phi_{n-1}, \phi_n)), \phi(M(\phi_{n-1}, \phi_n))). \end{cases}$$

From (4.1) we have
$||\phi_n - \phi_{n+1}||_{E_0} \leq H_E(T\phi_{n-1}, T\phi_n)$, which implies that

$$\begin{aligned} \psi(||\phi_n - \phi_{n+1}||_{E_0}) &\leq \psi(H_E(T\phi_{n-1}, T\phi_n)) \\ &\leq F(\psi(M(\phi_{n-1}, \phi_n)), \phi(M(\phi_{n-1}, \phi_n))). \qquad (4.2) \end{aligned}$$

Now we consider

$$M(\phi_{n-1}, \phi_n) = max\{||\phi_{n-1} - \phi_n||_{E_0}, d(\phi_{n-1}(c), T\phi_{n-1}), d(\phi_n(c), T\phi_n), \frac{1}{2}[d(\phi_{n-1}(c), T\phi_n) + d(\phi_n(c), T\phi_{n-1})]\}$$

$= max\{||\phi_{n-1} - \phi_n||_{E_0}, d(\phi_n(c), T\phi_n)\}.$
Suppose that $M(\phi_{n-1}, \phi_n) = d(\phi_n(c), T\phi_n)$.
From (4.2) we have

$$\psi(||\phi_n - \phi_{n+1}||_{E_0}) \leq F(\psi(d(\phi_n(c), T\phi_n)), \phi(d(\phi_n(c), T\phi_n))) \leq \psi(d(\phi_n(c), T\phi_n))$$

and hence
$||\phi_n - \phi_{n+1}||_{E_0} = ||\phi_n(c) - \phi_{n+1}(c)||_E \leq d(\phi_n(c), T\phi_n)$, a contradiction.
Therefore $M(\phi_{n-1}, \phi_n) = ||\phi_{n-1} - \phi_n||_{E_0}$.
From (4.2) $\psi(||\phi_n - \phi_{n+1}||_{E_0}) \leq F(\psi(||\phi_{n-1} - \phi_n||_{E_0}), \phi(||\phi_{n-1} - \phi_n||_{E_0})) \leq \psi(||\phi_{n-1} - \phi_n||_{E_0})$.
Since ψ is strictly monotonically increasing we have
$||\phi_n - \phi_{n+1}||_{E_0} \leq ||\phi_{n-1} - \phi_n||_{E_0}$.
Therefore the sequence $\{||\phi_n - \phi_{n+1}||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$ and hence it is convergent.
Let $\lim_{n\to\infty} ||\phi_n - \phi_{n+1}||_{E_0} = r$. We now show that $r = 0$.
From (4.2), $\psi(||\phi_n - \phi_{n+1}||_{E_0}) \leq F(\psi(||\phi_{n-1} - \phi_n||_{E_0}), \phi(||\phi_{n-1} - \phi_n||_{E_0}))$.
On applying limits as $n \to \infty$, we get $\psi(r) \leq F(\psi(r), \phi(r)) \leq \psi(r)$, which implies that either $\psi(r) = 0$ or $\phi(r) = 0$. Therefore $r = 0$ and hence

$$\lim_{n\to\infty} ||\phi_n - \phi_{n+1}||_{E_0} = 0. \tag{4.3}$$

Now we show that $\{\phi_n\}$ is a Cauchy sequence in R_c.
Suppose that the sequence $\{\phi_n\}$ is not a Cauchy sequence. By Lemma 2.30, there exist an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_k-1}||_{E_0} < \epsilon$ and

$$\lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon. \tag{4.4}$$

As in the proof of Theorem 3.1, we get

$$\begin{cases} \lim_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon \text{ and} \\ \lim_{k\to\infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon = \lim_{k\to\infty} ||\phi_{m_k} - \phi_{n_k+1}||_{E_0}. \end{cases} \tag{4.5}$$

