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Fuzzy (h, β)-contractions in non-Archimedean fuzzy metric spaces

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Abstract. In this work, we introduce the new concepts of fuzzy (h, β)-contractive mapping via triangular (h, β)-admissible mappings. Later, we prove some fixed point results for some mappings that provide fuzzy (h, β)-contractibility and triangular (h, β)-admissibility in complete non-Archimedean fuzzy metric spaces. Some examples are supplied in order to support the applicability of our results. Our main results substantially generalize and extend some known results in the existing literature.

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1. Introduction and Preliminaries

It is well known the fixed point theory has an important role in mathematical analysis. The concept of fuzzy metric space was introduced in different ways by some authors [5, 14] and the fixed point theory in this kind of spaces has been intensively studied [6, 10]. Gregori and Sapena [10] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. Later, Mihet [16] enlarged the class of fuzzy contractive mappings of Gregori and Sapena, considered these mappings in fuzzy metric spaces in the sense of Kramosil and Michalek and obtained a fixed point theorem for fuzzy contractive mappings. For more details on fixed point theory for contraction type mappings in fuzzy metric spaces, we refer the interested reader to [1, 15, 19, 21, 28, 29] and the references cited therein. On the other hand, one of the most popular theorem in the fixed point theory is the Banach fixed point theorem [2]. By using this theorem, most authors have proved several fixed point theorems for various mappings [3, 4, 11, 18, 17, 20, 24, 25, 26]. Recently, Dinarvand [7] has introduced the new concept of fuzzy $\beta-\varphi$-contractive mapping via triangular β-admissible mappings.

In this work, we prove some fixed point results in non-Archimedean fuzzy metric spaces. Motivated by Dinarvand, we introduce the new concepts of fuzzy

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(h,β)-contractive mapping via triangular (h,β)-admissible mappings. Later, we derive several sufficient conditions which ensure the existence and uniqueness of fixed points for these classes of mappings in the setup of complete non-Archimedean fuzzy metric spaces. Some examples are supplied in order to support the applicability of our results. We present some fixed point results in G-complete fuzzy metric spaces and some cyclic results. Our main results substantially generalize and extend some known results in the existing literature.

Definition 1.1. [22] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous $t-$norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative,

(TN-2) $*$ is continuous,

(TN-3) $*(a,1) = a$ for every $a \in [0,1]$,

(TN-4) $*(a,b) \le *(c,d)$ whenever $a \le c$, $b \le d$ and $a,b,c,d \in [0,1]$.

An arbitrary $t-$norm $*$ can be extended (by associativity) in a unique way to an nary operator taking for $(x_1,x_2,...,x_n) \in [0,1]^n$, $n \in \mathbb{N}$, the value $*(x_1,x_2,...,x_n)$ is defined, in [9], by $*_{I=1}^{0} x_i = 1$, $*_{I=1}^{n} x_i = *(*_{I=1}^{n-1} x_i, x_n) = *(x_1,x_2,...,x_n)$.

Definition 1.2. [8] A fuzzy metric space is an ordered triple $(X,M,*)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions, for all $x,y,z \in X$, $s,t > 0$:

(FM-1) $M(x,y,t) > 0$,

(FM-2) $M(x,y,t) = 1$ iff $x = y$,

(FM-3) $M(x,y,t) = M(y,x,t)$,

(FM-4) $M(x,z,t+s) \ge M(x,y,t) * M(y,z,s)$,

(FM-5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

If, in the above definition, the triangular inequality (FM-4) is replaced by (NA) $M(x,z,max\{t,s\}) \ge M(x,y,t) * M(y,z,s)$ for all $x,y,z \in X$, $s,t > 0$, or equivalently,
$M(x,z,t) \ge M(x,y,t) * M(y,z,t)$
then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space [12].

Definition 1.3. Let $(X,M,*)$ be a fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to converge to x in X, denoted by $x_n \to x$, if and only if $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$, i.e. for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x,t) > 1-r$ for all $n \ge n_0$ [14, 23].

(ii) A sequence $\{x_n\}$ is a M-Cauchy sequence if and only if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) \geq 1 - \varepsilon$ for all $m > n \geq n_0$ [23, 8]. A sequence $\{x_n\}$ is a G-Cauchy sequence if and only if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for any $p > 0$ and $t > 0$ [9, 10, 27].

(iii) The fuzzy metric space $(X, M, *)$ is called M-complete (G-complete) if every M-Cauchy (G-Cauchy) sequence is convergent.

Definition 1.4. [10] Let Ψ be the class of all mappings $\psi : [0,1] \longrightarrow [0,1]$ such that

(i) ψ is continuous and nondecreasing,

(ii) $\psi(t) > t$ for all $t \in (0,1)$.

