Novi Sad J. Math. Vol. 50, No. 1, 2020, 129-155
https://doi.org/10.30755/NSJOM.09449

Disjoint reiterative m_n-distributional chaos[1]

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Abstract. In this paper, we introduce and analyze the notion of disjoint (m_n, i)-distributional chaos, where $1 \leq i \leq 12$, as well as the notions of disjoint m_n-distributional chaos of type 2 and disjoint reiterative m_n-distributional chaos of types 1+ and 2^{Bd} for general sequences of multivalued linear operators in Fréchet spaces. We reconsider and slightly improve our recent results regarding disjoint distributional chaos in Fréchet spaces.

AMS Mathematics Subject Classification (2010): 47A06; 47A16; 54H20

Key words and phrases: disjoint m_n-distributional chaos; disjoint reiterative m_n-distributional chaos; Fréchet spaces

1. Introduction and preliminaries

Linear chaos is an enormous and rapidly growing field of research, with a great number of recent results that cannot be easily summarized. For more details about the subject, we refer the reader to the monographs [1] by F. Bayart, E. Matheron, [10] by K.-G. Grosse-Erdmann, A. Peris, and [14] by the author of this paper.

Distributional chaos for interval maps was introduced by B. Schweizer and J. Smítal in [20] (1994), while in the setting of linear continuous operators distributional chaos was first considered in the analyses of quantum harmonic oscillator, by J. Duan et al [9] (1999); cf. also [2], [4], [5], [19] and references cited therein.

Disjoint hypercyclicity in linear topological dynamics was introduced independently by L. Bernal–González [3] (2007) and J. Bès, A. Peris [6] (2007); see also [12]-[14] and references cited therein. In our recent research paper [15], we have analyzed a great number of different types of disjoint distributional chaos for general sequences of multivalued linear operators in Fréchet spaces (cf. [8],[7] for the initial studies of hypercyclicity and chaos in multi-valued setting). As mentioned in the abstract, the main aim of this paper is to continue and slightly improve the results obtained in [15] for disjoint (m_n, i)-distributional chaos, where $1 \leq i \leq 12$ and (m_n) is an increasing sequence of positive reals satisfying $\liminf_{n\to\infty} \frac{m_n}{n} > 0$. We also consider disjoint m_n-distributional chaos of type 2 and disjoint reiterative m_n-distributional chaos of types 1+ and

[1]The author is partially supported by the grant of the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-68/2020/14/200156)

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2^{Bd}. Albeit our main structural results are given for linear single-valued operators, we have decided to introduce the above-mentioned notion for general sequences of multivalued linear operators because a great number of theoretical results regarding (reiterative) m_n-distributionally irregular vectors and manifolds holds in this framework. Concerning densely disjoint Li-Yorke chaotic operators and abstract PDEs, we refer the reader to [16].

The organization and main ideas of this paper can be plainly described as follows. After collecting some preliminaries, we consider disjoint (m_n, i)-distributional chaos ($1 \leq i \leq 12$), disjoint m_n-distributional chaos of type 2 and disjoint reiterative m_n-distributional chaos of types 1+ and 2^{Bd} for sequences of multivalued linear operators in Fréchet spaces (Section 2). Disjoint (reiteratively) m_n-distributionally irregular vectors (of type 1+, 2 or 2^{Bd}) and associated irregular manifolds are investigated in Section 3. Concerning dense (reiterative) distributional chaos of the above types, it is worth noting that the construction of distributionally irregular vectors established in [4, Theorem 15] provides the basis of our examinations; in Section 4, we state a great number of results, proving only Theorem 4.1 and Proposition 4.6. In the final section of the paper, we briefly explain how the introduced notion can be further generalized, with unclear perspectives.

Hereafter we assume that X and Y are two non-trivial Fréchet spaces over the same field of scalars $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ as well as that the topologies of X and Y are induced by the fundamental systems $(p_n)_{n\in\mathbb{N}}$ and $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms, respectively (separability is not our standing assumption). The translation invariant metric $d : X \times X \to [0, \infty)$, defined by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \ x, \ y \in X, \tag{1.1}$$

satisfies the following properties: $d(x+u, y+v) \leq d(x, y)+d(u, v)$, $x, \ y, \ u, \ v \in X$, $d(cx, cy) \leq (|c|+1)d(x, y)$, $c \in \mathbb{K}$, $x, \ y \in X$, and $d(\alpha x, \beta x) \geq \frac{|\alpha-\beta|}{1+|\alpha-\beta|} d(0, x)$, $x \in X$, $\alpha, \ \beta \in \mathbb{K}$. We define the translation invariant metric $d_Y : Y \times Y \to [0, \infty)$ by replacing $p_n(\cdot)$ with $p_n^Y(\cdot)$ in (1.1). If $(X, \|\cdot\|)$ or $(Y, \|\cdot\|_Y)$ is a Banach space, then we assume that the distance of two elements $x, \ y \in X$ ($x, \ y \in Y$) is given by $d(x, y) := \|x-y\|$ ($d_Y(x, y) := \|x-y\|_Y$). With this terminological change, our results clarified in Fréchet spaces continue to hold in Banach spaces.

It will be assumed that $N \in \mathbb{N}$ and $N \geq 2$. Then the fundamental system of increasing seminorms $(\mathbf{p}_n^{Y^N})_{n\in\mathbb{N}}$, where $\mathbf{p}_n^{Y^N}(x_1, \cdot \cdot \cdot, x_N) := \sum_{j=1}^{N} p_n^Y(x_j)$, $n \in \mathbb{N}$ ($x_j \in Y$ for $1 \leq j \leq N$), induces the topology on the Fréchet space Y^N. We endow the space Y^N with the translation invariant metric

$$\mathrm{d}_{Y^N}(\vec{x}, \vec{y}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mathbf{p}_n(\vec{x}-\vec{y})}{1+\mathbf{p}_n(\vec{x}-\vec{y})}, \quad \vec{x}, \ \vec{y} \in Y^N,$$

while in the case that Y is a Banach space, the metric on Y^N is given by $d_{Y^N}(\vec{x}, \vec{y}) := \max_{1\leq j\leq N} \|x_j - y_j\|_Y$, $\vec{x} \in Y^N$, $\vec{y} \in Y^N$.

Suppose that $C \in L(X)$ is injective and $p_n^C(x) := p_n(C^{-1}x)$, $n \in \mathbb{N}$, $x \in R(C)$. Then $p_n^C(\cdot)$ is a seminorm on $R(C)$ and the system $(p_n^C)_{n\in\mathbb{N}}$ induces a Fréchet locally convex topology on $R(C)$; we denote this space by $[R(C)]$. Set $\mathbb{N}_n := \{1, \cdot \cdot \cdot, n\}$ $(n \in \mathbb{N})$.

Let X, Y, Z and T be given non-empty sets. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and

$$\sigma \circ \rho := \big\{(x, t) \in X \times T : \exists y \in Y \cap Z \text{ such that } (x, y) \in \rho \text{ and } (y, t) \in \sigma\big\},$$

respectively. Domain and range of ρ are introduced by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in \rho\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in \rho\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$. If ρ is a binary relation on X and $n \in \mathbb{N}$, then we define ρ^n inductively. Set $D_\infty(\rho) := \bigcap_{n\in\mathbb{N}} D(\rho^n)$, $\rho(X') := \{y : y \in \rho(x) \text{ for some } x \in X'\}$ $(X' \subseteq X)$.

For any mapping $\mathcal{A} : X \to P(Y)$, we set $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}),\ y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is a multivalued linear operator (MLO) iff the associated binary relation $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., iff $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$. In our work, we will identify $\mathcal{A}$ and its associated linear relation $\check{\mathcal{A}}$, so that the notion of $D(\mathcal{A})$, which is a linear subspace of X, as well as the sets $R(\mathcal{A})$ and $D_\infty(\mathcal{A})$ are understood. For more details about multivalued linear operators, we refer the reader to the references cited in [15].

We will use the following notions of lower and upper densities for a subset $A \subseteq \mathbb{N}$:

Definition 1.1. ([13]) Let $q \in [1, \infty)$, and let (m_n) be an increasing sequence in $[1, \infty)$. Then:

(i) The lower (m_n)-density of A, denoted by $\underline{d}_{m_n}(A)$, is defined through:

$$\underline{d}_{m_n}(A) := \liminf_{n\to\infty} \frac{|A \cap [1, m_n]|}{n};$$

(ii) The upper (m_n)-density of A, denoted by $\overline{d}_{m_n}(A)$, is defined through:

$$\overline{d}_{m_n}(A) := \limsup_{n\to\infty} \frac{|A \cap [1, m_n]|}{n};$$

(iii) The lower $l;(m_n)$-Banach density of A, denoted shortly by $\underline{Bd}_{l;m_n}(A)$, as follows

$$\underline{Bd}_{l;m_n}(A) := \liminf_{s\to+\infty} \liminf_{n\to\infty} \frac{|A \cap [n+1, n+m_s]|}{s};$$

(iv) The (upper) $l;(m_n)$-Banach density of A, denoted shortly by $\overline{Bd}_{l;m_n}(A)$, as follows

$$\overline{Bd}_{l;m_n}(A) := \liminf_{s\to+\infty} \limsup_{n\to\infty} \frac{|A \cap [n+1, n+m_s]|}{s};$$

(v) The (upper) $l:(m_n)$-Banach density of A, denoted shortly by $\overline{Bd}_{l:m_n}(A)$, by

$$\overline{Bd}_{l:m_n}(A) := \liminf_{s\to+\infty} \sup_{n\in\mathbb{N}} \frac{|A\cap[n+1, n+m_s]|}{s}.$$

Denote by R the class consisting of all increasing sequences (m_n) of positive reals satisfying $\liminf_{n\to\infty} \frac{m_n}{n} > 0$, i.e., there exists a finite constant $L > 0$ such that $n \leq Lm_n$, $n \in \mathbb{N}$. Unless stated otherwise, we will always assume that $(m_n) \in$ R henceforth. The assumption $m_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ can be made.

Assume that $\sigma > 0$, $\epsilon > 0$ and $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ are two given sequences in Y. Consider the following condition:

$$\begin{aligned} &\underline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_k, y_k\big) < \sigma\big\}\Big) = 0, \\ &\underline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_k, y_k\big) \geq \epsilon\big\}\Big) = 0. \end{aligned} \tag{1.2}$$

We will use the following special case of [18, Definition 2.2]:

Definition 1.2. Suppose that, for every $k \in \mathbb{N}$, $\mathcal{A}_k : D(\rho_k) \subseteq X \to Y$ is an MLO and $\tilde{X}$ is a closed subspace of X. If there exist an uncountable set $S \subseteq \bigcap_{k=1}^{\infty} D(\mathcal{A}_k) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $k \in \mathbb{N}$ there exist elements $x_k \in \mathcal{A}_k x$ and $y_k \in \mathcal{A}_k y$ such that (1.2) holds, then it is said that the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is $(m_n, \tilde{X})$-reiteratively distributionally chaotic.

The sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is said to be densely $(m_n, \tilde{X})$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. An MLO $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ is said to be (densely) $(m_n, \tilde{X})$-distributionally chaotic iff the sequence $(\mathcal{A}_k \equiv \mathcal{A}^k)_{k\in\mathbb{N}}$ is. The set S is said to be $(m_n, \sigma_{\tilde{X}})$-scrambled set ($(m_n, \sigma)$-scrambled set of type s in the case that $\tilde{X} = X$) of the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ (the MLO $\mathcal{A}$); in the case that $\tilde{X} = X$, then we also say that the sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ (the MLO $\mathcal{A}$) is m_n-distributionally chaotic.

Let $\lambda \in (0,1]$ and $m_n \equiv n^{1/\lambda}$. Then the (dense) $(m_n, \tilde{X})$-distributional chaos is also called (dense) $(\lambda, \tilde{X})$-distributional chaos [$\tilde{X}$-distributional chaos, provided that $\lambda = 1$], the (dense) m_n-distributional chaos is also called (dense) λ-distributional chaos [distributional chaos, provided that $\lambda = 1$] and the $(m_n, \sigma_{\tilde{X}})$-scrambled set S is also called $(\lambda, \sigma_{\tilde{X}})$-scrambled set [$\sigma_{\tilde{X}}$-scrambled set, provided that $\lambda = 1$].

2. Disjoint reiteratively m_n-distributionally chaotic properties of type s for MLOs

Let $\sigma > 0$, let $\epsilon > 0$, and let $(x_{j,k})_{k\in\mathbb{N}}$ and $(y_{j,k})_{k\in\mathbb{N}}$ be sequences in X $(1 \leq j \leq N)$. Consider the following conditions:

$$\begin{aligned} &\underline{d}_{m_n}\Bigg(\bigcup_{j\in\mathbb{N}_N}\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Bigg)=0, \text{ and}\\ &\underline{d}_{m_n}\Bigg(\bigcup_{j\in\mathbb{N}_N}\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Bigg)=0; \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\underline{d}_{m_n}\Bigg(\bigcup_{j\in\mathbb{N}_N}\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Bigg)=0, \text{ and}\\ &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Big)=0; \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Big)=0, \text{ and}\\ &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Big)=0; \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Big)=0, \text{ and}\\ &\big(\exists j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Big)=0; \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\big(\exists j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Big)=0, \text{ and}\\ &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Big)=0; \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Big)=0, \text{ and}\\ &\big(\exists j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Big)=0; \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\big(\forall j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Big)=0, \text{ and}\\ &\underline{d}_{m_n}\Bigg(\bigcup_{j\in\mathbb{N}_N}\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Bigg)=0; \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\big(\exists j\in\mathbb{N}_N\big)\ \underline{d}_{m_n}\Big(\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)<\sigma\big\}\Big)=0, \text{ and}\\ &\underline{d}_{m_n}\Bigg(\bigcup_{j\in\mathbb{N}_N}\big\{k\in\mathbb{N} : d_Y\big(x_{j,k},y_{j,k}\big)\geq\epsilon\big\}\Bigg)=0; \end{aligned} \tag{2.8}$$

$$\text{(2.9)}\qquad \begin{aligned} &\underline{d}_{m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Biggr)=0, \text{ and}\\ &\underline{d}_{m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0; \end{aligned}$$

$$\text{(2.10)}\qquad \begin{aligned} &\underline{d}_{m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Biggr)=0, \text{ and}\\ &\bigl(\forall j\in\mathbb{N}_N\bigr)\ \underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Bigr)=0; \end{aligned}$$

$$\text{(2.11)}\qquad \begin{aligned} &\underline{d}_{m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Biggr)=0, \text{ and}\\ &\underline{d}_{m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0; \end{aligned}$$

$$\text{(2.12)}\qquad \begin{aligned} &\bigl(\forall j\in\mathbb{N}_N\bigr)\ \underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Bigr)=0, \text{ and}\\ &\underline{d}_{m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}: d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0. \end{aligned}$$

Now we are ready to introduce the following notion:

Definition 2.1. Let $i\in\mathbb{N}_{12}$ and $(m_n)\in\mathrm{R}$. Suppose that, for every $j\in\mathbb{N}_N$ and $k\in\mathbb{N}$, $\mathcal{A}_{j,k}:D(\mathcal{A}_{j,k})\subseteq X\to Y$ is an MLO and $\tilde{X}$ is a closed linear subspace of X. Then we say that the sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is disjoint $(\tilde{X},m_n,i)$-distributionally chaotic, $(d,\tilde{X},m_n,i)$-distributionally chaotic for short, iff there exist an uncountable set $S\subseteq\bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty}D(\mathcal{A}_{j,k})\cap\tilde{X}$ and $\sigma>0$ such that for each $\epsilon>0$ and for each pair $x,\ y\in S$ of distinct points we have that for each $j\in\mathbb{N}_N$ and $k\in\mathbb{N}$ there exist elements $x_{j,k}\in\mathcal{A}_{j,k}x$ and $y_{j,k}\in\mathcal{A}_{j,k}y$ such that (2.i) holds.

The sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is said to be densely $(d,\tilde{X},m_n,i)$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. A finite sequence $(\mathcal{A}_j)_{1\leq j\leq N}$ of MLOs on X is said to be (densely) $(\tilde{X},m_n,i)$-distributionally chaotic iff the sequence $((\mathcal{A}_{j,k}\equiv\mathcal{A}_j^k)_{k\in\mathbb{N}})_{1\leq j\leq N}$ is. The set S is said to be $(d,\sigma_{\tilde{X}},m_n,i)$-scrambled set ($(d,\sigma,m_n,i)$-scrambled set in the case that $\tilde{X}=X$) of $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ $((\mathcal{A}_j)_{1\leq j\leq N})$; in the case that $\tilde{X}=X$, then we also say that the sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ $((\mathcal{A}_j)_{1\leq j\leq N})$ is disjoint (m_n,i)-distributionally chaotic, (d,m_n,i)-distributionally chaotic for short.

If $m_n\equiv n$, then the notion introduced above has been analyzed for the first time in [15]. Then we simply say that the sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$

is disjoint $(\tilde{X}, i)$-distributionally chaotic, $(d, \tilde{X}, i)$-distributionally chaotic for short. Similar terminological agreements will be accepted for all other terms from Definition 2.1.

Since there exists a finite constant $l > 0$ such that $\underline{d}_{m_n}(A) \geq l\underline{d}(A)$ for any subset $A \subseteq \mathbb{N}$, it is clear that for each $i \in \mathbb{N}_{12}$ and $(m_n) \in \mathrm{R}$, the supposition that the sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is disjoint $(\tilde{X}, m_n, i)$-distributionally chaotic implies that $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is disjoint $(\tilde{X}, i)$-distributionally chaotic. Hence, the notion of $(d, \tilde{X}, m_n, i)$-distributional chaos is stronger than that of $(d, \tilde{X}, i)$-distributional chaos, so that we actually further specify here the notion introduced in [15].

Directly from definition, it readily follows that, if $i \in \{1, 2, 3, 7\}$, resp. $i \in \{4, 5, 6, 8\}$, and $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is (densely) $(d, \tilde{X}, m_n, i)$-distributionally chaotic, then for each $j \in \mathbb{N}_N$, resp. there exists $j \in \mathbb{N}_N$, such that the component $(\mathcal{A}_{j,k})_{k\in\mathbb{N}}$ is (densely) $(\tilde{X}, m_n, i)$-distributionally chaotic.

As already mentioned, besides the notion introduced in Definition 2.1, we can analyze a great number of other types of disjoint m_n-distributional chaos. From the space and time limitations, we will consider here only disjoint analogues of the notion analyzed in [11, Section 4], with $i = 1$. Consider the following conditions:

$$\begin{aligned} &\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) < \sigma\right\}\right) = 0, \text{ and} \\ &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\overline{d}_{m_n}\left(\bigcap_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \sigma\right\}\right) > 0, \text{ and} \\ &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\overline{Bd}_{l:m_n}\left(\bigcap_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \sigma\right\}\right) > 0, \text{ and} \\ &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0. \end{aligned} \tag{2.15}$$

Definition 2.2. Let $(m_n) \in \mathrm{R}$. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\mathcal{A}_{j,k} : D(\mathcal{A}_{j,k}) \subseteq X \to Y$ is an MLO and $\tilde{X}$ is a closed linear subspace of X. Then we say that the sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is reiteratively disjoint $(\tilde{X}, m_n)$-distributionally chaotic of type 1+ $[2^{Bd}]$, resp. disjoint $(\tilde{X}, m_n)$-distributionally chaotic of type 2, reiteratively $(d, \tilde{X}, m_n)$-distributionally chaotic of type 1+ $[2^{Bd}]$, $(d, \tilde{X}, m_n)$-distributionally chaotic of type 2 for short, iff there exist an uncountable set $S \subseteq \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k}) \cap \tilde{X}$ and $\sigma > 0$ [an uncountable set $S \subseteq \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k}) \cap \tilde{X}$], resp. an uncountable set $S \subseteq \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k}) \cap \tilde{X}$, such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$ there exist elements $x_{j,k} \in \mathcal{A}_{j,k}x$ and $y_{j,k} \in \mathcal{A}_{j,k}y$ such that (2.13) holds [there exist elements $x_{j,k} \in \mathcal{A}_{j,k}x$, $y_{j,k} \in \mathcal{A}_{j,k}y$ and number $\sigma > 0$ such that (2.14) holds], resp. there exist elements $x_{j,k} \in \mathcal{A}_{j,k}x$, $y_{j,k} \in \mathcal{A}_{j,k}y$ and number $\sigma > 0$ such that (2.15) holds.

The sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is said to be densely reiteratively disjoint $(\tilde{X}, m_n)$-distributionally chaotic of type $1+$ $[2^{Bd}]$, resp. densely disjoint $(\tilde{X}, m_n)$-distributionally chaotic of type 2, iff S can be chosen to be dense in $\tilde{X}$. A finite sequence $(\mathcal{A}_j)_{1\leq j\leq N}$ of MLOs on X is said to be (densely) reiteratively disjoint $(\tilde{X}, m_n)$-distributionally chaotic of type $1+$ $[2^{Bd}]$, resp. disjoint $(\tilde{X}, m_n)$-distributionally chaotic of type 2, iff the sequence $((\mathcal{A}_{j,k} \equiv \mathcal{A}_j^k)_{k\in\mathbb{N}})_{1\leq j\leq N}$ is. In the case that $\tilde{X} = X$, then we also say that the sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ $((\mathcal{A}_j)_{1\leq j\leq N})$ is disjoint reiteratively m_n-distributionally chaotic of type $1+$ $[2^{Bd}]$, resp. disjoint m_n-distributionally chaotic of type 2.

The use of any strongly equivalent metric $d'_Y(\cdot,\cdot)$ with $d_Y(\cdot,\cdot)$ in the above definitions leads to the same notion of disjoint (reiterative) m_n-distributional chaos.

