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Cubic spline scheme on variable mesh for singularly perturbed periodical boundary value problem

A. Puvaneswari[1], A. Ramesh Babu[2] and T. Valanarasu [3][4]

Abstract. In this paper, a numerical method is suggested to solve singularly perturbed periodical boundary value problem for linear second order ordinary differential equation with a small parameter multiplying the first and second derivatives. This method involves a cubic spline scheme along with non-uniform meshes to the above said problem so as to derive the scheme is second order accurate in the maximum norm. The theoretical results are validated through numerical experiments.

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1. Introduction

Singularly Perturbed Differential Equations (SPDEs) arise in diverse areas of applied mathematics, including linearized Navier-Stokes equation at high Reynolds number, heat transport problem with large Peclet numbers, magneto-hydrodynamics duct problems at Hartman number and drift diffusion equation of semiconductor device modeling. In particular, singularly perturbed periodical boundary value problem arises in geophysical fluid dynamics, oceanic and atmospheric circulation [3, 10]. The numerical solution of such a problem exhibits significant difficulties, particularly when the diffusion coefficient is small and it corresponds to a high Reynolds number, Peclet number etc. This implies that sharp boundary and/or interior layers may degrade the accuracy of standard schemes. Therefore, the interest in developing and analyzing efficient numerical methods for singularly perturbed problems has increased enormously. Parameter-uniform numerical methods, with maximum norm errors independent of the singular perturbation parameter, have been developed over thirty years (see [5, 9, 10, 11] and the reference are therein). Therefore these types of SPDEs have to be dealt with separately and diligently.

Motivated by the works of [1, 2, 3, 4, 6, 7, 8], a numerical method involving cubic spline scheme is suggested to solve the following class of singularly

[1]Department of Mathematics, University College of Engineering, Anna University, Tiruchirappalli-620 024, Tamilnadu, India. e-mail: apuvaneswari@gmail.com

[2]Department of Mathematics, School of Engineering, Amrita Vishwa Vidhyapeetham, Coimbatore - 641 112, Tamilnadu, India. e-mail: matramesh2k5@yahoo.co.in

[3]Department of Mathematics, Govt. Arts and Science College (formerly Bharathidasan University College), Perambalur- 621 107, Tamilnadu, India. e-mail: valanarasu@bdu.ac.in

[4]Corresponding author

perturbed periodical boundary value problem:

$$(1.1)\quad Lu(x) \equiv -\varepsilon^2 u''(x) - \varepsilon a(x)u'(x) + b(x)u(x) = f(x),\ x \in \Omega = (0,1),$$
$$(1.2)\quad B_L\, u \equiv u(0) - u(1) = 0,$$
$$(1.3)\quad B_R\, u \equiv \varepsilon(u'(1) - u'(0)) = A,$$

where $\varepsilon\,(0 < \varepsilon \ll 1)$ is a singular perturbation parameter, $\alpha^* \geq a(x) \geq \alpha > 0$, $\beta^* \geq b(x) \geq \beta > 0$ and A is a given constant. We assume that $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions and besides $a(0) = a(1)$, $b(0) = b(1)$, $f(0) = f(1)$ such that the Boundary Value Problem (BVP) (1.1)-(1.3) has a unique solution $u \in C^4(\bar{\Omega})$, $\bar{\Omega} = [0,1]$. The solution $u(x)$ exhibits boundary layer at both end points $x = 0$ and $x = 1$.

Amiraliyev et al.,[1], constructed a difference scheme based on the method of integral identities by employing exponential basis function and interpolating quadrature rules with the weight and remainder terms in the integral form on uniform mesh for the problem (1.1)-(1.3). They proved that the method is pointwise first order convergent, uniformly in ε. Further, a hybrid difference scheme on Shishkin mesh is developed by Zhongdi Cen [4], for the BVP (1.1)-(1.3). He proved that the scheme is almost second-order convergent, uniformly in ε. In this article, we constructed a non-uniform mesh in the boundary layer regions and uniform mesh outside these regions. On this mesh, continuous problem (1.1)-(1.3)(differential equation and periodical boundary conditions) is discretized by employing the cubic spline scheme.

Throughout the paper, C (sometimes subscripted) is a generic constant independent of the nodes, mesh sizes and the perturbation parameter ε. Let $y : D \to \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\parallel y \parallel_D = \sup_{x \in D} |y(x)|$.

2. Preliminaries

In this section, we present maximum principle, stability result and the bounds for derivatives of the solution of the BVP (1.1)-(1.3). The approach to derive derivative bounds is completely different from both [1] and [4] and it is based on [8], thus

Let us consider the following BVP:

$$(2.1)\quad Lu = g(x,\varepsilon), \quad B_L\, u = 0, \quad B_R\, u = A.$$

Definition 2.1. A function $g(x,\varepsilon)$ is said to be of class (K,j) if the derivatives of g with respect to x satisfy

$$(2.2)\quad |g^{(i)}(x,\varepsilon)| \leq K\varepsilon\Big[1 + \varepsilon^{-i-1}\Big(\exp\Big(\frac{-c_0 x}{2\varepsilon}\Big) + \exp\Big(\frac{-c_0(1-x)}{2\varepsilon}\Big)\Big)\Big], \quad 0 \leq i \leq j.$$

Lemma 2.2. (Maximum Principle) *[4] Let v be any smooth function satisfying $B_L\, v = 0$, $B_R\, v \geq 0$ and $Lv(x) \geq 0, \forall x \in \Omega$. Then $v(x) \geq 0, \forall x \in \bar{\Omega}$.*

Lemma 2.3. (Stability Result) *[4] If g is of class $(K, 0)$, then for the solution $u(x)$ of BVPs (2.1) satisfies*

$$||u||_{\bar{\Omega}} \leq \beta^{-1}||f|| + \bar{\beta}|A|,$$

where $\bar{\beta} = c_0^{-1}\coth(c_0/4), \quad c_0 = -\alpha^* + \sqrt{\alpha^{*2} + 4\beta}.$