We now show that $\lim_{k\to\infty} ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} = \epsilon$ for any $l_1, l_2 \in \mathbb{N}$.
Let $l_1, l_2 \in \mathbb{N}$. We now consider

$$\begin{aligned} ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} \\ \leq \quad & ||\phi_{m_k+l_1} - \phi_{m_k+l_1-1}||_{E_0} + ||\phi_{m_k+l_1-1} - \phi_{m_k+l_1-2}||_{E_0} \\ & + \ldots + ||\phi_{m_k+1} - \phi_{m_k}||_{E_0} + ||\phi_{m_k} - \phi_{n_k+1}||_{E_0} \\ & + ||\phi_{n_k+1} - \phi_{n_k+2}||_{E_0} + \ldots + ||\phi_{n_k+l_2-1} - \phi_{n_k+l_2}||_{E_0}. \end{aligned}$$

On applying limit superior as $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} \leq \epsilon. \tag{4.6}$$

Now we consider

$$\begin{aligned}
&||\phi_{m_k} - \phi_{n_k+1}||_{E_0} \\
&\quad \leq \quad ||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + ||\phi_{m_k+1} - \phi_{m_k+2}||_{E_0} + ... \\
&\qquad\quad +||\phi_{m_k+l_1-1} - \phi_{m_k+l_1}||_{E_0} + ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} \\
&\qquad\quad +||\phi_{n_k+l_2} - \phi_{n_k+l_2-1}||_{E_0} + ... + ||\phi_{n_k+2} - \phi_{n_k+1}||_{E_0}.
\end{aligned}$$

Now by applying Proposition 1.1 with

$$\begin{aligned}
a_k &= ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} \text{ and} \\
b_k &= (||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + ||\phi_{m_k+1} - \phi_{m_k+2}||_{E_0} \\
&\quad +... + ||\phi_{m_k+l_1-1} - \phi_{m_k+l_1}||_{E_0} \\
&\quad +||\phi_{n_k+l_2} - \phi_{n_k+l_2-1}||_{E_0} + ... + ||\phi_{n_k+2} - \phi_{n_k+1}||_{E_0})
\end{aligned}$$

we have

$$\begin{aligned}
\epsilon \quad \leq \quad & \liminf_{k\to\infty} ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} + \limsup_{k\to\infty}(||\phi_{m_k} - \phi_{m_k+1}||_{E_0} \\
&+||\phi_{m_k+1} - \phi_{m_k+2}||_{E_0} + ... + ||\phi_{m_k+l_1-1} - \phi_{m_k+l_1}||_{E_0} \\
&+||\phi_{n_k+l_2} - \phi_{n_k+l_2-1}||_{E_0} + ... + ||\phi_{n_k+2} - \phi_{n_k+1}||_{E_0}).
\end{aligned}$$

Hence

$$\epsilon \leq \liminf_{k\to\infty} ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0}. \tag{4.7}$$

From (4.6) and (4.7), we get

$$\lim_{k\to\infty} ||\phi_{m_k+l_1} - \phi_{n_k+l_2}||_{E_0} = \epsilon \text{ for any } l_1, l_2 \in \mathbb{N}. \tag{4.8}$$

We choose $l_1, l_2 \in \mathbb{N}$ such that $(n_k + l_2) - (m_k + l_1) = 1$.
From (4.1) we get

$$\begin{aligned}
&\psi(||\phi_{n_k+l_2} - \phi_{m_k+l_1}||_{E_0}) \\
&\quad \leq \quad \psi(H_E(T\phi_{n_k+l_2-1}, T\phi_{m_k+l_1-1})) \\
&\quad \leq \quad F(\psi(M(\phi_{n_k+l_2-1}, \phi_{m_k+l_1-1})), \phi(M(\phi_{n_k+l_2-1}, \phi_{m_k+l_1-1}))).
\end{aligned}$$

