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Commutative rings whose proper homomorphic images are nil clean

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Abstract. As defined by Diesl a (noncommutative) ring R is called nil clean if every element of R is a sum of a nilpotent and an idempotent. The purpose of this paper is to study and investigate a new class of rings called nil neat rings, which is presented in [7, Problem 4]. Actually, these rings are a natural generalization of the notion of neat rings, as rings for which any proper homomorphic images are nil clean. It is well-known that any homomorphic image of a nil clean ring is again nil clean. In this paper, it is proved that a nil neat ring which is not nil clean is either a field that is not isomorphic to $\mathbb{Z}_2$ or a one-dimensional domain. We also show that a ring R is nil neat if and only if every nonzero prime ideal of R is maximal, and that for all nonzero maximal ideals M of R, $R/M \cong \mathbb{Z}_2$.

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1. Introduction

Throughout this article we assume that all rings are commutative and possess an identity. The letters Nil(R), Id(R) and U(R) will stand for the set of nilpotents, the set of idempotents and the set of units of R, respectively. The ring R is said to be clean if every element of R can be written as a sum of a unit and an idempotent. The notion of a clean ring was first introduced by Nicholson[11]. Later in [1], the notion of commutative clean rings is discussed and several important results were obtained. All commutative von Neumann regular rings (Boolean rings) and local rings provided the earliest nontrivial examples of clean rings. The notion of a nil clean ring, that is, a (not necessarily commutative) ring in which every element can be written as a sum of a nilpotent and an idempotent is introduced and discussed in detail by Diesl [8]. For other recent articles related to nil clean rings see [2, 4, 5, 9]. A basic property of clean (nil clean) rings is that any homomorphic image of a clean (a nil clean) ring is again clean (nil clean). The ring R is said to be a neat ring provided that every nontrivial homomorphic image of R is clean. W. Wm. McGovern [10] introduced and investigated the notion of a neat ring. The ring of integers, $\mathbb{Z}$, and any nonlocal PID are examples of neat rings which are not clean. We say a ring R is a nil neat ring if every nontrivial homomorphic image

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is nil clean. One can see that this concept is a modification of the notion of neat ring. We first characterize local nil clean rings (Theorem 2.5), and then we shift our attention to the relationship between nil clean and nil neat rings. Finally, we provide a complete classification of the nil neat rings.

2. Main results

As it was mentioned earlier, a ring R is called nil clean provided that every element of R can be written as the sum of an idempotent and a nilpotent. Also, a ring R is said to be indecomposable when the only idempotents of R are 0 and 1. Otherwise, the ring is called decomposable. It is easy to see that local rings are indecomposable. Indeed, let (R, m) be a local ring and $e \in \mathrm{Id(R)}$. It follows easily that e or $e-1$ are units, since, if $e \in m$ and $e-1 \in m$, then $1 \in m$. It is a contradiction. Thus e is 0 or 1, since $e(e-1) = 0$. In the following, we give a list of characterizations of commutative nil clean rings.

Theorem 2.1. *Let R be a ring. Then the following statements are true:*

(1) The class of nil clean rings is closed under homomorphic images.

(2) [8, Proposition 3.15] Let I be any nil ideal of R. Then R is nil clean if and only if R/I is nil clean.

(3) [8, Proposition 3.13] Any finite direct product of nil clean rings is nil clean.

(4) A reduced indecomposable ring is nil clean if and only if it is isomorphic to $\mathbb{Z}_2$. In particular, any nil clean domain is isomorphic to $\mathbb{Z}_2$.

(5) A nil clean ring is zero-dimensional, and hence a clean ring.

Proof. The part (1) is clear since the homomorphic image of a nilpotent (resp. an idempotent) element is again a nilpotent (resp. an idempotent).

For (4), notice that we are saying that 0 is the only nilpotent element and 0 and 1 are the only idempotents. That a ring is nil clean in this case only leaves us with two possibilities for elements in R: 0, 1.