Lemma 2.4. *Let u be the solution of (2.1) and g be of class (K, j). Then*

$$|u^{(i)}(p)| \leq C\varepsilon^{-i}, \quad p = \{0, 1\}, \quad 1 \leq i \leq j + 2. \tag{2.3}$$

Proof. Equation (2.1) can be rewritten as

$$\varepsilon^2 u'' + \varepsilon a u' = bu - g \equiv h \tag{2.4}$$

Let $P(x)$ be an indefinite integral of $a(x)$. Then the solution of (2.4) is given by

$$u(x) = \int_0^x z(t)dt + c_1 + c_2 \int_0^x \exp(-\varepsilon^{-1}(P(t) - P(0)))dt, \tag{2.5}$$

where $z(x) = \int_0^x \varepsilon^{-2}h(t)\exp(-\varepsilon^{-1}(P(x) - P(t)))dt$ and the constants c_1 and c_2 are determined by the boundary conditions. Using the inequality

$$\exp(-\varepsilon^{-1}(P(x) - P(t))) \leq \exp(-\varepsilon^{-1}c_0(x - t)), \quad t \leq x \tag{2.6}$$

and (2.2), we get

$$|z(x)| \leq C\Big[1 + \exp\Big(\frac{-c_0 x}{2\varepsilon}\Big) + \exp\Big(\frac{-c_0(1 - x)}{2\varepsilon}\Big)\Big].$$

Therefore $\int_0^x |z(t)|dt \leq C$. Also, from (2.5), we have

$$u(1) = \int_0^1 z(t)dt + c_1 + c_2 \int_0^1 \exp(-\varepsilon^{-1}(P(t) - P(0)))dt,$$
$$u(0) = c_1, \quad u'(0) = c_2.$$

Then applying the boundary condition (1.2), we conclude that

$$c_2 = \frac{-\int_0^1 z(t)dt}{\int_0^1 \exp\big(-\varepsilon^{-1}(P(t) - P(0))\big)dt}.$$

Using that $a(x)$ is bounded on Ω, $P(t) - P(0) \leq Ct$ and $\int_0^1 \exp(-\varepsilon^{-1}[P(t) - P(0)])dt \geq C\varepsilon$, we find that $|u'(0)| \leq C\varepsilon^{-1}$ and from (1.3), we get $|u'(1)| \leq C\varepsilon^{-1}$.

The inequality is true for $i = 1$. For $i > 1$, we obtain the result by induction and repeated differentiation process on the equation (2.1). □

Lemma 2.5. *Let u be the solution of (2.1) and g be of class (K, j). Then*

$$|u^{(i)}(x)| \leq C\Big[1 + \varepsilon^{-i}\Big(\exp\Big(\frac{-c_0 x}{\varepsilon}\Big) + \exp\Big(\frac{-c_0(1 - x)}{\varepsilon}\Big)\Big)\Big], \quad 0 \leq i \leq j + 1.$$

Proof. We prove the theorem by mathematical induction. From the Stability result, the inequality holds for $i = 0$. Differentiating (2.4) on both sides $(i-1)$ times, and setting $z(x) = u^{(i)}(x)$, we get

$$\varepsilon^2 z'(x) + \varepsilon a(x) z(x) = h(x),$$

where $h(x) = \varepsilon h_1(x) + h_2(x)$, $h_1(x)$ depends on a, u and $h_2(x)$ depends on b, u, g and their derivatives up to and including $(i-1)$. Using (2.2) and the inductive hypothesis, we have

$$h(x) \le C\varepsilon\Big[1 + \varepsilon^{-i}\Big(\exp\Big(\frac{-c_0 x}{2\varepsilon}\Big) + \exp\Big(\frac{-c_0(1-x)}{2\varepsilon}\Big)\Big)\Big]. \tag{2.7}$$

Let $P(x)$ be an indefinite integral of $a(x)$. Then

$$z(x) = z(0)\exp\left(\frac{-[P(x)-P(0)]}{\varepsilon}\right) + \varepsilon^{-2}\int_0^x h(t)\exp\left(\frac{-[P(x)-P(t)]}{\varepsilon}\right)dt.$$

From (2.6), (2.7) and Lemma 2.4, we have

$$\begin{aligned}|z(x)| \le & C\varepsilon^{-i}\exp(\frac{-c_0 x}{\varepsilon}) \\ & + C\varepsilon^{-1}\int_0^x \Big(\exp(\frac{-c_0(x-t)}{\varepsilon}) \\ & + \varepsilon^{-i}\{\exp(\frac{-c_0(2x-t)}{2\varepsilon}) + \exp(\frac{-c_0(1+x-3t)}{2\varepsilon})\}\Big)dt,\end{aligned}$$

and the desired result follows from the above inequality. □

Theorem 2.6. *Let u satisfy (1.1), then $u(x) = \gamma_1 v(x) + \gamma_2 w(x) + z(x)$,*

$$|z^{(i)}(x)| \le C\Big[1 + \varepsilon^{-i+1}\Big(\exp\Big(\frac{-c_0 x}{\varepsilon}\Big) + \exp\Big(\frac{-c_0(1-x)}{\varepsilon}\Big)\Big)\Big],$$

where $v(x) = \exp(-c_0\varepsilon^{-1}x), w(x) = \exp(-c_0\varepsilon^{-1}(1-x))$, and $|\gamma_1| \le C_1, |\gamma_2| \le C_2$.