On applying limits as $k \to \infty$, we get $\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon)$ and hence $F(\psi(\epsilon), \phi(\epsilon)) = \psi(\epsilon)$. Therefore $\epsilon = 0$, a contradiction.
Therefore the sequence $\{\phi_n\}$ is a Cauchy sequence in $R_c \subseteq E_0$.
Since E_0 is complete, we have $\phi_n \to \phi^*$ as $n \to \infty$.
Since R_c is topologically closed, we have $\phi^* \in R_c$.
Clearly,

$$\begin{aligned}
d(\phi^*(c), T\phi^*) &\leq M(\phi_n, \phi^*) \\
&= max\{||\phi_n - \phi^*||_{E_0}, d(\phi_n(c), T\phi_n), d(\phi^*(c), T\phi^*), \\
&\qquad \tfrac{1}{2}[d(\phi_n(c), T\phi^*) + d(\phi^*(c), T\phi_n)]\} \\
&\leq max\{||\phi_n - \phi^*||_{E_0}, ||\phi_n(c) - \phi_{n+1}(c)||_E, d(\phi^*(c), T\phi^*),
\end{aligned}$$

$\frac{1}{2}[d(\phi_n(c), T\phi^*) + ||\phi^* - \phi_{n+1}||_{E_0}]\}.$
On applying limits as $n \to \infty$ we get
$d(\phi^*(c), T\phi^*) \leq \lim_{n\to\infty} M(\phi_n, \phi^*) \leq d(\phi^*(c), T\phi^*)$ and hence

$$\lim_{n\to\infty} M(\phi_n, \phi^*) = d(\phi^*(c), T\phi^*). \tag{4.9}$$

Since $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq \alpha^*(T\phi_{n-1}, T\phi_n) \geq 1$ and from (v), we have $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Since T is α^*-admissible, we have $\alpha^*(T\phi_n, T\phi^*) \geq 1$.
Clearly,
$d(\phi_{n+1}(c), T\phi^*) \leq H_E(T\phi_n, T\phi^*)$, which implies that
$\psi(d(\phi_{n+1}(c), T\phi^*)) \leq \psi(H_E(T\phi_n, T\phi^*))$
$\leq F(\psi(M(\phi_n, \phi^*)), \phi(M(\phi_n, \phi^*))).$
On applying limits as $n \to \infty$ and by using (4.9) we get
$\psi(d(\phi^*(c), T\phi^*)) \leq F(\psi(d(\phi^*(c), T\phi^*)), \phi(d(\phi^*(c), T\phi^*))) \leq \psi(d(\phi^*(c), T\phi^*)).$
Therefore, either $\psi(d(\phi^*(c), T\phi^*)) = 0$ or $\phi(d(\phi^*(c), T\phi^*)) = 0$, and hence $d(\phi^*(c), T\phi^*) = 0$. Therefore $\phi^*(c) \in T\phi^*$ and hence ϕ^* is a PPF dependent fixed point of T. □

Theorem 4.2. *Let $c \in I$. Let $T : E_0 \to K(E)$, $\alpha : E \times E \to \mathbb{R}^+$, $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ and $f : E_0 \to E_0$ be four functions satisfying the following conditions:*

i) there exist functions $\psi, \phi \in \Psi$, with ψ strictly monotonically increasing, such that

$$\begin{cases} \alpha(f\gamma(c), f\eta(c)) \geq 1 \implies \\ \qquad \psi(H_E(T\gamma, T\eta)) \leq F(\psi(||f\gamma - f\eta||_{E_0}), \phi(||f\gamma - f\eta||_{E_0})) \end{cases} \tag{4.10}$$

for any $\gamma, \eta \in E_0$,
ii) T is an $f - \alpha^*-$admissible mapping,
iii) R_c is algebraically closed with respect to the difference and $f(R_c) \subseteq R_c$,
iv) $T\phi \subseteq f(R_c)(c) = \{f\psi(c)/\psi \in R_c\}$ for any $\phi \in E_0$,
v) if $\{f\phi_n\}$ is a sequence in E_0 such that $f\phi_n \to f\phi^*$ as $n \to \infty$ and $\alpha(f\phi_n(c), f\phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(f\phi_n(c), f\phi^*(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$,
vi) $f(R_c)$ is complete, and
vii) there exist $\phi_0 \in R_c$ and $f\phi_1(c) \in T\phi_0$ such that $\alpha(f\phi_0(c), f\phi_1(c)) \geq 1$.
Then T and f have a PPF dependent coincidence point in R_c.