As for (5) let R be a nil clean ring. If R is a domain, we are done, by (4). Now, let P be a nonzero prime ideal of R. Then by (2) and (4), the quotient R/P is isomorphic to $\mathbb{Z}_2$, and so P is a maximal ideal. The second half is true by [1, Corollary 11]. □

In light of Theorem 2.1(4), it is of interest to give a description of indecomposable nil clean rings (which are not necessarily reduced). Recall that an element in a ring, say $r \in R$, is called unipotent if it can be written as $1 + b$ for some nilpotent $b \in R$. It is a well-known fact that every unipotent of R is unit. The set of unipotent elements of R is denoted by $\mathrm{Uni(R)}$. A ring R is said to be a UU ring if all units are unipotent. The class of UU rings has been extensively investigated in [6]. We start with the following easy observation.

Lemma 2.2. *A ring R is indecomposable nil clean if and only if for every element x in R, either $x \in \mathrm{Nil(R)}$ or $x \in \mathrm{Uni(R)}$.*

Proof. The proof is routine by using the definitions. □

Lemma 2.3. *If R is an indecomposable nil clean, then so is every homomorphic image of R.*

Proof. Suppose that R is an indecomposable nil clean ring and S an arbitrary ring such that $f : R \to S$ is an epimorphism. Let $b \in S$. Then $b = f(a)$, for some $a \in R$ and hence either $b = f(n)$ or $b = 1 + f(n')$ for some $n, n' \in \mathrm{Nil(R)}$, by Lemma 2.2. Again, by using Lemma 2.2, we conclude that S is an indecomposable nil clean ring. □

Theorem 2.4. *A ring R is an indecomposable nil clean if and only if R is a* UU *ring and it has exactly one prime ideal.*

Proof. Supoose that R is an indecomposable nil clean ring. It is clear that every unit of R is unipotent. Assume that R is a nil clean domain. Theorem 2.1(4) shows that R is isomorphic to $\mathbb{Z}_2$, and so we are done. Now, suppose, on the contrary, that P_1 and P_2 are two nonzero prime ideals of R. It follows from Chinese reminder theorem and Theorem 2.1(4) that $R/P_1P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Combining this with Lemma 2.3, we conclude that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is an indecomposable nil clean ring. It is a contradiction, because $(\bar{0}, \bar{1}) \in \mathrm{Id}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Conversely, suppose R is a ring with exactly one prime ideal M and every unit of R is unipotent. Let $a \in R$. It is clear that either $a \in \mathrm{Nil(R)}$ or $a \in \mathrm{Uni(R)}$. It follows from Lemma 2.2 that R is an indecomposable nil clean ring. □

We collect the above results of indecomposable nil clean rings in the following theorem.

Theorem 2.5. *Let R be a ring. The following statements are equivalent.*

(1) R is a local nil clean ring.

(2) R is an indecomposable nil clean ring.

(3) Every element of R is either a nilpotent or an unipotent.

(4) R is a UU *ring and it has exactly one prime ideal.*

One of the fundamental properties of clean rings is that every homomorphic image of a clean ring is clean. As mentioned before a neat ring is defined as a ring in which every proper homomorphic image is clean. In particular, the ring of integers, $\mathbb{Z}$, and any nonlocal PID are examples of neat rings which are not clean [10, Proposition 2.4]. In analogy to the above definition, we restrict our attention to a new class of rings as follows:

Definition 2.6. *The ring R is called nil neat if every proper homomorphic image of R is nil clean.*

By the above definition it is clear that every nil neat ring is neat, but the converse is not true in general. For instance, it is well-known that every nontrivial factor of $\mathbb{Z}$ is a product of local rings and hence clean [1, Proposition 2(3)]. Thus, $\mathbb{Z}$ is neat while the ring $\mathbb{Z}_6$ is not nil clean. Consequently, the ring of integers is not nil neat.