Proof. Set $\gamma_1 = \dfrac{\varepsilon u'(0)}{c_0}$, $\gamma_2 = \dfrac{\varepsilon u'(1)}{c_0}$. We have $|\gamma_1| \le C_1$ and $|\gamma_2| \le C_2$. Further set $z(x) = u(x) - \gamma_1 v(x) - \gamma_2 w(x)$. We see that

$$Lz = f - bu - \gamma_1\varepsilon\big(c_0 - a(x)\big)v'(x) + \gamma_2\varepsilon\big(c_0 + a(x)\big)w'(x) + bz = g.$$

Differentiating once, we get $Lz' = g' + a'(\varepsilon z') - b'z$. Using a similar argument as in [8] and by Lemma 2.5 with u replaced by z', we arrive at the desired result. □

3. Construction of Mesh

Let us consider the BVP (1.1)-(1.3). Since there are boundary layers at $x = 0$ and at $x = 1$, we decompose $\bar{\Omega}$ into three subdomains

$$[0, \sigma], \quad [\sigma, 1 - \sigma], \quad [1 - \sigma, 1],$$

where $\sigma = \varepsilon \log(\frac{1}{\varepsilon})$ denotes the width of the boundary layer. Let n_1, n_2 and n_3 be the number of points in the subdomains $[0, \sigma]$, $[\sigma, 1 - \sigma]$ and $[1 - \sigma, 1]$, respectively, such that $n_1 + n_2 + n_3 = N$ and $n_1 = n_3$. Further, let the positive constants $\tilde{h}_1$ and K be known. Then we generate the mesh as follows:

In the subdomain $[0, \sigma]$, the grid is non-uniform and is defined thus:

$$\tilde{h}_j = \tilde{h}_{j-1} + K[\frac{\tilde{h}_{j-1}}{\varepsilon}] \min\left(\tilde{h}^2_{j-1}, \varepsilon\right), \quad j = 2, ..., n_1.$$

Now, let

$$\tilde{q} = \sum_{j=1}^{n_1} \tilde{h}_j$$
$$q = \frac{\sigma}{\tilde{q}}$$
$$h_j = q\tilde{h}_j, \quad j = 1, ..., n_1$$

In the subdomain $[\sigma, 1 - \sigma]$, the grid is uniform and is defined as follows:

$$h_j = \frac{1 - 2\sigma}{n_2}, \quad j = n_1 + 1, ..., n_1 + n_2.$$

In the subdomain $[1 - \sigma, 1]$ the grid is the mirror image of the grid on $[0, \sigma]$, and therfore will be given by

$$h_j = h_{N+1-j}, \quad j = n_1 + n_2 + 1, ..., N$$

and hence we define $x_0 = 0, \quad x_j = x_{j-1} + h_j, \quad j = 1, ..., N.$

4. Derivation of the Difference Scheme

In this section, we derive cubic spline difference scheme which will be used to approximate the differential equations (1.1) and the periodical boundary condition (1.3).

Let $x_0 = 0$, $x_N = 1$, $\quad x_j = x_0 + \sum\limits_{k=1}^{j} h_k, \quad j = 1, ..., N$, $h_j = x_j - x_{j-1}$ be the mesh. For the given values $u(x_0), u(x_1), ..., u(x_N)$ of a function $u(x)$ at the nodal points $x_0, x_1, ..., x_N$ there exists an interpolating cubic spline function $S_j(x)$ with the following properties:

(i) $S_j(x)$ coincides with a polynomial of degree three on each subintervals $[x_{j-1}, x_j]$, $j = 1, ..., N$

(ii) $S_j(x) \in C^2(\bar{\Omega})$
(iii) $S_j(x_j) = u(x_j), \quad j = 0, 1, ..., N.$
Then the cubic spline function can be written as follows:

(4.1)
$$S_j(x) = \frac{(x_j - x)^3}{6h_j} M_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} M_j + (u(x_{j-1}) - \frac{h_j^2}{6} M_{j-1})(\frac{x_j - x}{h_j}) + (u(x_j) - \frac{h_j^2}{6} M_j)(\frac{x - x_{j-1}}{h_j}),$$

where $x \in [x_{j-1}, x_j]$ and $M_j = S_j''(x_j)$, $j = 0, ..., N$.

We will derive the difference scheme by employing the above spline function with a view to gaining the approximate solution of $u(x)$.

Differentiating (4.1) and denoting the nodal interpolants of $u(x)$ by u_j's, we get

(4.2)
$$S_j'(x) = -\frac{(x_j - x)^2}{2h_j} M_{j-1} + \frac{(x - x_{j-1})^2}{2h_j} M_j + (\frac{u_j - u_{j-1}}{h_j}) - (\frac{M_j - M_{j-1}}{6}) h_j$$

Since $S_j(x) \in C^2(\bar{\Omega})$, we have

$$S_j'(x_j) = S_{j+1}'(x_j).$$

This gives

$$\frac{h_j}{6} M_{j-1} + \frac{h_j + h_{j+1}}{3} M_j + \frac{h_{j+1}}{6} M_{j+1} = \frac{u_{j+1} - u_j}{h_{j+1}} - \frac{u_j - u_{j-1}}{h_j}, \tag{4.3}$$

where

$$M_j = \frac{1}{\varepsilon^2}(-f_j - \varepsilon a_j u_j' + b_j u_j). \tag{4.4}$$

The second order approximations for the first order derivative of $u(x)$ are obtained by employing Taylor series expansion for u around x_j in order to get the following approximations for u_{j+1} and u_{j-1}

$$u_{j+1} \simeq u_j + h_{j+1} u_j' + \frac{h_{j+1}^2}{2} u_j'' \tag{4.5}$$

$$u_{j-1} \simeq u_j - h_j u_j' + \frac{h_j^2}{2} u_j''. \tag{4.6}$$

From equations (4.5) and (4.6), we get

$$u_j' \simeq \frac{1}{h_j h_{j+1}(h_j + h_{j+1})}(-h_{j+1}^2 u_{j-1} + (h_{j+1}^2 - h_j^2) u_j + h_j^2 u_{j+1}) \tag{4.7}$$

$$u_j'' \simeq \frac{2}{h_j h_{j+1}(h_j + h_{j+1})}(h_{j+1} u_{j-1} - (h_j + h_{j+1}) u_j + h_j u_{j+1}). \tag{4.8}$$

Also, we have

$$u'_{j+1} \simeq u'_j + h_{j+1}u''_j \tag{4.9}$$

and

$$u'_{j-1} \simeq u'_j - h_j u''_j. \tag{4.10}$$

Substituting (4.7) and (4.8) into (4.9), we get

(4.11)
$$u'_{j+1} \simeq \frac{1}{h_j h_{j+1}(h_j + h_{j+1})}[h_{j+1}^2 u_{j-1} - (h_j + h_{j+1})^2 u_j + (h_j^2 + 2h_j h_{j+1})u_{j+1}]$$

and again from (4.10), we have

(4.12)
$$u'_{j-1} \simeq \frac{1}{h_j h_{j+1}(h_j + h_{j+1})}[-(h_{j+1}^2 + 2h_j h_{j+1})u_{j-1} + (h_j + h_{j+1})^2 u_j - h_j^2 u_{j+1}].$$