Proof. Let $\phi_0 \in R_c$ and $x_1 = f\phi_1(c) \in T\phi_0$ be such that $\alpha(f\phi_0(c), f\phi_1(c)) \geq 1$. If $f\phi_1 = f\phi_0$ then ϕ_0 is a PPF dependent coincidence point of T and f.
Suppose that $f\phi_1 \neq f\phi_0$.
From (4.10), we get

$$\psi(H_E(T\phi_0, T\phi_1)) \leq F(\psi(||f\phi_0 - f\phi_1||_{E_0}), \phi(||f\phi_0 - f\phi_1||_{E_0})).$$

Since $x_1 \in T\phi_0$, by Lemma 2.23, there exists $x_2 \in T\phi_1$ such that $||x_1 - x_2||_E \leq H(T\phi_0, T\phi_1)$. Since $x_2 \in T\phi_1$ and $T\phi_1 \subseteq f(R_c)(c)$,

we choose $\phi_2 \in R_c$ such that $x_2 = f\phi_2(c) \in T\phi_1$.
If $f\phi_2 = f\phi_1$ then ϕ_1 is a PPF dependent coincidence point of T and f.
Suppose that $f\phi_2 \neq f\phi_1$.
Since $\alpha(f\phi_0(c), f\phi_1(c)) \geq 1$ and T is $f - \alpha^*-$admissible, we have $\alpha^*(T\phi_0, T\phi_1) \geq 1$.
Clearly $\alpha(f\phi_1(c), f\phi_2(c)) \geq \alpha^*(T\phi_0, T\phi_1) \geq 1$ and hence $\alpha(f\phi_1(c), f\phi_2(c)) \geq 1$.
From (4.10), we get

$$\psi(H_E(T\phi_1, T\phi_2)) \leq F(\psi(||f\phi_1 - f\phi_2||_{E_0}), \phi(||f\phi_1 - f\phi_2||_{E_0})).$$

Since $x_2 \in T\phi_1$, by Lemma 2.23 there exists $x_3 \in T\phi_2$ such that $||x_2 - x_3||_E \leq H_E(T\phi_1, T\phi_2)$. Continuing this process, we get a sequence $\{f\phi_n\}$ in $f(R_c)$ satisfying the following:
for any $n \in \mathbb{N}$,

$$\begin{cases} f\phi_n \neq f\phi_{n-1}, \\ x_n = f\phi_n(c) \in T\phi_{n-1}, \\ ||f\phi_n - f\phi_{n+1}||_{E_0} = ||f\phi_n(c) - f\phi_{n+1}(c)||_E \\ \qquad\qquad = ||x_n - x_{n+1}||_E \leq H_E(T\phi_{n-1}, T\phi_n), \\ \alpha(f\phi_{n-1}(c), f\phi_n(c)) \geq 1 \text{ and hence} \\ \psi(H_E(T\phi_{n-1}, T\phi_n)) \leq F(\psi(||f\phi_{n-1} - f\phi_n||_{E_0}), \\ \qquad\qquad \phi(||f\phi_{n-1} - f\phi_n||_{E_0})). \end{cases} \tag{4.11}$$

From (4.11), we have $||f\phi_n - f\phi_{n+1}||_{E_0} \leq H_E(T\phi_{n-1}, T\phi_n)$, which implies that

$$\begin{aligned} &\psi(||f\phi_n - f\phi_{n+1}||_{E_0}) \\ &\quad \leq \psi(H_E(T\phi_{n-1}, T\phi_n)) \\ &\quad \leq F(\psi(||f\phi_{n-1} - f\phi_n||_{E_0}), \phi(||f\phi_{n-1} - f\phi_n||_{E_0})) \\ &\quad \leq \psi(||f\phi_{n-1} - f\phi_n||_{E_0}). \end{aligned} \tag{4.12}$$