Proposition 2.7. *If R is a nil neat ring which is not nil clean, then R is reduced.*

Proof. Suppose, on the contrary, that R is a nil neat ring which is not nil clean and $\mathrm{Nil(R)} \neq 0$. Thus $R/\mathrm{Nil(R)}$ is nil clean and hence R is also nil clean by Theorem 2.1(2). It is a contradiction. □

Theorem 2.8. *Let R be a decomposable ring. Then, R is a nil neat ring if and only if R is nil clean.*

Proof. Assume that R is a decomposable ring. Then there are ideals I and J such that $R = I \oplus J$. Now, if R is nil neat, then $J \cong R/I$(resp. $I \cong R/J$) is also nil clean. Thus, R being a direct product of nil clean rings is nil clean, by Theorem 2.1(2).

□

Theorem 2.9. *The following are equivalent for a ring R.*

(1) R is nil neat.

(2) R/aR is nil clean for every nonzero $a \in R$.

(3) For any collection of nonzero prime ideals $\{P_j\}_{j\in J}$ of R with $I = \cap_{j\in J} P_j$ different than 0 we have R/I is nil clean.

(4) R/aR is nil neat for every $a \in R$.

(5) R/I is nil clean for every nonzero semiprime ideal.

(6) R/I is a Boolean ring for every nonzero semiprime ideal.

(7) $R/P \cong \mathbb{Z}_2$ for every nonzero prime ideal.

Moreover, a homomorphic image of a nil neat ring is nil neat.

Proof. (1) ⇔ (2) follows from the standard fact that a homomorphic image of a nil clean ring is nil clean and the fact that any nontrivial ideal contains a principal nontrivial ideal.
(1) ⇒ (4). We know that R/aR is a nil clean ring where a is a nonzero element of R and so R/aR is nil neat for every $a \in R$.
(4) ⇒ (1) is clear by using $a = 0$.
(1) ⇒ (5) is obvious.
(5) ⇒ (1). Suppose that I is a nonzero ideal of R. Thus, $R/\sqrt{I}$ is nil clean since $\sqrt{I}$ is a nonzero semiprime ideal of R. It follows that $\frac{R/I}{\sqrt{I}/I}$ is nil clean and so does R/I, by [8, Proposition 3.15]. Thus R is a nil neat ring.

(3) $\Leftrightarrow$ (5) are straightforward.
(5) $\Rightarrow$ (6). Assume that I is a nonzero semiprime ideal of R and R/I a nil clean ring. It follows from $\mathrm{Nil(R/I)} = 0$ that R/I is a Boolean ring.
(6) $\Rightarrow$ (7) is clear
(7) $\Rightarrow$ (5). Suppose that I is a nonzero semiprime ideal of R. Then, by an easy verification, we can show that R/I is a subring of the Boolean ring $\prod \mathbb{Z}_2$. Thus R/I is Boolean and hence nil clean. □

While every nil clean ring is nil neat, the following example shows that the two notions are not equivalent in general.

Example 2.10. *Consider the ring $R = \mathbb{Z}_{(2)}$, the localization of the integers at the prime 2. It is clear that $0_{(2)}$ and $2_{(2)}$ are the only prime ideals of $\mathbb{Z}_{(2)}$ and hence $\mathbb{Z}_{(2)}$ is not nil clean by Theorem 2.1(5). To show that $\mathbb{Z}_{(2)}$ is nil neat, we use the homomorphism $f : \mathbb{Z}_{(2)} \to \mathbb{Z}_2$ with $f(m/n) = \bar{1}$ when $(m, 2) = 1$ and otherwise $f(m/n) = \bar{0}$. It is easy to check that f is an epimorphism with the kernel $2_{(2)}$ and so $\mathbb{Z}_{(2)}/2_{(2)} \cong \mathbb{Z}_2$. It follows from Theorem 2.9(7) that R is a nil neat ring.*

Corollary 2.11. *A ring R is nil neat if and only if*

(i) Every nonzero prime ideal of R is maximal, and

(ii) $R/M \cong \mathbb{Z}_2$ for every nonzero maximal ideal.