From the above detailed computations, we arrive at the following linear system of equations:

$$L^N u_j = Qf_j, \quad j = 1, ..., N-1, \tag{4.13}$$

where

$$\begin{aligned} L^N u_j &= r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}, \\ Qf_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}. \end{aligned}$$

(4.14)
$$\begin{aligned} r_j^- &= \frac{(h_{j+1} + 2h_j)}{6(h_j + h_{j+1})}\varepsilon a_{j-1} + \frac{h_{j+1}}{3h_j}\varepsilon a_j - \frac{h_{j+1}^2}{6h_j(h_j + h_{j+1})}\varepsilon a_{j+1} + \frac{h_j}{6}b_{j-1} - \frac{\varepsilon^2}{h_j}, \\ r_j^c &= -\frac{(h_j + h_{j+1})}{6h_{j+1}}\varepsilon a_{j-1} - \frac{(h_{j+1}^2 - h_j^2)}{3h_j h_{j+1}}\varepsilon a_j + \frac{(h_j + h_{j+1})}{6h_j}\varepsilon a_{j+1} \\ &\quad + \frac{(h_j + h_{j+1})}{3}b_j + \frac{\varepsilon^2}{h_j} + \frac{\varepsilon^2}{h_{j+1}}, \\ r_j^+ &= \frac{h_j^2}{6h_{j+1}(h_j + h_{j+1})}\varepsilon a_{j-1} - \frac{h_j}{3h_{j+1}}\varepsilon a_j - \frac{(2h_{j+1} + h_j)}{6(h_j + h_{j+1})}\varepsilon a_{j+1} + \frac{h_{j+1}}{6}b_{j+1} - \frac{\varepsilon^2}{h_{j+1}}, \\ q_j^- &= \frac{h_j}{6}, \quad q_j^c = \frac{(h_j + h_{j+1})}{3}, \quad q_j^+ = \frac{h_{j+1}}{6}. \end{aligned}$$

Now, we approximate the periodical boundary condition (1.3) in the following manner. From (4.1), we can find one sided limits of the first order derivative as

$$S'_j(x_j-) = \frac{h_j}{6}M_{j-1} + \frac{h_j}{3}M_j + \frac{u_j - u_{j-1}}{h_j} \tag{4.15}$$

and

$$S'_j(x_j+) = -\frac{h_{j+1}}{3}M_j - \frac{h_{j+1}}{6}M_{j+1} + \frac{u_{j+1}-u_j}{h_{j+1}}. \tag{4.16}$$

Substituting M_j to (4.15) and (4.16), we get an approximation of the one sided first order derivatives at the boundary condition (1.3). It is noted that the mesh widths of the subintervals $[0,\sigma]$ and $[1-\sigma,1]$ are the same and therefore let it be h_N. Hence, discretization of the periodical boundary condition (1.3) reduces to

(4.17)

$$\begin{aligned}
&[\frac{\varepsilon}{12}a_{N-1} - \frac{\varepsilon}{6}a_N]u_{N-2} + [\frac{2\varepsilon}{3}a_N + \frac{h_N}{6}b_{N-1} - \frac{\varepsilon^2}{h_N}]u_{N-1} \\
&+[-\frac{\varepsilon}{12}a_{N-1} - \frac{\varepsilon}{2}a_N + \frac{h_N}{3}b_N + \frac{\varepsilon^2}{h_N}]u_N \\
&+[\frac{\varepsilon}{2}a_0 + \frac{\varepsilon}{12}a_1 + \frac{h_N}{3}b_0 + \frac{\varepsilon^2}{h_N}]u_0 + [-\frac{2\varepsilon}{3}a_0 + \frac{h_N}{6}b_1 - \frac{\varepsilon^2}{h_N}]u_1 + [\frac{\varepsilon}{6}a_0 - \frac{\varepsilon}{12}a_1]u_2 \\
&= A\varepsilon + \frac{h_N}{6}f_{N-1} + \frac{h_N}{3}f_N + \frac{h_N}{3}f_0 + \frac{h_N}{6}f_1.
\end{aligned}$$

From equation (4.13), we have

$$u_2 = \frac{-r_1^- u_0 - r_1^c u_1 + \frac{h_N}{6}f_0 + \frac{2h_N}{3}f_1 + \frac{h_N}{6}f_2}{r_1^+}$$

and

$$u_{N-2} = \frac{-r_{N-1}^c u_{N-1} - r_{N-1}^+ u_N + \frac{h_N}{6}f_{N-2} + \frac{2h_N}{3}f_{N-1} + \frac{h_N}{6}f_N}{r_{N-1}^-}.$$

We obtain the following difference scheme by eliminating u_0, u_2, u_{N-2} from equation (4.17),

$$B_R^N u_N = Qf_N,$$

where

$$\begin{aligned}
B_R^N u_N =& r_N^- u_{N-1} + r_N^c u_N + r_N^+ u_1, \\
Qf_N =& A\varepsilon + \frac{h_N}{6}f_{N-1} + \frac{h_N}{3}f_N + \frac{h_N}{3}f_0 + \frac{h_N}{6}f_1 \\
&- (\frac{k_1}{B_1})[\frac{h_N}{6}f_{N-2} + \frac{2h_N}{3}f_{N-1} + \frac{h_N}{6}f_N] \\
&- (\frac{k_2}{B_2})[\frac{h_N}{6}f_0 + \frac{2h_N}{3}f_1 + \frac{h_N}{6}f_2],
\end{aligned}$$