If $m_n \equiv n^{1/\lambda}$ for some $\lambda \in (0, 1]$, then the above notions are also called (dense, reiterative) disjoint $\tilde{X}_\lambda$-distributional chaos of type s, $(d, \tilde{X}, \lambda, i)$-distributional chaos for short, (dense, reiterative) disjoint $(\tilde{X}, \lambda)$-scrambled set of type s, etc.

Let $(m_n) \in \mathrm{R}$ be arbitrary. If a linear continuous operator $T \in L(X)$ satisfies the requirements of Godefroy-Schapiro Criterion (see e.g. [10]), then T is (m_n)-distributionally chaotic [11] so that it is very simple to construct two disjoint subsets A and B of $\mathbb{N}$ such that $\mathbb{N} = A \cup B$ and $\underline{d}_{m_n}(A) = \underline{d}_{m_n}(B) = 0$. Therefore, arguing as in [15], we can show that the multivalued linear operators $X \times X, \cdot\cdot\cdot, X \times X$, totally counted N times, are densely $(d, m_n, 1)$-distributionally chaotic. In particular, the previous example shows that dense (full, moreover, with meaning clear) $(d, m_n, 1)$-distributional chaos occurs in finite-dimensional spaces for the sequences of MLOs, and the same thing holds for the sequences of linear continuous operators; for any integer $i \in \mathbb{N}_8$ we have that the $(d, \tilde{X}, m_n, i)$-distributional chaos of operators $T_1 \in L(X), \cdot\cdot\cdot, T_N \in L(X)$ implies that there exists an index $j \in \mathbb{N}_N$ such that T_j is $(\tilde{X}, m_n)$-distributionally chaotic. On the other hand, if $i \in \{9, 10, 11, 12\}$ then the $(d, \tilde{X}, m_n, i)$-distributional chaos of operators $T_1 \in L(X), \cdot\cdot\cdot, T_N \in L(X)$ implies that there exists an index $j \in \mathbb{N}_N$ such that T_j is Li-Yorke chaotic ([15]).

The following proposition with $m_n \equiv n$ has been clarified in [15]. It is unquestionably the most intriguing statement concerning relations between various types of $(d, \tilde{X}, m_n, i)$-distributional chaos introduced above:

Proposition 2.3. *Let* $(m_n) \in \mathrm{R}$. *For any sequence* $\mathbb{A} \equiv ((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ *of* MLOs*, the following holds:*

1. $(d, \tilde{X}, m_n, 1)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, i)$-*distributional chaos of* $\mathbb{A}$ *for all* $i \in \mathbb{N}_{12}$;

2. $(d, \tilde{X}, m_n, 2)$-*distributional chaos implies* $(d, \tilde{X}, m_n, i)$-*distributional chaos for all* $i \in \{3, 4, 5, 6, 10, 11, 12\}$;

3. $(d, \tilde{X}, m_n, 3)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, i)$-*distributional chaos of* $\mathbb{A}$ *for all* $i \in \{4, 5, 10, 12\}$;

4. $(d, \tilde{X}, m_n, 4)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, 12)$-*distributional chaos of* $\mathbb{A}$;

5. $(d, \tilde{X}, m_n, 5)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, 10)$-*distributional chaos of* $\mathbb{A}$;

6. $(d, \tilde{X}, m_n, 6)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, i)$-*distributional chaos of* $\mathbb{A}$ *for all* $i \in \{4, 11, 12\}$;

7. $(d, \tilde{X}, m_n, 7)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, i)$-*distributional chaos of* $\mathbb{A}$ *for all* $i \in \{3, 4, 5, 8, 9, 10, 12\}$;

8. $(d, \tilde{X}, m_n, 8)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, i)$-*distributional chaos of* $\mathbb{A}$ *for all* $i \in \{5, 9, 10\}$;

9. $(d, \tilde{X}, m_n, 9)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, 10)$-*distributional chaos of* $\mathbb{A}$;

10. $(d, \tilde{X}, m_n, 10)$-*distributional chaos of* $\mathbb{A}$ *does not imply anything, in general;*

11. $(d, \tilde{X}, m_n, 11)$-*distributional chaos of* $\mathbb{A}$ *implies* $(d, \tilde{X}, m_n, 12)$-*distributional chaos of* $\mathbb{A}$;

12. $(d, \tilde{X}, m_n, 12)$-*distributional chaos of* $\mathbb{A}$ *does not imply anything, in general.*

Hence, $(d, \tilde{X}, m_n, 1)$-distributional chaos implies all others; the notion of $(d, \tilde{X}, m_n, 9)$-distributional chaos is incredibly important, as well ([15]):

Proposition 2.4. *Suppose* $(m_n) \in \mathrm{R}$ *and that, for every* $j \in \mathbb{N}_N$ *and* $k \in \mathbb{N}$, $\mathcal{A}_{j,k} : D(\mathcal{A}_k) \subseteq X \to Y$ *is an* MLO, *and also that* $\tilde{X}$ *is a closed linear subspace of* X. *Define, for every* $k \in \mathbb{N}$, *the* MLO $\mathbb{A}_k : D(\mathbb{A}_k) \subseteq X \to Y^N$ *by* $D(\mathbb{A}_k) := \bigcap_{1 \leq j \leq N} D(\mathcal{A}_{j,k})$ *and* $\mathbb{A}_k x := \{(x_{1,k}, \cdot \cdot \cdot, x_{N,k}) : x_{j,k} \in \mathcal{A}_{j,k}x$ *for all* $j \in \mathbb{N}_N\}$. *Then the sequence* $((\mathcal{A}_{j,k})_{k \in \mathbb{N}})_{1 \leq j \leq N}$ *is disjoint* $(\tilde{X}, m_n, 9)$-*distributionally chaotic iff the sequence* $(\mathbb{A}_k)_{k \in \mathbb{N}}$ *is* $(\tilde{X}, m_n)$-*distributionally chaotic.*

We need the following lemma.

Lemma 2.5. *Let* $(m_n) \in \mathrm{R}$, $B \subseteq \mathbb{N}$ *and* $\underline{d}_{m_n}(B^c) = 0$. *Then there exist pairwise disjoint subsets* $B_1, \cdot \cdot \cdot, B_N$ *of* B *such that* $B = B_1 \cup \cdot \cdot \cdot \cup B_N$ *and:*

(i) $\underline{d}_{m_n}(B_j^c) = 0$ *for all* $j \in \mathbb{N}_N$.

(ii) $\underline{d}_{m_n}(B_j^c) > 0$ *for all* $j \in \mathbb{N}_N$.

Proof. To prove (i), observe that there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that the segment $[1, m_{n_k}]$ contains at least $m_{n_k} - (n_k k^{-2})$ elements of the set B. Moreover, we can choose the sequence (n_k) such that n_{k+1} is enormously greater than m_{n_k} for all $k \in \mathbb{N}$, more precisely, such that $(n_{k+1}(k+1)^{-2}) + m_{n_k} \leq n_k/k$ for all $k \in \mathbb{N}$. Set $n_0 := 0$, $m_0 := 0$ and $B_j := B \cap \bigcup_{k \in \mathbb{N}_0} (m_{n_{kN+j-1}}, m_{n_{kN+j}}]$ $(j \in \mathbb{N}_N)$. Then it is easy to see that the segment $[1, m_{n_{kN+j}}]$ contains at least $m_{n_{kN+j}} - (n_{kN+j}(kN+j)^{-2}) - m_{n_{kN+j-1}} \geq m_{n_{kN+j}} - (n_{kN+j}(kN+j)^{-1})$ for all $k \in \mathbb{N}$, which implies $\underline{d}_{m_n}(B_j^c) = 0$ $(j \in \mathbb{N}_N)$. The proof of (ii) is much simpler. We list the elements of set B one by one in the sets $B_1, \cdots B_N$, respectively. Then it is clear that for each $n \in \mathbb{N}$ the segment $[1, n]$ contains at most n/N elements of the set B_j $(j \in \mathbb{N}_N)$. If $n \leq L m_n$ for all $n \in \mathbb{N}$, then it can be easily seen from our construction that $\underline{d}_{m_n}(B_j^c) \geq (N-1)/NL$ for all $j \in \mathbb{N}_N$ (observe that the supposition $\underline{d}_{m_n}(B^c) = 0$ is even superfluos for (ii)). □

Keeping in mind Lemma 2.5 and the argumentation from [15], we can simply show that for each sequence $(m_n) \in \mathrm{R}$ the notions of (d, m_n, i_1)-distributional chaos and (d, m_n, i_2)-distributional chaos differ for the sequences of linear continuous operators on finite-dimensional spaces, provided that $i_1,\ i_2 \in \mathbb{N}_{12}$ and $i_1 \neq i_2$. If a pair $i_1,\ i_2 \in \mathbb{N}_{12}$ of different indexes and a sequence $(m_n) \in \mathrm{R}$ are given, then it is not trivial to construct an example of linear continuous operators $T_1 \in L(X), \cdots, T_N \in L(X)$ showing that the notions of (d, m_n, i_1)-distributional chaos and (d, m_n, i_2)-distributional chaos do not coincide. Concerning this question, we would like to note that our analyses from [11, Theorem 2.8] and [15, Example 3.24] enable one to simply deduce the following theorem:

Theorem 2.6. *Suppose that* $X := c_0(\mathbb{N})$ *or* $X := l^p(\mathbb{N})$ *for some* $p \in [1, \infty)$. *Then there exist two continuous linear operators* T_1 *and* T_2 *on* X *which are* λ*-distributionally chaotic for any number* $\lambda \in (0,1]$, *satisfying that the tuple* (T_1, T_2) *is* $(d, span\{e_1\}, \lambda, i)$*-distributionally chaotic for any* $i \in \{3,4,5,10,12\}$, *the tuple* (T_1, T_2) *is not* (d, X, λ, i)*-distributionally chaotic for any* $i \in \{1,7,8,9\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \|T_i^j x\| = +\infty \quad \textit{for all} \quad x \in X \setminus \{0\},\ 1 \leq i \leq 2, \tag{2.16}$$

as well as $\lim_{j\to\infty} T_i^j x = 0$ *for some* $x \in X$ *iff* $x = 0$ $(1 \leq i \leq 2)$.

Based on our considerations carried out in [11, Theorem 2.11] and [15, Example 3.24], we can also deduce the following result:

Theorem 2.7. *Suppose that* $X := c_0(\mathbb{N})$ *or* $X := l^p(\mathbb{N})$ *for some* $p \in [1, \infty)$. *Then for each number* $\lambda \in (0,1]$ *there exist continuous linear operators* $T_1,\ T_2$ *on* X *satisfying* (2.16), $\lim_{j\to\infty} T_i^j x = 0$ *for some* $x \in X$ *iff* $x = 0$ $(1 \leq i \leq 2)$, *which are both* λ*-distributionally chaotic, not* λ'*-distributionally chaotic for any* $\lambda' \in (0, \lambda)$ *and which satisfies that the tuple* (T_1, T_2) *is* $(d, span\{e_1\}, \lambda, i)$*-distributionally chaotic for* $i \in \{3,4,5,10,12\}$ *and not* $(d, X, 1, i)$*-distributionally chaotic for* $i \in \{1,7,8,9\}$.