Proof. Suppose that R is a nil neat ring. (i) can be obtained by applying Theorem 2.1(4) to any nonzero prime ideal of R and (ii) is clear by Theorem 2.9(7). The converse is obvious by using Theorem 2.9.

□

Corollary 2.12. *Let R be a ring. Then R is nil neat if and only if either R is a field or $R/\mathrm{J(R)}$ is isomorphic to a subring of a product of copies of $\mathbb{Z}_2$ and, moreover every nonzero prime ideal of R is maximal.*

Proof. Suppose that R is a nil neat ring which is not a field. Then $R/\mathrm{J(R)}$ is embeddable inside of $\prod_{M \in Max(R)}(R/M)$; which is isomorphic to a product of copies of $\mathbb{Z}_2$ by Corollary 2.11. It follows that $R/\mathrm{J(R)}$ is also isomorphic to a subring of product of copies of $\mathbb{Z}_2$. Conversely, it is clear that R is a nil neat ring when R is a field. Now, assume that R is a non-field and $\varphi : R/\mathrm{J(R)} \to \prod \mathbb{Z}_2$ is a monomorphism. We know that the order of the element $1_{R/\mathrm{J(R)}}$ divides the order of $1_{\varphi(R/\mathrm{J(R)})}$. This implies that $o(1_{R/\mathrm{J(R)}}) = 2$, since $\prod \mathbb{Z}_2$ has characteristic 2. Now let M_j be a nonzero prime ideal of R, and consider the epimorphism $\pi_j : R/\mathrm{J(R)} \to \mathrm{R/M_j}$. By our hypothesis, M_j is a nonzero maximal ideal and it is clear that $\pi_j(1_{R/\mathrm{J(R)}}) = 1_{R/M_j}$ and so, 2 divides the order of the element $1_{R/M_j}$. We conclude that the field R/M_j has characteristic 2 and hence $R/M_j \cong \mathbb{Z}_2$. Combining this fact and Theorem 2.9(7), we deduce that R is a nil neat ring. □

Recall that a ring is called uniquely (nil)clean if every element is uniquely the sum of an idempotent and a (nilpotent)unit.

Theorem 2.13. *For any ring R, the following are equivalent:*

(1) R is a clean UU *ring.*

(2) R is a nil clean ring;

(3) R is a uniquely nil clean ring;

(4) R is a uniquely clean ring such that every prime ideal of R is maximal;

(5) $\mathrm{J(R)}$ *is a nil ideal, and* $R/\mathrm{J(R)}$ *is a Boolean ring;*

(6) R is an exchange UU *ring.*

Moreover, if R is not a domain, then the above six statements are equivalent to:

(7) R is a nil neat ring.

Proof. (1) $\Rightarrow$ (2). If x is an element of a clean ring R, then $x+1 = u+e$ for some $u \in \mathrm{U(R)}$ and $e \in \mathrm{Id(R)}$. This shows, by our hypothesis, that $x+1 = (1+n)+e$ for some $n \in \mathrm{Nil(R)}$, and so $x = n + e$; i.e., R is a nil clean ring.
(2) $\Rightarrow$ (1). Suppose that R is a nil clean ring. We conclude from Theorem 2.1(5) that R is a clean ring. Moreover, the second half can be deduced from [8, Corollary 3.10].
(2) $\Leftrightarrow$ (3). By definition, a uniquely nil clean ring is nil clean and the converse is true by [7, Proposition 1.6].
(3) $\Leftrightarrow$ (4) is true by [3, Corollary 4.2].
(1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows by [6, Theorem 4.3].
Now, let R be a ring which is not a domain.
(7) $\Rightarrow$ (1). Let R be a nil neat ring which is not a domain. The first half is clear from Corollary 2.11 (i) and [1, Corollary 11]. To see the second half, consider a nonzero prime ideal P of R. It follows from Theorem 2.9(7) that $R/P \cong \mathbb{Z}_2$ and so $R = P \cup (1+P)$. We conclude that $\mathrm{U(R)} \subseteq 1 + \mathrm{P}$ for every prime ideal P of R. Thus $\mathrm{U(R)} = 1 + \mathrm{Nil(R)}$.
(2) $\Rightarrow$ (7) is trivially true. □