Since ψ is strictly monotonically increasing, we have

$$||f\phi_n - f\phi_{n+1}||_{E_0} \leq ||f\phi_{n-1} - f\phi_n||_{E_0}.$$

Therefore the sequence $\{||f\phi_n - f\phi_{n+1}||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$ and hence it is convergent. Let $\lim_{n\to\infty} ||f\phi_n - f\phi_{n+1}||_{E_0} = r$.
We now show that $r = 0$.
From (4.12), we have

$$\begin{aligned} \psi(||f\phi_n - f\phi_{n+1}||_{E_0}) &\leq F(\psi(||f\phi_{n-1} - f\phi_n||_{E_0}), \phi(||f\phi_{n-1} - f\phi_n||_{E_0})) \\ &\leq \psi(||f\phi_{n-1} - f\phi_n||_{E_0}). \end{aligned}$$

On applying limits as $n \to \infty$ we obtain that $\psi(r) \leq F(\psi(r), \phi(r)) \leq \psi(r)$.
This implies that either $\psi(r) = 0$ or $\phi(r) = 0$ and hence $r = 0$.
Therefore

$$\lim_{n\to\infty} ||f\phi_n - f\phi_{n+1}||_{E_0} = 0. \tag{4.13}$$

We now show that $\{f\phi_n\}$ is a Cauchy sequence in $f(R_c)$.
Suppose that the sequence $\{f\phi_n\}$ is not a Cauchy sequence.
By Lemma 2.30, there exist an $\epsilon > 0$ and two subsequences $\{f\phi_{m_k}\}$ and $\{f\phi_{n_k}\}$ of $\{f\phi_n\}$ with $m_k > n_k > k$ such that $||f\phi_{n_k} - f\phi_{m_k}||_{E_0} \geq \epsilon$, $||f\phi_{n_k} - f\phi_{m_k-1}||_{E_0} < \epsilon$ and $\lim\limits_{k\to\infty} ||f\phi_{n_k} - f\phi_{m_k}||_{E_0} = \epsilon$.
As in the proof of the Theorem 4.1, we get
$\lim\limits_{k\to\infty} ||f\phi_{m_k+l_1} - f\phi_{n_k+l_2}||_{E_0} = \epsilon$ for any $l_1, l_2 \in \mathbb{N}$.
We choose $l_1, l_2 \in \mathbb{N}$ such that $(n_k + l_2) - (m_k + l_1) = 1$.
From (4.11),

$$\begin{aligned} \psi(||f\phi_{n_k+l_2} - f\phi_{m_k+l_1}||_{E_0}) &\leq \psi(H_E(T\phi_{n_k+l_2-1}, T\phi_{m_k+l_1-1})) \\ &\leq F(\psi(||f\phi_{n_k+l_2-1} - f\phi_{m_k+l_1-1})||_{E_0}, \\ &\quad \phi(||f\phi_{n_k+l_2-1} - f\phi_{m_k+l_1-1}||_{E_0})) \\ &\leq \psi(||f\phi_{n_k+l_2-1} - f\phi_{m_k+l_1-1}||_{E_0}). \end{aligned}$$

On applying limits as $k \to \infty$, we get $\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon)$.
This implies that $F(\psi(\epsilon), \phi(\epsilon)) = \psi(\epsilon)$ and hence $\epsilon = 0$, a contradiction.
Therefore the sequence $\{f\phi_n\}$ is a Cauchy sequence in $f(R_c)$.
Since $f(R_c)$ is complete, we have $f\phi_n \to \phi^*$ as $n \to \infty$ for some $\phi^* \in f(R_c)$.
Since $\phi^* \in f(R_c)$, there exists $\eta \in R_c$ such that $\phi^* = f\eta$ and hence $\lim\limits_{n\to\infty} f\phi_n = f\eta$. From (4.11), we have $\alpha(f\phi_n(c), f\phi_{n+1}(c)) \geq 1$.
From(v), $\alpha(f\phi_n(c), f\eta(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
Clearly $d(f\phi_{n+1}(c), T\eta) \leq H_E(T\phi_n, T\eta)$ and hence

$$\begin{aligned} \psi(d(f\phi_{n+1}(c), T\eta)) &\leq \psi(H_E(T\phi_n, T\eta)) \\ &\leq F(\psi(||f\phi_n - f\eta||_{E_0}), \phi(||f\phi_n - f\eta||_{E_0})) \\ &\leq \psi(||f\phi_n - f\eta||_{E_0}). \end{aligned}$$