(4.18)

$$r_N^- = (\frac{k_1}{B_1})[\frac{\varepsilon}{3}a_{N-2} - \frac{\varepsilon}{3}a_N - \frac{2h_N}{3}b_{N-1} - \frac{2\varepsilon^2}{h_N}] + [\frac{2\varepsilon}{3}a_N + \frac{h_N}{6}b_{N-1} - \frac{\varepsilon^2}{h_N}],$$

$$r_N^c = (\frac{k_1}{B_1})[\frac{-\varepsilon}{12}a_{N-2} + \frac{\varepsilon}{3}a_{N-1} + \frac{\varepsilon}{4}a_N - \frac{h_N}{6}b_N + \frac{\varepsilon^2}{h_N}]$$
$$+ [\frac{-\varepsilon}{12}a_{N-1} - \frac{\varepsilon}{2}a_N + \frac{h_N}{3}b_N + \frac{\varepsilon^2}{h_N}],$$
$$+ [\frac{\varepsilon}{2}a_0 + \frac{\varepsilon}{12}a_1 + \frac{h_N}{3}b_0 + \frac{\varepsilon^2}{h_N}]$$
$$+ (\frac{k_2}{B_2})[\frac{-\varepsilon}{4}a_0 - \frac{\varepsilon}{3}a_1 + \frac{\varepsilon}{12}a_2 - \frac{h_N}{6}b_0 + \frac{\varepsilon^2}{h_N}],$$

$$r_N^+ = (\frac{k_2}{B_2})[\frac{\varepsilon}{3}a_0 - \frac{\varepsilon}{3}a_2 - \frac{2h_N}{3}b_1 - \frac{2\varepsilon^2}{h_N}] + [-\frac{2\varepsilon}{3}a_0 + \frac{h_N}{6}b_1 - \frac{\varepsilon^2}{h_N}],$$

$$k_1 = [\frac{\varepsilon}{12}a_{N-1} - \frac{\varepsilon}{6}a_N],$$

$$k_2 = [\frac{\varepsilon}{6}a_0 - \frac{\varepsilon}{12}a_1],$$

$$B_1 = [\frac{\varepsilon}{4}a_{N-2} + \frac{\varepsilon}{3}a_{N-1} - \frac{\varepsilon}{12}a_N + \frac{h_N}{6}b_{N-2} - \frac{\varepsilon^2}{h_N}],$$

$$B_2 = [\frac{\varepsilon}{12}a_0 - \frac{\varepsilon}{3}a_1 - \frac{\varepsilon}{4}a_2 + \frac{h_N}{6}b_2 - \frac{\varepsilon^2}{h_N}]$$

and

$$B_L^0 u_0 = u_0 - u_N = 0.$$

Thus, we obtain the difference scheme as

$$L^N u_j = Qf_j, \quad j = 1, ..., N-1$$
$$B_L^0 u_0 = 0, \quad B_R^N u_N = Qf_N.$$

Remark 4.1. Since the boundary condition (1.3) contains the first derivative at both end points $x = 0$ and $x = 1$, the discretization of (1.3) by spline difference scheme will not yield a three point difference scheme. Therefore, we made a scheme to be a three point difference scheme by using the difference scheme (4.13).

5. Error Estimate

We will make use of comparison functions and discrete maximum principle in order to derive the error estimate of the solution. Now we state the following lemmas as in [8, 2] which are indispensable segments in the proof of the final result.

Lemma 5.1. *(Discrete Maximum Principle) Let* $\{u_j\}$ *be a set of values at the grid points* x_j*, satisfying* $B_L^0 u_0 = 0, B_R^N u_N \geq 0$ *and* $L^N u_j \geq 0, j = 1, ..., N-1$*, then* $u_j \geq 0, j = 0, 1, ..., N$

Lemma 5.2. *If* $F_1(h, \varepsilon) \geq 0$ *and* $F_2(h, \varepsilon) \geq 0$ *are such that*

$$L^N(F_1(h,\varepsilon)\phi_j + F_2(h,\varepsilon)\psi_j) \geq L^N(\pm e_j) = \pm\tau_j(u), \quad j = 1, ..., N-1$$
$$B_R^N(F_1(h,\varepsilon)\phi_N + F_2(h,\varepsilon)\psi_N) \geq L^N(\pm e_N) = \pm\tau_N(u),$$

then the discrete maximum principle implies that

$$|e_j| \leq F_1(h,\varepsilon)|\phi_j| + F_2(h,\varepsilon)|\psi_j|,$$

where $e_j = |u(x_j) - u_j|$ *for each* j *and* ϕ *and* ψ *are two comparison functions.*

We use the following two comparison functions

$$\phi(x) = C\exp\left[-\frac{2C_1 x}{\varepsilon}\right] \quad \text{and} \quad \psi(x) = C\exp\left[-\frac{2C_2(1-x)}{\varepsilon}\right]$$

Remark 5.3. The following inequalities hold for $\Lambda_j \in \{\phi_j, \psi_j\}$

$$L^N\Lambda_j \geq M, \quad \text{when} \quad Ch_c^2 \leq \varepsilon$$

and

$$L^N\Lambda_j \geq Mh_c, \quad \text{when} \quad Ch_c^2 \geq \varepsilon$$

where $h_c = \max_j h_j$ (= a constant).

Now, we estimate the truncation error of the scheme (4.13). At first, let us consider the case in which $Ch_c^2 \leq \varepsilon$. We have

$$\tau_j(u) = T_{0,j}u_j + T_{1,j}u_j' + T_{2,j}u_j'' + T_{3,j}u_j''' + \text{remainder terms}, \quad j = 1, ..., N-1$$

where

$$T_{0,j} = (r_j^- + r_j^c + r_j^+) - (q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}),$$
$$T_{1,j} = (h_{j+1}r_j^+ - h_j r_j^-) + \{q_j^-(\varepsilon a_{j-1} + h_j b_{j-1}) + q_j^c \varepsilon a_j + q_j^+(\varepsilon a_{j+1} - h_{j+1}b_{j+1})\},$$
$$T_{2,j} = \left(\frac{h_j^2}{2}r_j^- + \frac{h_{j+1}^2}{2}r_j^+\right) + \varepsilon^2(q_j^- + q_j^c + q_j^+)$$
$$- \left[q_j^-\left(h_j\varepsilon a_{j-1} + \frac{h_j^2}{2}b_{j-1}\right) + q_j^+\left(-h_{j+1}\varepsilon a_{j+1} + \frac{h_{j+1}^2}{2}b_{j+1}\right)\right]$$

and

$$T_{3,j} = \left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right) + \varepsilon^2(q_j^- h_j - q_j^+ h_{j+1})$$
$$+ q_j^-\left(\frac{h_j^2}{2}\varepsilon a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right) + q_j^+\left(\frac{h_{j+1}^2}{2}\varepsilon a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right).$$