Proof. We will include the most relevant details for the sake of completeness. Let $\lambda \in (0,1]$ and $\lambda' \in (0,\lambda)$. Without loss of generality, we may assume that $X = l^2$. Set

$$a_n := \frac{1}{2}\Big[\lfloor (n+1)^{4/\lambda}\ln(n+1)\rfloor - \lfloor n^{4/\lambda}\ln n\rfloor\Big], \quad n \in \mathbb{N}$$

and

$$b_n := a_{n-1} + \frac{1}{2}\big(3n^2 - 3n + 1\big), \quad n \in \mathbb{N}.$$

Then we know that the weighted forward shift $T_1 \equiv F_\omega : l^2 \to l^2$, defined by $F_\omega(x_1, x_2, \cdots) \mapsto (0, \omega_1 x_1, \omega_2 x_2, \cdots)$, where the sequence of weights $\omega = (\omega_k)_{k\in\mathbb{N}}$ consists of sufficiently large blocks of 2's of lengths $b_1,\ b_2, \cdots$, and sufficiently large blocks of (1/2)'s of lengths $a_1,\ a_2, \cdots$ is λ-distibutionally chaotic and not λ'-distributionally chaotic. We define $T_2 \equiv F_{\omega'} : l^2 \to l^2$, defined by $F_{\omega'}(x_1, x_2, \cdots) \mapsto (0, \omega'_1 x_1, \omega'_2 x_2, \cdots)$, where $\omega'_n := 1/\omega_n$ for all $n \in \mathbb{N}$. Since $\|T_2^j e_1\| = 1/\|T_1^j e_1\|$ for all $j \in \mathbb{N}$, the argumentation contained in the proof of [11, Theorem 2.11] shows that T_2 is likewise λ-distributionally chaotic with e_1 being the corresponding λ-distributionally irregular vector. This simply implies that the tuple (T_1, T_2) is $(d, span\{e_1\}, \lambda, i)$-distributionally chaotic for $i \in \{3,4,5,10,12\}$. To see that the tuple (T_1, T_2) is not $(d, X, 1, i)$-distibutionally chaotic for $i \in \{1,7,8,9\}$, we can argue as in [15, Example 3.24], because for each $j \in \mathbb{N}$ we have that $\|T_1^j e_1\| + \|T_2^j e_1\| \geq 2$. We already know that (2.16) holds for $i = 1$ and now we will prove that (2.16) holds for $i = 2$, with $x = e_1$. Let $F_{\omega''}(x_1, x_2, \cdots) \mapsto (0, \omega''_1 x_1, \omega''_2 x_2, \cdots)$ be the weighted forward shift, where the sequence of weights $\omega'' = (\omega''_k)_{k\in\mathbb{N}}$ consists of sufficiently large blocks of 2's of lengths $a_1,\ a_2, \cdots$, and sufficiently large blocks of (1/2)'s of lengths $b_1,\ b_2, \cdots$. Then for each integer $j \geq a_1 + 1$, we have that $\|T_2^j e_1\| = 2^{-a_1}\|F_{\omega''}^j e_1\|$ so that there exists $n_0 \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\big\|T_i^j x\big\| \geq \frac{1}{2(N-a_1)}\sum_{j=1}^{N-a_1}\big\|F_{\omega''}^j x\big\|, \quad N \geq n_0.$$

But, repeating literally the corresponding arguments from the proof of [11, Theorem 2.11] and taking into account that

$$\lim_{n\to\infty} \frac{2^{a_n - b_{n-1}} - 1}{n\min(a_n, b_n)} = +\infty,$$

we may deduce that $N^{-1}\sum_{j=1}^{N}\big\|F_{\omega''}^j x\big\| \to +\infty$ as $N \to +\infty$. Therefore, it remains to be proved that the operator T_2 is not λ'-distributionally chaotic, which immediately follows if we prove that $\underline{d}_{1/\lambda'}\big(\{j \in \mathbb{N} : \|T^j e_1\| < 2^k\}\big) = +\infty$. A direct computation shows that this holds if the upper $1/\lambda'$-density of the set

$$\bigcup_{n\geq n_1}\Big[\lfloor n^{4/\lambda}\ln n\rfloor + n^3, \lfloor (n+1)^{4/\lambda}\ln(n+1)\rfloor\Big]$$

is $+\infty$ for each positive integer n_1. This follows from corresponding definition of the upper $1/\lambda'$-density with the sequence $(c_n \equiv \lfloor (n+1)^{4/\lambda}\ln(n+1)\rfloor)$, since the interval $[1, c_n]$ has at least $c_n - cn^4$ elements for a certain constant $c \in (0, 1/5)$ and all $n \in \mathbb{N}$. $\square$

Given an MLO $\mathcal{A}$ in X and a sequence $(m_n) \in \mathrm{R}$, we can introduce and analyze the sets

$$S_{m_n}(\mathcal{A}) := \{\lambda > 0 : \lambda\mathcal{A} \text{ is } m_n - \text{distributionally chaotic}\}$$

and

$$DDC_{\mathcal{A},m_n,i,\vec{r}} := \Big\{\vec{\lambda} = (\lambda_1, \lambda_2, \cdot\cdot\cdot, \lambda_N) \in \mathbb{K}^N : \text{the tuple } (\lambda_1\mathcal{A}^{r_1}, \lambda_2\mathcal{A}^{r_2}, \cdot\cdot\cdot, \lambda_N\mathcal{A}^{r_N}) \text{ is } (d, m_n, i)\text{-distributionally chaotic}\Big\}.$$

The analysis of these sets is outside scope of this paper.

3. Disjoint (reiteratively) m_n-distributionally irregular vectors

Depending on the use of quantifiers $\cap$, $\cup$, $\forall$ and $\exists$, we recognize four different types of disjoint m_n-distributionally unbounded vectors and four different types of disjoint m_n-distributionally near to 0 vectors:

Definition 3.1. Let $(m_n) \in \mathrm{R}$. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\mathcal{A}_{j,k} : D(\mathcal{A}_{j,k}) \subseteq X \to Y$ is an MLO and $x \in \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})$, $x \neq 0$. Then we say that:

(i) x is (d, m_n)-distributionally near to 0 of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff there exists $A \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A^c) = 0$ as well as for each $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$ there exists $x_{j,k} \in \mathcal{A}_{j,k}x$ such that $\lim_{k\in A,k\to\infty} x_{j,k} = 0$, $j \in \mathbb{N}_N$;

(ii) x is (d, m_n)-distributionally near to 0 of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff for each $\epsilon > 0$, $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$ there exists $x_{j,k} \in \mathcal{A}_{j,k}x$ such that the set $\bigcap_{j\in\mathbb{N}_N}\{k \in \mathbb{N} : d_Y(x_{j,k}, y_{j,k}) \geq \epsilon\}$ has the lower m_n-density 0;

(iii) x is (d, m_n)-distributionally near to 0 of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff for every $j \in \mathbb{N}_N$ there exists a set $A_j \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A_j^c) = 0$ as well as for each $k \in A_j$ there exists $x_{j,k} \in \mathcal{A}_{j,k}x$ such that $\lim_{k\in A_j,k\to\infty} x_{j,k} = 0$;

(iv) x is (d, m_n)-distributionally near to 0 of type 4 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff there exist an integer $j \in \mathbb{N}_N$ and a set $A_j \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A_j^c) = 0$ as well as for each $k \in A_j$ there exists $x_{j,k} \in \mathcal{A}_{j,k}x$ such that $\lim_{k\in A_j,k\to\infty} x_{j,k} = 0$.

Definition 3.2. Let $(m_n) \in \mathrm{R}$. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\mathcal{A}_{j,k} : D(\mathcal{A}_{j,k}) \subseteq X \to Y$ is an MLO, $x \in \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})$, $x \neq 0$, $i \in \mathbb{N}_4$ and $m \in \mathbb{N}$. Then we say that:

(i) x is (d, m_n)-distributionally m-unbounded of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff there exists $B \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(B^c) = 0$ as well as for each $j \in \mathbb{N}_N$ and $k \in B$ there exists $x'_{j,k} \in \mathcal{A}_{j,k}x$ such that $\lim_{k\in B,k\to\infty} p_m^Y(x'_{j,k}) = \infty$, $j \in \mathbb{N}_N$;

(ii) x is (d, m_n)-distributionally m-unbounded of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff there exists $B \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(B^c) = 0$ as well as for each $j \in \mathbb{N}_N$ and $k \in B$ there exists $x'_{j,k} \in \mathcal{A}_{j,k}x$ such that

$$\lim_{k\in B,k\to\infty} \sum_{j\in\mathbb{N}_N} p_m^Y(x'_{j,k}) = \infty;$$

(iii) x is (d, m_n)-distributionally m-unbounded of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff for every $j \in \mathbb{N}_N$ there exists $B_j \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(B_j^c) = 0$ as well as for each $k \in B_j$ there exists $x'_{j,k} \in \mathcal{A}_{j,k}x$ such that

$$\lim_{k\in B_j,k\to\infty} p_m^Y(x'_{j,k}) = \infty;$$

(iv) x is (d, m_n)-distributionally m-unbounded of type 4 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff there exist an integer $j \in \mathbb{N}_N$ and a set $B_j \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(B_j^c) = 0$ as well as for each $k \in B_j$ there exists $x'_{j,k} \in \mathcal{A}_{j,k}x$ such that

$$\lim_{k\in B_j,k\to\infty} p_m^Y(x'_{j,k}) = \infty.$$

It is said that x is (d, m_n)-distributionally unbounded of type i for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff there exists $q \in \mathbb{N}$ such that x is (d, m_n)-distributionally q-unbounded of type i for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.

For each type od $(\tilde{X}, m_n)$-disjoint distributional chaos, we can introduce the notion of corresponding $(d, m_n, \tilde{X})$-distributionally irregular vectors, as follows:

Definition 3.3. Let $(m_n) \in \mathrm{R}$. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\mathcal{A}_{j,k} : D(\mathcal{A}_{j,k}) \subseteq X \to Y$ is an MLO and $x \in \tilde{X} \cap \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})$, $x \neq 0$. Then we say that:

(i) x is a $(d, \tilde{X}, m_n, 1)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, x is (d, m_n)-distributionally unbounded of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and the requirements of the last condition hold with $x'_{j,k} = x_{j,k}$, i.e., the sequences in definitions of d-distributionally nearness to 0 of type 1 and d-distributionally unboundedness of type 1 must be the same (*for the sake of brevity, in all remaining parts of this definition and Definition 3.4 below, we will assume a priori this condition)*;

(ii) x is a $(d, \tilde{X}, m_n, 2)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(iii) x is a $(d, \tilde{X}, m_n, 3)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(iv) x is a $(d, \tilde{X}, m_n, 4)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 4 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(v) x is a $(d, \tilde{X}, m_n, 5)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 4 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(vi) x is a $(d, \tilde{X}, m_n, 6)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 4 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(vii) x is a $(d, \tilde{X}, m_n, 7)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(viii) x is a $(d, \tilde{X}, m_n, 8)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 4 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(ix) x is a $(d, \tilde{X}, m_n, 9)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(x) x is a $(d, \tilde{X}, m_n, 10)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(xi) x is a $(d, \tilde{X}, m_n, 11)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 1 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$;

(xii) x is a $(d, \tilde{X}, m_n, 12)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to 0 of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ and x is (d, m_n)-distributionally unbounded of type 3 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.