On applying limits as $n \to \infty$ on both sides, we get $\psi(d(f\eta(c), T\eta)) \leq \psi(0) = 0$.
Therefore $\psi(d(f\eta(c), T\eta)) = 0$ and hence $f\eta(c) \in T\eta$.
Therefore T and f have a PPF dependent coincidence point in R_c. □

Corollary 4.3. *Let $c \in I$. Let $T : E_0 \to K(E)$, $\alpha : E \times E \to \mathbb{R}^+$ and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ be three functions satisfying the following conditions:*
i) there exist functions $\psi, \phi \in \Psi$, with ψ strictly monotonically increasing, such that

$$\alpha(\gamma(c), \eta(c)) \geq 1 \implies \psi(H_E(T\gamma, T\eta)) \leq F(\psi(||\gamma - \eta||_{E_0}), \phi(||\gamma - \eta||_{E_0}))$$

for any $\gamma, \eta \in E_0$,
ii) T is an α^*-admissible mapping,
iii) R_c is algebraically closed with respect to the difference,
iv) $T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,
v) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, and
vi) there exist $\phi_0 \in R_c$ and $\phi_1(c) \in T\phi_0$ such that $\alpha(\phi_0(c), \phi_1(c)) \geq 1$.
Then T has a PPF dependent fixed point in R_c.

Proof. By taking f = identity mapping in Theorem 4.2, we obtain the desired result. $\square$

The following corollary can be obtained directly from Corollary 4.3 by taking $\psi(t) = t$ and $F(s,t) = s - t$ for any $s, t \in \mathbb{R}^+$.

Corollary 4.4. *Let $c \in I$. Let $T : E_0 \to K(E), \alpha : E \times E \to \mathbb{R}^+$ and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ be three functions satisfying the following conditions*:

i) there exists $\phi \in \Psi$ such that
$$\alpha(\gamma(c), \eta(c)) \geq 1 \implies H_E(T\gamma, T\eta) \leq ||\gamma - \eta||_{E_0} - \phi(||\gamma - \eta||_{E_0})$$
for any $\gamma, \eta \in E_0$,

ii) T is an α^*−admissible mapping,

iii) R_c is algebraically closed with respect to the difference,

iv) $T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,

v) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$, and

vi) there exist $\phi_0 \in R_c$ and $\phi_1(c) \in T\phi_0$ such that $\alpha(\phi_0(c), \phi_1(c)) \geq 1$.

Then T has a PPF dependent fixed point in R_c.

If we take $\phi(t) = (1 - k)t$ for any $t \in \mathbb{R}^+$ and $k \in [0, 1)$ in Corollary 4.4, we get the following.

Corollary 4.5. *Let $c \in I$. Let $T : E_0 \to K(E), \alpha : E \times E \to \mathbb{R}^+$ and $\alpha^* : K(E) \times K(E) \to \mathbb{R}^+$ be three functions satisfying the following conditions*:

i) for any $\gamma, \eta \in E_0$,
$$\alpha(\gamma(c), \eta(c)) \geq 1 \implies H_E(T\gamma, T\eta) \leq k||\gamma - \eta||_{E_0}$$
where $k \in [0, 1)$,

ii) T is an α^*−admissible mapping,

iii) R_c is algebraically closed with respect to the difference,

iv) $T\phi \subseteq R_c(c)$ for any $\phi \in E_0$,

v) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, and

vi) there exist $\phi_0 \in R_c$ and $\phi_1(c) \in T\phi_0$ such that $\alpha(\phi_0(c), \phi_1(c)) \geq 1$.

Then T has a PPF dependent fixed point in R_c.

5. An application

Jachymski[21] introduced the following notation on a Banach space endowed with a graph.