Using (4.14), we find that $T_{0,j} = 0, T_{1,j} = 0, T_{2,j} = 0$ and $|T_{3,j}| \le C\varepsilon h_c^3$. Also, by using $u_0 = u_N, f_0 = f_N$, the truncation error at the boundary point is given by

$$\tau_N(u) = T_{0,N}u_N + T_{1,N}u'_N + T_{2,N}u''_N + T_{3,N}u'''_N + \text{remainder terms},$$

where

$$\begin{aligned}
T_{0,N} =&(r_N^- + r_N^c + r_N^+) - \Big(\frac{h_N}{6}b_{N-1} + \frac{h_N}{3}b_N + \frac{h_N}{3}b_N + \frac{h_N}{6}b_1\Big)\\
&+ \frac{k_1}{B_1}\Big(\frac{h_N}{6}b_{N-2}\\
&+ \frac{2h_N}{3}b_{N-1} + \frac{h_N}{6}b_N\Big) + \frac{k_2}{B_2}\Big(\frac{h_N}{6}b_N + \frac{2h_N}{3}b_1 + \frac{h_N}{6}b_2\Big),\\
T_{1,N} =&(-h_N r_N^- + h_N r_N^+) + \Big(\frac{h_N}{6}\varepsilon a_{N-1} + \frac{h_N^2}{6}b_{N-1} + \frac{2h_N}{3}\varepsilon a_N + \frac{h_N}{6}\varepsilon a_1 - \frac{h_N^2}{6}b_1\Big)\\
&+ \Big(\frac{k_1}{B_1}\Big)\Big(-\frac{h_N}{6}\varepsilon a_{N-2} - \frac{h_N^2}{3}b_{N-2} - \frac{2h_N}{3}\varepsilon a_{N-1} - \frac{h_N}{6}\varepsilon a_N - \frac{2h_N^2}{3}b_{N-1}\Big)\\
&+ \Big(\frac{k_2}{B_2}\Big)\Big(-\frac{h_N}{6}\varepsilon a_0 + \frac{2h_N^2}{3}b_1 - \frac{2h_N}{3}\varepsilon a_1 - \frac{h_N}{6}\varepsilon a_2 + \frac{h_N^2}{3}b_2\Big),\\
T_{2,N} =&\frac{h_N^2}{2}(r_N^- + r_N^+) - \left(-h_N\varepsilon^2 + \frac{h_N^2}{6}\varepsilon a_{N-1} + \frac{h_N^3}{12}b_{N-1} - \frac{h_N^2}{6}\varepsilon a_1 + \frac{h_N^3}{12}b_1\right)\\
&+ \left(\frac{k_1}{B_1}\right)\left(-h_N\varepsilon^2 + \frac{h_N^2}{3}\varepsilon a_{N-2} + \frac{h_N^3}{3}b_{N-2} + \frac{2h_N^2}{3}\varepsilon a_{N-1} + \frac{h_N^3}{3}b_{N-1}\right)\\
&\quad + \left(\frac{k_2}{B_2}\right)\left(-h_N\varepsilon^2 - \frac{2h_N^2}{3}\varepsilon a_1 + \frac{h_N^3}{3}b_1 - \frac{h_N^2}{3}\varepsilon a_2 + \frac{h_N^3}{3}b_2\right),\\
T_{3,N} =&\frac{h_N^3}{6}(r_N^+ - r_N^-) - \frac{h_N}{6}\left(-\frac{h_N^2}{2}\varepsilon a_{N-1} - \frac{h_N^3}{6}b_{N-1} - \frac{h_N^2}{2}\varepsilon a_1 + \frac{h_N^3}{6}b_1\right) +\\
&\left(\frac{k_1}{B_1}\right)\left(\frac{h_N}{6}\left(2h_N\varepsilon^2 - 2h_N^2\varepsilon a_{N-2} - \frac{4h_N^3}{3}b_{N-2}\right) +\right.\\
&\left.\frac{2h_N}{3}\left(h_N\varepsilon^2 - \frac{h_N^2}{2}\varepsilon a_{N-1} - \frac{h_N^3}{6}b_{N-1}\right)\right) +\\
&\left(\frac{k_2}{B_2}\right)\left(\frac{h_N}{6}\left(-2h_N\varepsilon^2 - 2h_N^2\varepsilon a_2 + \frac{4h_N^3}{3}b_2\right) +\right.\\
&\left.\frac{2h_N}{3}\left(-h_N\varepsilon^2 - \frac{h_N^2}{2}\varepsilon a_1 + \frac{h_N^3}{6}b_1\right)\right).
\end{aligned}$$

Using (4.18), we find that $T_{0,N} = 0, T_{1,N} = 0, T_{2,N} = 0$ and $|T_{3,N}| \le C\varepsilon h_c^3$. Now, from Theorem 2.6 and the inequality (2.3), we have

$$v_j''' = -\Big(\frac{c_0}{\varepsilon}\Big)^2 u'(0)\exp\Big(-\frac{c_0}{\varepsilon}x_j\Big), \quad w_j''' = \Big(\frac{c_0}{\varepsilon}\Big)^2 u'(1)\exp\Big(-\frac{c_0}{\varepsilon}(1-x_j)\Big),$$

and

$$|z'''(x)| \le C\left[1 + \varepsilon^{-2}\left(\exp\Big(-\frac{c_0 x_j}{\varepsilon}\Big) + \exp\left(-\frac{c_0(1-x_j)}{\varepsilon}\right)\right)\right].$$

Therefore, for $Ch_c^2 \leq \varepsilon$, we have the following estimates.