In the case that $\tilde{X} = X$, then we also say that x is a (d, m_n, i)-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ $(i \in \mathbb{N}_{12})$. We similarly define the notion of a (d, m_n, i)-distributionally near to 0 ((d, m_n, i)-distributionally m-unbounded, (d, m_n, i)-distributionally unbounded, $(d, \tilde{X}, m_n, i)$-distributionally irregular, (d, m_n, i)-distributionally irregular) vector for tuple $(\mathcal{A}_j)_{1\leq j\leq N}$ of MLOs.

Concerning distributionally m_n-unbounded vectors, big differences exist between Banach spaces and Fréchet spaces, as already observed in [15] for the case that $m_n \equiv n$.

Let $\{0\} \neq X' \subseteq \tilde{X}$ be a linear manifold, and let $i \in \mathbb{N}_{12}$. Then we say that:

d1. X' is a $(d, \tilde{X}, m_n, i)$-distributionally irregular manifold for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ ((d, m_n, i)-distributionally irregular manifold in the case that $\tilde{X} = X$) iff any element $x \in (X' \cap \bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})) \setminus \{0\}$ is a $(d, \tilde{X}, m_n, i)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$; the notion of a ($(d, m_n, i)$-, $(d, \tilde{X}, m_n, i)$-)distributionally irregular manifold for $(\mathcal{A}_j)_{1\leq j\leq N}$ is defined similarly.

d2. X' is a uniformly $(d, \tilde{X}, m_n, i)$-distributionally irregular manifold for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ (uniformly (d, m_n, i)-distributionally irregular manifold in the case that $\tilde{X} = X$) iff there exists $m \in \mathbb{N}$ such that any vector $x \in (X' \cap \bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})) \setminus \{0\}$ is both (d, m_n, i)-distributionally m-unbounded (with the meaning clear) and (d, m_n, i)-distributionally near to 0 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$. In this case, X' is $(2_{\tilde{X}}^{-m}, m_n)$-scrambled set for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.

We have the following:

d3. Suppose that $0 \neq x \in \tilde{X} \cap \bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})$ is a $(d, \tilde{X}, m_n, i)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$. Then $X' \equiv span\{x\}$ is a uniformly $(d, \tilde{X}, m_n, i)$-distributionally irregular manifold for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.

If X' is dense in $\tilde{X}$, then the notions of dense ((d, m_n, i)-, $(d, \tilde{X}, m_n, i)$-)distributionally irregular manifolds, dense uniformly ((d, m_n, i)-, $(d, \tilde{X}, m_n, i)$-)distributionally irregular manifolds, etc., are defined analogously. It will be said that $(\mathcal{A}_{1,k})_{k\in\mathbb{N}}, (\mathcal{A}_{2,k})_{k\in\mathbb{N}}, \cdots, (\mathcal{A}_{N,k})_{k\in\mathbb{N}}$ are $(d, \tilde{X}, m_n, i)$-distributionally chaotic iff the tuple $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is $(d, \tilde{X}, m_n, i)$-distributionally chaotic; a similar terminological agreement will be accepted for operators.

The conclusions obtained in [15, Remark 3.12] can be formulated for $(d, \tilde{X}, m_n, i)$-distributionally irregular vectors and associated manifolds. Concerning disjoint m_n-distributional chaos of type 2 and disjoint reiterative m_n-distributional chaos of types 1+ and 2^{Bd}, we introduce the following notion:

Definition 3.4. Let $(m_n) \in$ R. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\mathcal{A}_{j,k} : D(\mathcal{A}_{j,k}) \subseteq X \to Y$ is an MLO and $x \in \tilde{X} \cap \bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})$, $x \neq 0$. Then we say that:

(ii) x is a reiteratively $(d, \tilde{X}, m_n)$-distributionally irregular vector of type 1+ for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to zero of type 1, with corresponding elements $x_{j,k} \in \mathcal{A}_{j,k}x$ satisfying additionally that there exists $\sigma > 0$ with

$$\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k \in \mathbb{N} : d_Y\big(x_{j,k}, 0\big) < \sigma\right\}\right) = 0;$$

(ii) x is a reiteratively $(d, \tilde{X}, m_n)$-distributionally irregular vector of type 2^{Bd} for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to zero of type

1, with corresponding elements $x_{j,k} \in \mathcal{A}_{j,k}x$ satisfying additionally that there exists $\sigma > 0$ with

$$\overline{Bd}_{l:m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k \in \mathbb{N} : d_Y\bigl(x_{j,k}, 0\bigr) \geq \sigma\bigr\}\Biggr) > 0;$$

(iii) x is a $(d, \tilde{X}, m_n)$-distributionally irregular vector of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is (d, m_n)-distributionally near to zero of type 1, with corresponding elements $x_{j,k} \in \mathcal{A}_{j,k}x$ satisfying additionally that there exists $\sigma > 0$ with

$$\overline{d}_{m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k \in \mathbb{N} : d_Y\bigl(x_{j,k}, 0\bigr) \geq \sigma\bigr\}\Biggr) > 0.$$

Let $\{0\} \neq X' \subseteq \tilde{X}$ be a linear manifold, and let $i \in \mathbb{N}_{12}$. Then we say that X' is a reiteratively $(d, \tilde{X}, m_n)$-distributionally irregular manifold of type 1+ $[2^{Bd}]$, resp. $(d, \tilde{X}, m_n)$-distributionally irregular manifold of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ (reiterative (d, m_n)-distributionally irregular manifold of type 1+ $[2^{Bd}]$, resp. (d, m_n)-distributionally irregular manifold of type 2 in the case that $\tilde{X} = X$) iff any element $x \in (X' \cap \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})) \setminus \{0\}$ is a reiteratively $(d, \tilde{X}, m_n)$-distributionally irregular vector of type 1+ $[2^{Bd}]$ for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$; the above notion is introduced for $(\mathcal{A}_j)_{1\leq j\leq N}$ similarly.

We have the following: Suppose that $0 \neq x \in \tilde{X} \cap \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\mathcal{A}_{j,k})$ is a reiteratively $(d, \tilde{X}, m_n)$-distributionally irregular vector of type 1+ $[2^{Bd}]$, resp. $(d, \tilde{X}, m_n)$-distributionally irregular vector of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ (and, therefore, reiterative $(d, \sigma_{\tilde{X}}, m_n)$-scrambled set of type 1+ $[2^{Bd}]$, resp. $(d, \sigma_{\tilde{X}}, m_n)$-scrambled set of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$). Then $X' \equiv span\{x\}$ is a reiteratively $(d, \tilde{X}, m_n)$-distributionally irregular manifold of type 1+ $[2^{Bd}]$, resp. $(d, \tilde{X}, m_n)$-distributionally irregular manifold of type 2 for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.

Let $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ be given in advance. Then we define $((\mathbb{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ by $\mathbb{A}_{j,k} := (\mathcal{A}_{j,k})_{|\tilde{X}}$ $(k \in \mathbb{N},\ 1 \leq j \leq N)$. The following simple result, which can also be formulated for disjoint m_n-distributional chaos of type 2 and disjoint reiterative m_n-distributional chaos of types 1+ and 2^{Bd}, almost directly follows from introduced definitions:

Proposition 3.5. *Let $(m_n) \in \mathrm{R}$, $i \in \mathbb{N}_{12}$, let $\tilde{X}$ be a closed linear subspace of X, and let $\{0\} \neq X'$ be a linear subspace of $\tilde{X}$.*

(i) The sequence $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is $(d, \tilde{X}, m_n, i)$-distributionally chaotic iff the sequence $((\mathbb{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is (d, m_n, i)-distributionally chaotic.

(ii) A vector x is a $(d, \tilde{X}, m_n, i)$-distributionally irregular vector for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ iff x is a (d, m_n, i)-distributionally irregular vector for $((\mathbb{A}_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.

(iii) A manifold X' is a (uniformly) $(d, \tilde{X}, m_n, i)$-distributionally irregular manifold for $((\mathcal{A}_{j,k})_{k\in\mathbb{N}})_{1\le j\le N}$ iff X' is a (uniformly) (d, m_n, i)-distributionally irregular manifold for the sequence $((\mathbb{A}_{j,k})_{k\in\mathbb{N}})_{1\le j\le N}$.

4. Main results

We start by stating the following theorem closely connected with [4, Proposition 7, Proposition 9]:

Theorem 4.1. *Suppose that $(m_n) \in$ R, $((T_{j,k})_{k\in\mathbb{N}})_{1\le j\le N}$ is a tuple of operators in $L(X, Y)$. If the following two conditions are satisfied:*

$(I_{0,\cap})$: *there exists a dense linear subspace X_0 of X satisfying that for each $x \in X_0$ there exists a set $A_x \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A_x^c) = 0$ and $\lim_{k\in A_x} T_{j,k}x = 0$, $1 \le j \le N$;*

$(I_{\infty,\cap})$: *there exist a zero sequence (y_l) in X, a number $\epsilon > 0$, a strictly increasing sequence (N_l) in $\mathbb{N}$ and an integer $m \in \mathbb{N}$ such that, for every $l \in \mathbb{N}$, we have*

$$card\Bigl(\Bigl\{1 \le k \le m_{N_l} : (\forall j \in \mathbb{N}_N)\, p_m^Y\bigl(T_{j,k}y_l\bigr) \ge \epsilon\Bigr\}\Bigr) \ge m_{N_l} - \frac{N_l}{l},$$

(for every $l \in \mathbb{N}$, $card\bigl(\{1 \le k \le m_{N_l} : (\forall j \in \mathbb{N}_N)\, \|T_{j,k}y_l\|_Y \ge \epsilon\}\bigr) \ge m_{N_l} - \frac{N_l}{l}$, in the case that Y is a Banach space),

then there exists a $(d, m_n, 1)$-distributionally irregular vector for $((T_{j,k})_{k\in\mathbb{N}})_{1\le j\le N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\le j\le N}$ is $(d, m_n, 1)$-distributionally chaotic.