Corollary 2.14. *If R is a nil neat ring which is not nil clean, then R is either a field that is not isomorphic to $\mathbb{Z}_2$ or a one-dimensional domain.*

Proof. Suppose that R is a nil neat ring which is not nil clean. Theorem 2.13 (7) $\Rightarrow$ (2) implies that R is a domain and so we can deduce from Corollary 2.11 (i) that R is either a one-dimensional domain or a field that by Theorem 2.1(4) is not isomorphic to $\mathbb{Z}_2$. □

Theorem 2.15. *A ring R is nil clean if and only if R is a zero-dimensional ring, and $R/M \cong \mathbb{Z}_2$ for every maximal ideal M.*

Proof. Suppose that R is a nil clean ring. Theorem 2.1(5) shows that $dim R = 0$ and the second part is clear by the fact that a homomorphic image of a nil clean ring is nil clean and Theorem 2.1(4). Conversely, assume that R is a zero-dimensional ring and $R/M \cong \mathbb{Z}_2$ for every maximal ideal M. It follows from Corollary 2.11 that R is a nil neat ring and so R is nil clean, by using Corollary 2.14. □

Corollary 2.16. *A ring R is nil neat if and only if R is either a field, or a zero-dimensional ring in which $R/M \cong \mathbb{Z}_2$ for every nonzero maximal ideal M, or a one-dimensional domain in which $R/N \cong \mathbb{Z}_2$ for every nonzero maximal ideal N.*

Proof. Suppose that R is a nil neat ring. If R is also nil clean, then R is a zero-dimensional ring and $R/M \cong \mathbb{Z}_2$ for every maximal ideal M by Theorem 2.15. Now, if R is not nil clean, then Corollary 2.14 shows that R is either a field which is not isomorphic to $\mathbb{Z}_2$ or a one-dimensional domain. Moreover, from Theorem 2.9 we conclude that $R/N \cong \mathbb{Z}_2$ for every nonzero maximal ideal N. The converse is clear by using Theorems 2.15 and 2.9. □

The following results provide some conditions under which the notions of UU, uniquely clean, nil clean and nil neat rings are the same.

Theorem 2.17. *Let (R, M) be a local ring which is not a field. The following statements are equivalent:*

(1) R is a uniquely clean ring and M is a nil ideal.

(2) R is a nil clean ring.

(3) R is a nil neat ring.

(4) R is a UU *ring.*

Proof. Assume that (R, M) is a local ring with the nonzero maximal ideal. The proof of (1) $\Leftrightarrow$ (2) is obtained by comparing Theorem 2.15 and [1, Theorem 22]. Also, (2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (2) follows by Corollary 2.16. Finally, Theorem 2.13 (7) $\Rightarrow$ (1) implies the implication (3) $\Rightarrow$ (4) and, the converse is already well covered by the fact that local rings are clean by [1, proposition 2(1)] and (1) $\Rightarrow$ (7) of Theorem 2.13. □

Theorem 2.18. *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent:*

(1) R is an uniquely clean ring.

(2) R is a Boolean ring.

(3) R is a nil clean ring.

(4) R is a nil neat ring.

(5) R is a UU *ring.*

Proof. First, suppose that R is a von Neumann regular ring which is not a field. We know that R is a reduced ring and a zero-dimensional ring. (1) $\Rightarrow$ (2). Suppose that R is a uniquely clean ring. It follows from Corollary 2.15 and [3, Corollary 2.3] that R is a reduced nil clean ring, and hence a Boolean ring. Second, (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear and (4) $\Rightarrow$ (1) can be obtained by Corollary 2.16, [3, Corollary 2.3] and the fact that R is a zero-dimensional ring. Finally, (5) $\Leftrightarrow$ (3) is true by applying (5) $\Leftrightarrow$ (3) in [6, Theorem 4.1]. □

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