Let (E, d) be a metric space where $d(x, y) = ||x - y||_E$ for all $x, y \in E$ and $\triangle$ denotes the diagonal of the cartesian product of $E \times E$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with E, and the set $E(G)$ of its edges contains all loops; that is $\triangle \subseteq E(G)$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G

from x to y of length $N(N \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, ...N$. A graph G is connected if there is a path between any two vertices, G is weakly connected if $\tilde{G}$ is connected (where $\tilde{G}$ is the induced undirected graph) and G is transitive if $(x, y) \in E(G)$ and $(y, z) \in E(G)$ then $(x, z) \in E(G)$, for more details we refer to [30].

Definition 5.1. ([21]) Let (X, d) be a metric space endowed with a graph G. We say that a self mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G; that is, for any $x, y \in X$,

$$(x, y) \in E(G) \implies (Tx, Ty) \in E(G)$$

and T decreases weights of the edges of G in the following way :
there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \implies d(Tx, Ty) \leq \alpha d(x, y).$$

Theorem 5.2. *Let $c \in I$. Let $T : E_0 \to E$ and E endowed with a graph G. Suppose that the following conditions are true.*

i) there exist functions $\psi, \phi \in \Psi$, with ψ strictly monotonically increasing, such that

$$(f(c), g(c)) \in E(G) \implies \psi(||Tf - Tg||_E) \leq F(\psi(M(f, g)), \phi(M(f, g))),$$

where $M(f, g) = max\{||f - g||_{E_0}, ||f(c) - Tf||_E, ||g(c) - Tg||_E, \frac{1}{2}[||f(c) - Tg||_E + ||g(c) - Tf||_E]\}$

for any $f, g \in E_0$,

ii) if $(f(c), g(c)) \in E(G)$ then $(Tf, Tg) \in E(G)$,

iii) if $(f(c), g(c)) \in E(G)$ and $(g(c), h(c)) \in E(G)$ then $(f(c), h(c)) \in E(G)$ (i.e. G is transitive),

iv) R_c is algebraically closed with respect to the difference,

v) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $(\phi_n(c), \phi_{n+1}(c)) \in E(G)$ for any $n \in \mathbb{N} \cup \{0\}$, then $(\phi_n(c), \phi^*(c)) \in E(G)$ for any $n \in \mathbb{N} \cup \{0\}$, and

vi) there exists $\phi_0 \in R_c$ such that $(\phi_0(c), T\phi_0) \in E(G)$.

Then T has a PPF dependent fixed point in R_c.

Proof. We define $\alpha : E \times E \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } (x, y) \in E(G) \\ \frac{1}{5} & \text{otherwise.} \end{cases}$$

First we show that T is triangular α_c-admissible mapping.

Let $\alpha(f(c), g(c)) \geq 1$. Then $(f(c), g(c)) \in E(G)$. From (ii), we have $(Tf, Tg) \in E(G)$ and hence $\alpha(Tf, Tg) = 2 \geq 1$. Let $\alpha(f(c), g(c)) \geq 1$ and $\alpha(g(c), h(c)) \geq 1$. Then $(f(c), g(c)) \in E(G)$ and $(g(c), h(c)) \in E(G)$. Since G is transitive, we have $(f(c), h(c)) \in E(G)$. Therefore $\alpha(f(c), h(c)) \geq 1$ and hence T is triangular α_c-admissible mapping. From (vi), we have that there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Let $\{\phi_n\}$ be a sequence in E_0 such that $\phi_n \to \phi^*$ as $n \to \infty$ and $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$. Then $(\phi_n(c), \phi_{n+1}(c)) \in E(G)$. From (v), we have $(\phi_n(c), \phi^*(c)) \in E(G)$ for any $n \in \mathbb{N} \cup \{0\}$ and hence $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ for any $n \in \mathbb{N} \cup \{0\}$. Let

$f, g \in E_0$ be such that $\alpha(f(c), g(c)) \geq 1$. Then $(f(c), g(c)) \in E(G)$. From (i), we have T is generalized $\alpha - \psi - \phi - F-$contraction type mapping. Therefore all conditions of Corollary 3.2 are satisfied and hence T has a PPF dependent fixed point in R_c. □

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