Proof. The proof of this theorem follows by combining the arguments contained in the proofs of [4, Propositions 7 and 9], [15, Theorem 4.1] and [11, Proposition 3.6]. Concerning the above-mentioned propositions from [4], the following should be noted. First of all, for each natural number $l \in \mathbb{N}$ we set

$$M_l := \Biggl\{x \in X : (\exists n \in \mathbb{N})\, (\forall j \in \mathbb{N}_N)\, \bigl|\{k \in \mathbb{N} : p_m^Y(T_{j,k}x) < l\} \cap [1, m_n]\bigr| \le \frac{n}{l}\Biggr\}.$$

Then, clearly, M_l is an open set for all $l \in \mathbb{N}$. Let $l \in \mathbb{N}$, $x \in X$, $m_1 \in \mathbb{N}$ and $\delta > 0$. Then there exist $u \in \{y_1, y_2, \cdots\}$ and $n \in \mathbb{N}$ such that $p_{m_1}(u) < \delta\epsilon/l^2 := c$ and $|\{k \in [1, m_n] \cap \mathbb{N} : (\forall j \in \mathbb{N}_N)\, p_m^Y(T_{j,k}u) \ge \epsilon\}| \le \frac{n}{l}$. Define $u_s := x + \delta s u / l^2 c$ for $s = 0, 1, \cdots, l-1$. If we replace the sets A and B_s throughout the proof of [4, Proposition 7] with the sets

$$A := \Bigl\{1 \le j \le m_n : (\forall j \in \mathbb{N}_N)\, p_m^Y(T_{j,k}u) > \epsilon\Bigr\},$$

$$B_s := \Bigl\{1 \le j \le m_n : (\exists j \in \mathbb{N}_N)\, p_m^Y(T_j u_s) \le k\Bigr\}, \quad s = 0, 1, \cdots, 2k(1 + m_n) - 1,$$

where

$$u_s := x + \frac{\delta s u}{2k(1+m_n)C}, \quad s = 0, 1, \cdots, 2k(1+m_n) - 1$$

for a sufficiently large integer n, we can show that the set M_l is dense. Hence, $\mathcal{M} := \bigcap_{l\in\mathbb{N}} M_l$ is a residual set and each element of $\mathcal{M}$ is a $(d, m_n, 1)$-distributionally m-unbounded vector for the sequence $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$. Concerning [4, Proposition 9], it is only worth noting that

$$\begin{aligned} X_0 \subseteq & M_{l,m} \\ := & \left\{ x \in X : (\exists n \in \mathbb{N})\, (\forall j \in \mathbb{N}_N) \left| \left\{ j \in \mathbb{N} : p_m^Y(T_j x) \geq 1/l \right\} \cap [1, m_n] \right| \leq \frac{n}{l} \right\} \end{aligned}$$

for all $l,\ m \in \mathbb{N}$ as well as that the set $M_{l,m}$ is an open and dense subset of X for all $l,\ m \in \mathbb{N}$, so that the set $\bigcap_{l,m\in\mathbb{N}} M_{l,m}$ is residual. □

For the sequences of single-valued linear operators, we suppose that the condition (P) holds, where:

(P) $T_{j,k} : D(T_{j,k}) \subseteq X \to X$ is a linear mapping, $C \in L(X)$ is an injective mapping, as well as $R(C) \subseteq D_\infty(T_{j,k})$, $T_{j,k}C \in L(X)$ $(k \in \mathbb{N},\ j \in \mathbb{N}_N)$.

Then, for every $k \in \mathbb{N}$ and $j \in \mathbb{N}_N$, the mapping $T_{j,k} : R(C) \to X$ is an element of the space $L([R(C)], X)$. By Theorem 4.1, we immediately obtain the following

Corollary 4.2. *Suppose that the condition* (P) *holds, as well as that the following two conditions hold:*

$(L_0, \cap)$: *there exists a dense linear subspace X_0 of X satisfying that for each $x \in X_0$ there exists a set $A_x \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A_x^c) = 0$ and $\lim_{k\in A_x} T_{j,k}Cx = 0$ $(j \in \mathbb{N}_N)$.*

$(L_\infty, \cap)$: *there exist a sequence (z_l) in X, a number $\epsilon > 0$, a strictly increasing sequence (N_l) in $\mathbb{N}$ and an integer $m \in \mathbb{N}$ such that, for every $l \in \mathbb{N}$, we have*

$$card\Big(\Big\{1 \leq k \leq m_{N_l} : (\forall j \in \mathbb{N}_N)\, p_m^Y\big(T_{j,k}Cy_l\big) \geq \epsilon\Big\}\Big) \geq m_{N_l} - \frac{N_l}{l},$$

(for every $l \in \mathbb{N}$, $card\big(\{1 \leq k \leq m_{N_l} : (\forall j \in \mathbb{N}_N)\, \|T_{j,k}Cy_l\|_Y \geq \epsilon\}\big) \geq m_{N_l} - \frac{N_l}{l}$, in the case that Y is a Banach space),

Then there exists a $(d, m_n, 1)$-distributionally irregular vector for $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is $(d, m_n, 1)$-distributionally chaotic. Moreover, the corresponding (d, σ)-scrambled set S for $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ can be chosen to be a linear submanifold of $R(C)$.

As in the case $m_n \equiv n$, Theorem 4.1 (Corollary 4.2) admits a reformulation for any other type of (d, m_n, i)-distributional chaos introduced above and we only need to replace the condition $(I_{0,\cap})$ $((L_{0,\cap}))$ with one of the following conditions:

$(I_{0,\cup})$: $= I_{0,\cap}$;

$(I_{0,\forall})$: for every $j \in \mathbb{N}_N$ there exist a dense linear subspace X_0 of X and a set $A_j \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A_j^c) = 0$ and $\lim_{k \in A_j, k \to \infty} T_{j,k}x = 0$;

$(I_{0,\exists})$: there exist an integer $j \in \mathbb{N}_N$, a dense linear subspace X_0 of X and a set $A_j \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(A_j^c) = 0$ and $\lim_{k \in A_j, k \to \infty} T_{j,k}x = 0$;

$(L_0, \cup)$: $= L_{0,\cap}$;

$(L_{0,\cup})$: the same as $(I_{0,\cup})$ with $T_{j,k} = T_{j,k}C$,

$(L_{0,\forall})$: the same as $(I_{0,\forall})$ with $T_{j,k} = T_{j,k}C$,

$(L_{0,\exists})$: the same as $(I_{0,\exists})$ with $T_{j,k} = T_{j,k}C$,

and the condition $(I_{\infty,\cap})$ ($(L_{\infty,\cap})$) with one of the following conditions:

$(I_{\infty,\cup})$: there exist a zero sequence (y_l) in X, a number $\epsilon > 0$, a strictly increasing sequence (N_l) in $\mathbb{N}$ and an integer $m \in \mathbb{N}$ such that, for every $l \in \mathbb{N}$, we have

$$card\Big(\Big\{1 \le k \le m_{N_l} : \max_{1 \le j \le N} d_Y\big(T_{j,k}y_l, 0\big) > \epsilon\Big\}\Big) \ge m_{N_l} - \frac{N_l}{l},$$

(for every $l \in \mathbb{N}$, $card\big(\{1 \le k \le m_{N_l} : \max_{1 \le j \le N} \|T_{j,k}y_l\|_Y > \epsilon\}\big) \ge m_{N_l} - \frac{N_l}{l}$, in the case that Y is a Banach space);

$(I_{\infty,\forall})$: for every $j \in \mathbb{N}_N$, there exist a zero sequence (y_l) in X, a number $\epsilon > 0$, a strictly increasing sequence (N_l) in $\mathbb{N}$ and an integer $m \in \mathbb{N}$ such that, for every $l \in \mathbb{N}$, we have

$$card\Big(\Big\{1 \le k \le m_{N_l} : d_Y\big(T_{j,k}y_l, 0\big) > \epsilon\Big\}\Big) \ge m_{N_l} - \frac{N_l}{l}, \tag{4.1}$$

(for every $l \in \mathbb{N}$,

$$card\big(\{1 \le k \le m_{N_l} : \|T_{j,k}y_l\|_Y > \epsilon\}\big) \ge m_{N_l} - \frac{N_l}{l}, \tag{4.2}$$

in the case that Y is a Banach space);

$(I_{\infty,\exists})$: there exist an integer $j \in \mathbb{N}_N$, a zero sequence (y_l) in X, a number $\epsilon > 0$, a strictly increasing sequence (N_l) in $\mathbb{N}$ and an integer $m \in \mathbb{N}$ such that, for every $l \in \mathbb{N}$, we have that (4.1) holds ((4.2) holds, in the case that Y is a Banach space);

$(L_{\infty,\cup})$: the same as $(I_{\infty,\cup})$ with $T_{j,k} = T_{j,k}C$.

$(L_{\infty,\forall})$: the same as $(I_{\infty,\forall})$ with $T_{j,k} = T_{j,k}C$.

$(L_{\infty,\exists})$: the same as $(I_{\infty,\exists})$ with $T_{j,k} = T_{j,k}C$.

The argumentation used in the proof of [11, Theorem 4.1] and the proof of implication (iv) $\Rightarrow$ (iii) in [4, Theorem 15], along with the process of 'renorming' described in the proof of second part of [7, Theorem 3.7], can be used to see that the following sufficient criterion for dense $(d, m_n, 1)$-distributional chaos of linear continuous operators holds:

Theorem 4.3. *Suppose that X is separable, X_0 is a dense linear subspace of X, $(T_{j,k})_{k\in\mathbb{N}}$ is a sequence in $L(X,Y)$ $(j \in \mathbb{N}_N)$ and the following holds:*

- *(a)* $\lim_{k\to\infty} T_{j,k}x = 0$, $x \in X_0$, $j \in \mathbb{N}_N$,
- *(b) there exists a $(d, m_n, 1)$-distributionally unbounded vector x for $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$.*

Then there exists a dense uniformly $(d, m_n, 1)$-distributionally irregular manifold for the sequence $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is densely $(d, m_n, 1)$-distributionally chaotic.

Applying Theorem 4.3, we may deduce the following important corollary:

Corollary 4.4. *Suppose that the condition* (P) *holds, X is separable, X_0 is a dense linear subspace of X and the following holds:*

- *(a)* $\lim_{k\to\infty} T_{j,k}Cx = 0$, $x \in X_0$, $j \in \mathbb{N}_N$,
- *(b) there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(B^c) = 0$, and $\lim_{k\to\infty, k\in B} p_m(T_{j,k}Cx) = \infty$, $j \in \mathbb{N}_N$, resp. $\lim_{k\to\infty, k\in B} \|T_{j,k}Cx\| = \infty$, $j \in \mathbb{N}_N$, if X is a Banach space.*

Then there exists a uniformly $(d, m_n, 1)$-distributionally irregular manifold W for $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ are $(d, m_n, 1)$-distributionally chaotic. Furthermore, if $R(C)$ is dense in X, then W can be chosen to be dense in X and $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ are densely $(d, m_n, 1)$-distributionally chaotic.

It is worth noting that Theorem 4.3 and Corollary 4.4 can be straightforwardly reformulated for $(d, m_n, 9)$-distributional chaos by using Proposition 2.4 and [11, Theorem 4.1], as well as for some other types of (d, m_n, i)-distributional chaos.