$$|\tau_j(v)| \leq \frac{Ch_c^3}{\varepsilon^2} \exp\left(-\frac{c_0}{\varepsilon} x_j\right), \quad |\tau_j(w)| \leq \frac{Ch_c^3}{\varepsilon^2} \exp\left(-\frac{c_0}{\varepsilon}(1-x_j)\right)$$

and

$$|\tau_j(z)| \leq C\varepsilon h_c^3 \left[1 + \varepsilon^{-2}\left(\exp\left(-\frac{c_0}{\varepsilon} x_j\right) + \exp\left(-\frac{c_0}{\varepsilon}(1-x_j)\right)\right)\right], \quad Ch_c^2 \leq \varepsilon.$$

Since

$$\tau_j(u) = \tau_j(v) + \tau_j(w) + \tau_j(z), \quad j = 1, 2, ..., N,$$

we have

$$|\tau_j(u)| \leq C\frac{h_c^3}{\varepsilon^2}\left[1 + \exp\left(-\frac{c_0 x_j}{\varepsilon}\right) + \exp\left(-\frac{c_0 x_j}{\varepsilon}\right)\right], \quad Ch_c^2 \leq \varepsilon.$$

On the other hand, when $Ch_c^2 \geq \varepsilon$, we use the following expression for truncation error

$$\begin{aligned}\tau_j(u) = &\left(\frac{h_{j+1}^3}{6} r_j^+ - \frac{h_j^3}{6} r_j^-\right) u'''(\zeta_1) + \varepsilon^2 (h_j q_j^- - q_j^+ h_{j+1}) u'''(\zeta_2) \\ &+ q_j^- \left(\frac{h_j^2}{2}\varepsilon a_{j-1} + \frac{h_j^3}{6} b_{j-1}\right) u'''(\zeta_3) + q_j^+ \left(\frac{h_{j+1}^2}{2}\varepsilon a_{j+1} - \frac{h_{j+1}^3}{6} b_{j+1}\right) u'''(\zeta_4),\end{aligned}$$

where $x_{j-1} < \zeta_i < x_{j+1}, \quad i = 1(1)4$. After some algebraic simplifications, we find the sharper estimates

$$\left|\frac{h_{j+1}^3}{6} r_j^+ - \frac{h_j^3}{6} r_j^-\right| \leq C\varepsilon h_c^3, \quad \left|\varepsilon^2 (q_j^- h_j - q_j^+ h_{j+1})\right| \leq C\varepsilon h_c^3,$$

$$\left|q_j^- \left(\frac{h_j^2}{2}\varepsilon a_{j-1} + \frac{h_j^3}{6} b_{j-1}\right)\right| \leq C\varepsilon h_c^3, \quad \left|q_j^+ \left(\frac{h_{j+1}^2}{2}\varepsilon a_{j+1} - \frac{h_{j+1}^3}{6} b_{j+1}\right)\right| \leq C\varepsilon h_c^3.$$

Also, at the boundary point, the truncation error is given by

$$\begin{aligned}\tau_N(u) = &\frac{h_N^3}{6}(r_N^+ - r_N^-) u'''(\zeta_N) - \\ &\frac{h_N}{6}\left(-\frac{h_N^2}{2}\varepsilon a_{N-1} - \frac{h_N^3}{6} b_{N-1} - \frac{h_N^2}{2}\varepsilon a_1 + \frac{h_N^3}{6} b_1\right) u'''(\zeta_N) + \\ &\left(\frac{k_1}{B_1}\right)\left(\frac{h_N}{6}\left(2h_N\varepsilon^2 - 2h_N^2\varepsilon a_{N-2} - \frac{4h_N^3}{3} b_{N-2}\right) + \right. \\ &\left.\frac{2h_N}{3}\left(h_N\varepsilon^2 - \frac{h_N^2}{2}\varepsilon a_{N-1} - \frac{h_N^3}{6} b_{N-1}\right)\right) u'''(\zeta_N) + \\ &\left(\frac{k_2}{B_2}\right)\left(\frac{h_N}{6}\left(-2h_N\varepsilon^2 - 2h_N^2\varepsilon a_2 + \frac{4h_N^3}{3} b_2\right) + \right. \\ &\left.\frac{2h_N}{3}\left(-h_N\varepsilon^2 - \frac{h_N^2}{2}\varepsilon a_1 + \frac{h_N^3}{6} b_1\right)\right) u'''(\zeta_N),\end{aligned}$$

where $x_{N-1} < \zeta_N < x_N$. Now

$$\left|\frac{h_N^3}{6}(r_N^+ - r_N^-)\right| \le C\varepsilon h_c^3,$$

$$\left|\frac{h_N}{6}\left(-\frac{h_N^2}{2}\varepsilon a_{N-1} - \frac{h_N^3}{6}b_{N-1} - \frac{h_N^2}{2}\varepsilon a_1 + \frac{h_N^3}{6}b_1\right)\right| \le C\varepsilon h_c^3,$$

$$\left|\left(\frac{k_1}{B_1}\right)\left(\frac{h_N}{6}\left(2h_N\varepsilon^2 - 2h_N^2\varepsilon a_{N-2} - \frac{4h_N^3}{3}b_{N-2}\right) + \frac{2h_N}{3}\left(h_N\varepsilon^2 - \frac{h_N^2}{2}\varepsilon a_{N-1} - \frac{h_N^3}{6}b_{N-1}\right)\right)\right| \le C\varepsilon h_c^3,$$

$$\left|\left(\frac{k_2}{B_2}\right)\left(\frac{h_N}{6}\left(-2h_N\varepsilon^2 - 2h_N^2\varepsilon a_2 + \frac{4h_N^3}{3}b_2\right) + \frac{2h_N}{3}\left(-h_N\varepsilon^2 - \frac{h_N^2}{2}\varepsilon a_1 + \frac{h_N^3}{6}b_1\right)\right)\right| \le C\varepsilon h_c^3.$$

Using the above estimates and the above expression for $\tau_j(u)$, we obtain the same estimates for $\tau_j(v), \tau_j(w)$ and $\tau_j(z)$ as similar to the case of $Ch_c^2 \le \varepsilon$. Finally we choose

$$F_1 = h_c^2 \exp\left(-\frac{2c_0 x_j}{\varepsilon}\right)$$

and

$$F_2 = h_c^2 \exp\left(-\frac{2c_0(1-x_j)}{\varepsilon}\right).$$

Since Lemma 5.2 is true for both the cases $Ch_c^2 \le \varepsilon$ and $Ch_c^2 \ge \varepsilon$, we have proved the following key result.