We continue by providing the following illustrative example:

Example 4.5. Let $X := L^2(\mathbb{R})$, $c > b/2 > 0$, $\Omega := \{\lambda \in \mathbb{C} : \Re\lambda < c - b/2\}$ and $\mathcal{A}_c u := u'' + 2bxu' + cu$ is the bounded perturbation of the one-dimensional Ornstein-Uhlenbeck operator acting with domain $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$. Assume that $((P_{j,k}(z))_{k\in\mathbb{N}})_{1\leq j\leq N}$ is a tuple consisting of sequence of non-zero complex polynomials such that there exists an open connected subset Ω' of Ω such that $\lim_{k\to\infty} P_{j,k}(\lambda) = 0$, $\lambda \in \Omega'$, $j \in \mathbb{N}_N$ and also that there exists a number $\lambda \in \Omega$ such that $|P_{j,k}(\lambda)| > 1$ for all $k \in \mathbb{N}$ and $j \in \mathbb{N}_N$. Using our analysis from [11, Example 4.6] and Corollary 4.4, we get that the sequence $((P_{j,k}(\mathcal{A}_c))_{k\in\mathbb{N}})_{1\leq j\leq N}$ is densely $(d, m_n, 1)$-distributionally chaotic for each sequence $(m_n) \in$ R.

Concerning examples and results presented in [15], we can simply verify that the operators $z_1A^{n_1}, \cdot\cdot\cdot, z_NA^{n_N}$ from [15, Example 4.5], the operators $\Phi_1(D), \cdot\cdot\cdot, \Phi_N(D)$ from [15, Example 4.6] and the operators $T_1, \cdot\cdot\cdot, T_N$ from [15, Theorem 4.11, Example 4.12, Example 5.2] are densely $(d, m_n, 1)$-distributionally chaotic for each sequence $(m_n) \in \mathrm{R}$. It is also worth noting that [15, Proposition 4.9] can be used to provide a great number of illustrative examples of sequences of unbounded backward shift operators on Banach sequence spaces that are densely $(d, m_n, 1)$-distributionally chaotic for each sequence $(m_n) \in \mathrm{R}$.

Concerning disjoint distributional chaos for weighted translations on locally compact groups considered in [15, Subsection 5.2], it is only worth noting that [15, Theorem 5.6, Theorem 5.7] can be straightforwardly reformulated for disjoint $(d, m_n, 1)$-distributional chaos by assuming that the sequence B in their formulations satisfies the condition $\underline{d}_{m_n}(B^c) = 0$. n

Suppose now that X is a Fréchet sequence space in which $(e_n)_{n\in\mathbb{N}}$ is a basis (see e.g. [10, Section 4.1]) and the unilateral weighted backward shift T, given by

$$T\langle x_n\rangle_{n\in\mathbb{N}} := \langle x_{n+1}\rangle_{n\in\mathbb{N}}, \quad \langle x_n\rangle_{n\in\mathbb{N}} \in X,$$

is a continuous linear operator on X. For the sake of completeness, we will provide the main details of proof of following slight generalization of [15, Proposition 5.4]:

Proposition 4.6. *Suppose that S is an infinite set of natural numbers such that the series $\sum_{n\in S} e_n$ converges in X, and there exist natural numbers $r_1, \cdot\cdot\cdot, r_N$ such that the set*

$$\mathrm{Q} := \Bigl\{k \in \mathbb{N} : \bigl(\forall j \in \mathbb{N}_N\bigr)\, r_jk \in S + 1\Bigr\}$$

satisifes $\underline{d}_{m_n}(\mathrm{Q}^c) = 0$. Then the operators $T^{r_1}, \cdot\cdot\cdot, T^{r_N}$ are densely $(d, m_n, 1)$-distributionally chaotic.

Proof. By Theorem 4.3, it suffices to show that there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ such that $\underline{d}_{m_n}(B^c) = 0$ and $\lim_{k\to\infty, k\in B} \|T^{r_jk}x\| = \infty$, $j \in \mathbb{N}_N$. For this, we employ Theorem 4.1: Put $y_l := \sum_{n\in S, n\geq l} e_n$, $l \in \mathbb{N}$. Then $\lim_{l\to\infty} y_l = 0$ and there exists a number $\epsilon > 0$ such that $d(\langle x_n\rangle_{n\in\mathbb{N}}, 0) < \epsilon$ implies $|x_1| < 1$. Hence, it suffices to construct a strictly increasing sequence (N_l) in $\mathbb{N}$ such that, for every $l \in \mathbb{N}$, we have

$$card\Bigl(\Bigl\{1 \leq k \leq m_{N_l} : (\forall j \in \mathbb{N}_N)\, d\bigl(T^{r_jk}y_l, 0\bigr) > \epsilon\Bigr\}\Bigr) \geq m_{N_l} - \frac{N_l}{l}.$$

But, this simply follows from our choice of number ϵ, the equality $\underline{d}_{m_n}(\mathrm{Q}^c) = 0$ and the fact that for a number $l \in \mathbb{N}$ given in advance we have $(T^{r_j})^k y_l = T^{r_jk}y_l = e_1 + \cdots$ for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$ such that $r_jk - 1 \in S \cap [l, \infty)$. $\square$

We can similarly reformulate [15, Proposition 5.5] by assuming that the set Q_g satisfies the condition $\underline{d}_{m_n}(\mathrm{Q}_g^c) = 0$.

Regarding disjoint m_n-distributional chaos of type 2 and disjoint reiterative m_n-distributional chaos of types 1+ and 2^{Bd}, it is worth noting that the proofs of our structural results from [11, Section 4] in connection with these types of dense (reiterative) m_n-distributional chaos and the arguments already used can be employed for proving the following results:

Theorem 4.7. *Suppose that X is separable, X_0 is a dense linear subspace of X, $(T_{j,k})_{k\in\mathbb{N}}$ is a sequence in $L(X,Y)$ $(j\in\mathbb{N}_N)$ and the following holds:*

(a) $\lim_{k\to\infty} T_{j,k}x=0$, $x\in X_0$, $j\in\mathbb{N}_N$,

(b) there exist $m\in\mathbb{N}$, $c>0$, $x\in X$ and set $B\subseteq\mathbb{N}$ such that $\underline{Bd}_{l;m_n}(B^c)=0$ and $\lim_{k\in B} p_m^Y(T_{j,k}x)=+\infty$ for all $j\in\mathbb{N}_N$.

Then there exists a dense (d,m_n)-distributionally irregular manifold of type 1+ for the sequence $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is densely (d,m_n)-distributionally chaotic of type 1+.

Theorem 4.8. *Suppose that X is separable, X_0 is a dense linear subspace of X, $(T_{j,k})_{k\in\mathbb{N}}$ is a sequence in $L(X,Y)$ $(j\in\mathbb{N}_N)$ and the following holds:*

(a) $\lim_{k\to\infty} T_{j,k}x=0$, $x\in X_0$, $j\in\mathbb{N}_N$,

(b) there exist $m\in\mathbb{N}$, $c>0$, $x\in X$ and set $B\subseteq\mathbb{N}$ such that $\overline{d}_{m_n}(B)=c$ and $\lim_{k\in B} p_m^Y(T_{j,k}x)=+\infty$ for all $j\in\mathbb{N}_N$.

Then there exists a dense (d,m_n)-distributionally irregular manifold of type 2 for the sequence $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is densely (d,m_n)-distributionally chaotic of type 2.

Theorem 4.9. *Suppose that X is separable, X_0 is a dense linear subspace of X, $(T_{j,k})_{k\in\mathbb{N}}$ is a sequence in $L(X,Y)$ $(j\in\mathbb{N}_N)$ and the following holds:*

(a) $\lim_{k\to\infty} T_{j,k}x=0$, $x\in X_0$, $j\in\mathbb{N}_N$,

(b) there exist $m\in\mathbb{N}$, $c>0$, $x\in X$ and set $B\subseteq\mathbb{N}$ such that $\overline{Bd}_{l:m_n}(B)=c$ and $\lim_{k\in B} p_m^Y(T_{j,k}x)=+\infty$ for all $j\in\mathbb{N}_N$.

Then there exists a dense (d,m_n)-distributionally irregular manifold of type 2^{Bd} for the sequence $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$, and particularly, $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is densely (d,m_n)-distributionally chaotic of type 2^{Bd}.

5. Conclusions and final remarks

We close the paper by providing a small heuristical study concerning disjoint reiteratively m_n-distributionally chaotic properties of type s for general binary relations over metric spaces.

Assume that (X,τ) is a topological space and (Y,d_Y) is a metric space, (m_n) is an increasing sequence in $[1,\infty)$ satisfying $\liminf_{n\to\infty}\frac{m_n}{n}>0$, that $\sigma>0$, $\epsilon>0$ and that $(x_{j,k})_{k\in\mathbb{N}}$, $(y_{j,k})_{k\in\mathbb{N}}$ are two given sequences in Y

($1 \leq j \leq N$). For any type of (d, m_n, i)-distributional chaos considered by now, where $1 \leq i \leq 12$, we can analyze a great number of new types of disjoint (reiterative) distributional chaos of type s for general binary relations (in what follows, eight new types concretly). We will briefly explain our idea for $i = 1$ and $i = 12$. Consider the following conditions:

$$(5.1)\qquad \begin{aligned} &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) < \sigma\right\}\right) = 0, \text{ and} \\ &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned}$$

$$(5.2)\qquad \begin{aligned} &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) < \sigma\right\}\right) = 0, \text{ and} \\ &\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned}$$

$$(5.3)\qquad \begin{aligned} &\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) < \sigma\right\}\right) = 0, \text{ and} \\ &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned}$$

$$(5.4)\qquad \begin{aligned} &\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) < \sigma\right\}\right) = 0, \text{ and} \\ &\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned}$$

$$(5.5)\qquad \begin{aligned} &\overline{d}_{m_n}\left(\bigcap_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \sigma\right\}\right) > 0, \text{ and} \\ &\underline{d}_{m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned}$$

$$(5.6)\qquad \begin{aligned} &\overline{d}_{m_n}\left(\bigcap_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \sigma\right\}\right) > 0, \text{ and} \\ &\underline{Bd}_{l;m_n}\left(\bigcup_{j\in\mathbb{N}_N}\left\{k\in\mathbb{N} : d_Y\left(x_{j,k}, y_{j,k}\right) \geq \epsilon\right\}\right) = 0; \end{aligned}$$

$$\begin{aligned}&\overline{Bd}_{l;m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)>\sigma\bigr\}\Biggr)>0,\ \text{and}\\&\underline{d}_{m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0;\end{aligned}\tag{5.7}$$

$$\begin{aligned}&\overline{Bd}_{l;m_n}\Biggl(\bigcap_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)>\sigma\bigr\}\Biggr)>0,\ \text{and}\\&\underline{Bd}_{l;m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0.\end{aligned}\tag{5.8}$$

Definition 5.1. Suppose that, for every $j\in\mathbb{N}_N$ and $k\in\mathbb{N}$, $\rho_{j,k}:D(\rho_{j,k})\subseteq X\to Y$ is a binary relation and $\tilde{X}$ is a non-empty subset of X. If there exist an uncountable set $S\subseteq\bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty}D(\rho_{j,k})\cap\tilde{X}$ and $\sigma>0$ such that for each $\epsilon>0$ and for each pair $x,\ y\in S$ of distinct points we have that for each $j\in\mathbb{N}_N$ and $k\in\mathbb{N}$ there exist elements $x_{j,k}\in\rho_{j,k}x$ and $y_{j,k}\in\rho_{j,k}y$ such that (5.1) [(5.2),(5.3),(5.4)] holds, resp. (5.5) [(5.6),(5.7),(5.8)] holds, then we say that the tuple of sequences $((\rho_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is $(d,m_n,\tilde{X},1)$-reiteratively distributionally chaotic of type 1, resp. $(d,m_n,\tilde{X},1)$-reiteratively distributionally chaotic of type 2.