Theorem 5.4. *If $u(x_j)$ is the solution of the BVP (1.1)-(1.3) and u_j, $j = 0, 1, ..., N$ is the numerical solution obtained by the cubic spline scheme, then we have*

$$\max_j |u(x_j) - u_j| \le Ch_c^2\left[\exp\left(\frac{-C_3 x_j}{\varepsilon}\right) + \exp\left(\frac{-C_4(1-x_j)}{\varepsilon}\right)\right].$$

6. Numerical Experiments

In this section, we present two numerical examples to illustrate the method discussed in this paper.

Example 6.1. [4]:

$$-\varepsilon^2 u''(x) - \varepsilon(1+x)u'(x) + 3u(x) = f(x),\ x \in (0,1),$$
$$u(0) = u(1), \quad \varepsilon(u'(1) - u'(0)) = 1 - 2\varepsilon,$$

where $f(x)$ is chosen such that the exact solution is given by

$$u(x) = \frac{e^{-\frac{x}{\varepsilon}} + e^{-\frac{(1-x)}{\varepsilon}}}{2(1-e^{-\frac{1}{\varepsilon}})} + x(1-x) + 1$$

Example 6.2.

$$\begin{aligned} &-\varepsilon^2 u''(x) - 2\varepsilon(2+\sin(2\pi x))u'(x) + (1+\cos(2\pi x))u(x) = f(x),\ x \in (0,1), \\ &\qquad u(0) = u(1), \quad \varepsilon(u'(1) - u'(0)) = 3, \end{aligned}$$

where $f(x)$ is chosen such that the exact solution is given by

$$u(x) = \frac{e^{-\frac{3x}{2\varepsilon}} + e^{-\frac{3(1-x)}{2\varepsilon}}}{(1-e^{-\frac{3}{2\varepsilon}})} + \cos(2\pi x)$$

Let u^N be a numerical approximation for the exact solution u on the mesh Ω^N and N is the number of mesh points. For a finite set of values $\varepsilon \in R_\varepsilon = \{2^{-1}, 2^{-2}, ..., 2^{-28}\}$, we compute the maximum pointwise errors by

$$E_\varepsilon^N = \max_{x_i \in \bar{\Omega}_\varepsilon^N} |(u^N - u)(x_i)| \quad \text{and} \quad E^N = \max_\varepsilon E_\varepsilon^N$$

From these quantities the orders of convergence are computed from

$$p^N = \log_2(\frac{E^N}{E^{2N}}).$$

The computed errors E^N and the computed orders of convergence p^N for the above BVPs are given in the Tables 1 and 2.

Table 1: *Computed errors E^N and the computed order of convergence p^N of Example 6.1.*

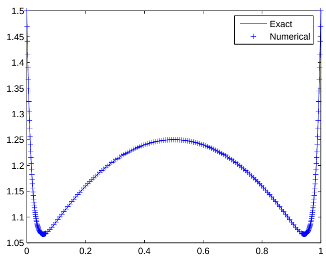
	Number of mesh points N					
	32	64	128	256	512	1024
	Cubic Spline Scheme					
E^N	5.9529e-2	1.8856e-2	4.4922e-3	9.8010e-4	2.1620e-4	4.9573e-5
p^N	1.6586	2.0695	2.1964	2.1806	2.1247	—
	Hybrid Difference Scheme in [4]					
E^N	1.9940e-2	7.4252e-3	2.6127e-3	8.7741e-4	2.8332e-4	8.8617e-5
p^N	1.4252	1.5069	1.5742	1.6308	1.6768	—

7. Conclusion

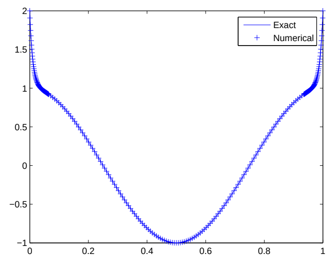
Thus we suggested a second order numerical method involving cubic spline scheme on non-uniform mesh to solve singularly perturbed convection diffusion problem with periodical boundary conditions, whose solutions exhibit boundary

Table 2: *Computed errors E^N and the computed order of convergence p^N of the Example 6.2.*

	Number of mesh points N					
	32	64	128	256	512	1024
	Cubic Spline Scheme					
E^N	4.2764e-1	8.5767e-2	2.2695e-2	5.7925e-3	1.4471e-3	3.5707e-4
p^N	2.3179	1.9180	1.9701	2.0010	2.0189	—
	Hybrid Difference Scheme in [4]					
E^N	8.1577e-1	3.2377e-1	1.1526e-1	3.6256e-2	1.1170e-2	3.4209e-3
p^N	1.3332	1.4901	1.6686	1.6986	1.7072	—



(a) Exact and numerical solution of Example 1 for $\varepsilon = 2^{-6}$ with $N = 256$.



(b) Exact and numerical solution of Example 2 for $\varepsilon = 2^{-6}$ with $N = 256$.

layers at both end points $x = 0$ and $x = 1$. In this method, both differential equations and periodical boundary conditions are discretized by cubic spline scheme. The method is of second order accurate by conducting two numerical examples and the error of the scheme is measured using the discrete maximum norm. From Tables 1 and 2, we also compared our method to hybrid difference scheme in [4], it shows a good results and exhibits the advantage of cubic spline scheme. In the construction of mesh, we have chosen $\tilde{h}_1 = 0.00001$ and $K = 1$ for both examples. We note that the increase in the value of K will lead to more concentration of points near the boundary regions and for a fixed K, the increase in the value of $\tilde{h}_1$ leads to the same result.

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