In the case that $i=12$, we consider the following conditions:

$$\begin{aligned}&\bigl(\forall j\in\mathbb{N}_N\bigr)\ \underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Bigr)=0,\ \text{and}\\&\underline{d}_{m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0;\end{aligned}\tag{5.9}$$

$$\begin{aligned}&\bigl(\forall j\in\mathbb{N}_N\bigr)\ \underline{d}_{m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Bigr)=0,\ \text{and}\\&\underline{Bd}_{l;m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0;\end{aligned}\tag{5.10}$$

$$\begin{aligned}&\bigl(\forall j\in\mathbb{N}_N\bigr)\ \underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Bigr)=0,\ \text{and}\\&\underline{d}_{m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0;\end{aligned}\tag{5.11}$$

$$\begin{aligned}&\bigl(\forall j\in\mathbb{N}_N\bigr)\ \underline{Bd}_{l;m_n}\Bigl(\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)<\sigma\bigr\}\Bigr)=0,\ \text{and}\\&\underline{Bd}_{l;m_n}\Biggl(\bigcup_{j\in\mathbb{N}_N}\bigl\{k\in\mathbb{N}:d_Y\bigl(x_{j,k},y_{j,k}\bigr)\geq\epsilon\bigr\}\Biggr)=0;\end{aligned}\tag{5.12}$$

$$(\forall j \in \mathbb{N}_N)\ \overline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) \geq \sigma\big\}\Big) > 0, \text{ and}$$
$$\underline{d}_{m_n}\Bigg(\bigcup_{j \in \mathbb{N}_N} \big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) \geq \epsilon\big\}\Bigg) = 0; \tag{5.13}$$

$$(\forall j \in \mathbb{N}_N)\ \overline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) \geq \sigma\big\}\Big) > 0, \text{ and}$$
$$\underline{Bd}_{l;m_n}\Bigg(\bigcup_{j \in \mathbb{N}_N} \big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) \geq \epsilon\big\}\Bigg) = 0; \tag{5.14}$$

$$(\forall j \in \mathbb{N}_N)\ \overline{Bd}_{l;m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) > \sigma\big\}\Big) > 0, \text{ and}$$
$$\underline{d}_{m_n}\Bigg(\bigcup_{j \in \mathbb{N}_N} \big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) \geq \epsilon\big\}\Bigg) = 0; \tag{5.15}$$

$$(\forall j \in \mathbb{N}_N)\ \overline{Bd}_{l;m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) > \sigma\big\}\Big) > 0, \text{ and}$$
$$\underline{Bd}_{l;m_n}\Bigg(\bigcup_{j \in \mathbb{N}_N} \big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) \geq \epsilon\big\}\Bigg) = 0; \tag{5.16}$$

Definition 5.2. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\rho_{j,k} : D(\rho_{j,k}) \subseteq X \to Y$ is a binary relation and $\tilde{X}$ is a non-empty subset of X. If there exist an uncountable set $S \subseteq \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(\rho_{j,k}) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$ there exist elements $x_{j,k} \in \rho_{j,k}x$ and $y_{j,k} \in \rho_{j,k}y$ such that (5.9) [(5.10),(5.11),(5.12)] holds, resp. (5.13) [(5.14),(5.15),(5.16)] holds, then we say that the tuple of sequences $((\rho_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is $(d, m_n, \tilde{X}, 12)$-reiteratively distributionally chaotic of type 1, resp. $(d, m_n, \tilde{X}, 12)$-reiteratively distributionally chaotic of type 2.

Concerning disjoint reiterative distributional chaos of types $2\frac{1}{2}$ and 3, we would like to note that for any type of reiterative $[\tilde{X}, m_n, i]$-distributional chaos, where $9 \leq i \leq 20$, we can consider two kinds of disjoint reiterative $[\tilde{X}, m_n, i]$-distributional chaos of types $2\frac{1}{2}$ and 3; see [18] for the notion. We will explain this only for $i = 9$. Consider the following conditions:

there exist $c > 0$ and $r > 0$ such that

$$\underline{d}_{m_n}\Bigg(\bigcap_{j \in \mathbb{N}_N} \big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) < \sigma\big\}\Bigg)$$
$$< c < \overline{d}_{m_n}\Bigg(\bigcap_{j \in \mathbb{N}_N} \big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) < \sigma\big\}\Bigg) \tag{5.17}$$

for $0 < \sigma < r$;
there exist $c > 0$ and $r > 0$ such that

$$(\forall j \in \mathbb{N}_N)\ \underline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) < \sigma\big\}\Big) < c < \overline{d}_{m_n}\Big(\big\{k \in \mathbb{N} : d_Y\big(x_{j,k}, y_{j,k}\big) < \sigma\big\}\Big) \tag{5.18}$$

for $0 < \sigma < r$;
there exist positive real numbers $a,\ b,\ c > 0$ such that (5.17) holds for $\sigma \in [a, b]$;
there exist positive real numbers $a,\ b,\ c > 0$ such that (5.18) holds for $\sigma \in [a, b]$.

We can further extend the notions introduced in Definition 5.1 and Definition 5.2 by allowing the parameter $\sigma > 0$ to depend on pairs $x,\ y \in S$. We will use this approach in the following

Definition 5.3. Let $i \in \mathbb{N}_2$. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $\rho_{j,k} : D(\rho_{j,k}) \subseteq X \to Y$ is a binary relation and $\tilde{X}$ is a non-empty subset of X. If there exists an uncountable set $S \subseteq \bigcap_{j=1}^{N}\bigcap_{k=1}^{\infty} D(\rho_{j,k}) \cap \tilde{X}$ such that for each pair $x,\ y \in S$ of distinct points we have that for each $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$ there exist elements $x_{j,k} \in \rho_{j,k}x$ and $y_{j,k} \in \rho_{j,k}y$ as well as positive real numbers $c > 0$ and $r > 0$, resp. there exist positive real numbers $a,\ b,\ c > 0$ such that (5.17) [(5.18)] holds, resp. (5.17) [(5.18)] holds, then we say that the tuple of sequences $((\rho_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is $[\tilde{X}, m_n, 9]$-reiteratively distributionally chaotic of type $i; 2\frac{1}{2}$, resp. $[\tilde{X}, m_n, 9]$-reiteratively distributionally chaotic of type $i; 3$.

We can introduce many other types of reiterative disjoint distributional chaos of type s by considering different types of (Banach) lower and upper m_n-densities for definitions, but all aspects of the introduced notions cannot be easily perceived. This is only an initial study of disjoint reiterative m_n-distributional chaos and we can propose a great deal of open problems and questions about this theme, especially disjoint analogues of questions from [11].

We close the paper with the observation that we have recently analyzed disjoint distributionally chaotic properties of abstract PDEs in [17]. The results established in this paper can be slightly generalized for disjoint m_n-distributional chaos; for more details about this topic, the reader may consult the forthcoming monograph [14].

References

[1] Bayart, F., and Matheron, E. *Dynamics of linear operators*, vol. 179 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2009.

[2] Bermúdez, T., Bonilla, A., Martínez-Giménez, F., and Peris, A. Li-Yorke and distributionally chaotic operators. *J. Math. Anal. Appl. 373*, 1 (2011), 83–93.

[3] Bernal-González, L. Disjoint hypercyclic operators. *Studia Math. 182*, 2 (2007), 113–131.

[4] Bernardes, Jr., N. C., Bonilla, A., Müller, V., and Peris, A. Distributional chaos for linear operators. *J. Funct. Anal. 265*, 9 (2013), 2143–2163.

[5] Bernardes, Jr., N. C., Bonilla, A., Peris, A., and Wu, X. Distributional chaos for operators on Banach spaces. *J. Math. Anal. Appl. 459*, 2 (2018), 797–821.

[6] Bès, J., and Peris, A. Disjointness in hypercyclicity. *J. Math. Anal. Appl. 336*, 1 (2007), 297–315.

[7] Chen, C.-C., Conejero, J. A., Kostić, M., and Murillo-Arcila, M. Dynamics of multivalued linear operators. *Open Math. 15*, 1 (2017), 948–958.

[8] Conejero, J. A., Chen, C.-C., Kostić, M., and Murillo-Arcila, M. Dynamics on binary relations over topological spaces. *Symmetry 10*, 6 (2011), 1–12.

[9] Duan, J., Fu, X.-C., Liu, P.-D., and Manning, A. A linear chaotic quantum harmonic oscillator. *Appl. Math. Lett. 12*, 1 (1999), 15–19.

[10] Grosse-Erdmann, K.-G., and Peris Manguillot, A. *Linear chaos.* Universitext. Springer, London, 2011.

[11] Kostić, M. Reiterative m_n-distributional chaos of type s in Fréchet spaces. in press.

[12] Kostić, M. *Abstract Volterra integro-differential equations.* CRC Press, Boca Raton, FL, 2015.

[13] Kostić, M. $\mathcal{F}$-Hypercyclic operators on Fréchet spaces. *Publ. Inst. Math., Nouv. Sér 106*, 120 (2019), 1–18.

[14] Kostić, M. *Chaos for linear operators and abstract differential equations.* Nova Science Publishers Inc., New York, 2020.

[15] Kostić, M. Disjoint distributional chaos in Fréchet spaces. *Results Math. 75* (2020), Art. No. 83.

[16] Kostić, M. Disjoint li-yorke chaos in Fréchet spaces. *Electronic J. Math. Anal. Appl. 8*, 1 (2020), 248–272.

[17] Kostić, M. Disjoint distributionally chaotic abstract pdes. *Novi Sad J. Math.* (accepted).

[18] Kostić, M., and Velinov, D. Reiterative (m_n)-distributional chaos for binary relations over metric spaces. *Mat. Bilten 43*, 1 (2019), 5–25.

[19] Oprocha, P. Distributional chaos revisited. *Trans. Amer. Math. Soc. 361*, 9 (2009), 4901–4925.

[20] Schweizer, B., and Smítal, J. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc. 344*, 2 (1994), 737–754.

Received by the editors May 1, 2019
First published online July 6, 2019