Lemma 1.5. *[10] If $\psi \in \Psi$, then $\psi(1) = 1$.*

Lemma 1.6. *[10] If $\psi \in \Psi$, then $\lim_{n\to+\infty} \psi^n(t) = 1$ for all $t \in (0,1)$.*

Definition 1.7. [7] Let $(X, M, *)$ be a fuzzy metric space and $f : X \to X$ be a given mapping. We say that f is a triangular β-admissible mapping if there exists a function $\beta : X \times X \times (0,\infty) \to (0,\infty)$ such that

(T_{β_1}) $\beta(x, y, t) \leq 1$ implies $\beta(fx, fy, t) \leq 1$ for all $x, y \in X$ and for all $t > 0$,

(T_{β_2}) $\beta(x, z, t) \leq 1$ and $\beta(z, y, t) \leq 1$ imply $\beta(x, z, t) \leq 1$ for all $x, y \in X$ and for all $t > 0$.

Definition 1.8. [13] Let X be a nonempty set and $f, T : X \to X$. The pair (f, T) is said to be weakly compatible if f and T commute at their coincidence points (i.e.$fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called a point of coincidence of f and T if there exists a point $x \in X$ such that $y = fx = Tx$.

2. Main results

We denote by Φ the set of lower semicontinuous functions $\varphi : [0,1] \to [0,1]$ such that:

(i) $\varphi(t) = 0$ iff $t = 0$ and $\varphi(1) = 1$,

(ii) $\varphi(t) > 0$ for all $t > 0$ and $\varphi(t) \leq t$ for all $t \in (0,1)$.

Let X be a nonempty set and let $f, h : X \to X$ be arbitrary two mappings. We denote by $Coin(f, h)$ the set of all fixed points of coincidence f and h.

Definition 2.1. Let $(X, M, *)$ be a non-Archimedean fuzzy metric space and $f, h : X \to X$ be given mappings. We say that f is a triangular (h, β)-admissible mapping if there exists a function $\beta : X \times X \times (0,\infty) \to (0,\infty)$ such that

(a) $\beta(hx, hy, t) \leq 1$ implies $\beta(fx, fy, t) \leq 1$ for all $x, y \in X$ and for all $t > 0$,

(b) $\beta(x, z, t) \leq 1$ and $\beta(z, y, t) \leq 1$ imply $\beta(x, y, t) \leq 1$ for all $x, y \in X$ and for all $t > 0$.

Example 2.2. Let $f, h : [0, +\infty) \to \mathbb{R}$ be defined by

$$fx = \begin{cases} 1 & , \quad x \in [0,1] \\ \frac{1}{2} & , \quad otherwise. \end{cases} \text{ and } hx = \begin{cases} 1 & , \quad x \in [0,1] \\ 3 & , \quad otherwise. \end{cases}$$

Suppose that $\beta : X \times X \times [0, +\infty) \to \mathbb{R}^+$ is given by

$$\beta(x, y, t) = \begin{cases} 1 & , \quad x, y \in [0,1] \\ 3 & , \quad otherwise. \end{cases}$$

f is a triangular (h, β)-admissible mapping. Indeed, if $\beta(hx, hy, t) \leq 1$, then $hx, hy \in [0,1]$. So $x, y \in [0,1]$. Thus $\beta(fx, fy, t) \leq 1$. Now assume that $\beta(x, z, t) \leq 1$ and $\beta(z, y, t) \leq 1$, so $x, z \in [0,1]$ and $z, y \in [0,1]$. Then, $x, y \in [0,1]$ and so $\beta(x, y, t) \leq 1$.

Lemma 2.3. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space and f be a triangular (h, β)-admissible mapping. Assume that there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$. Define a sequence $\{x_n\}$ and $\{y_n\}$ by $y_n = fx_n = hx_{n+1}$ for all $n \in \mathbb{N}$. Then*

$$\beta(y_m, y_n, t) \leq 1 \textit{ for all } m, n \in \mathbb{N} \textit{ with } m < n.$$

Proof. Since there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$, it follows that $\beta(hx_0, fx_0, t) = \beta(hx_0, hx_1, t) \leq 1$. Now, by using (a) in Definition 2.1 we obtain

$$\beta(hx_1, hx_2, t) = \beta(fx_0, fx_1, t) \leq 1 \Rightarrow \beta(hx_2, hx_3, t) = \beta(fx_1, fx_2, t) \leq 1.$$

By continuing the process as above, we get

$$\beta(y_n, y_{n+1}, t) = \beta(hx_{n+1}, hx_{n+2}, t) \leq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Let $m, n \in \mathbb{N}$ with $m < n$. Because $\beta(y_m, y_{m+1}, t) \leq 1$ and $\beta(y_{m+1}, y_{m+2}, t) \leq 1$, it follows by using (b) in Definition 2.1 that $\beta(y_m, y_{m+2}, t) \leq 1$. Again, since $\beta(y_m, y_{m+2}, t) \leq 1$ and $\beta(y_{m+2}, y_{m+3}, t) \leq 1$ by applying (b) in Definition 2.1, we have $\beta(y_m, y_{m+3}, t) \leq 1$. By continuing this process inductively, we get $\beta(y_n, y_m, t) \leq 1$. □

Definition 2.4. Let $(X, M, *)$ be a non-Archimedean fuzzy metric space and f be a triangular (h, β)-admissible mapping. We say that f is a fuzzy (h, β)-contractive mapping if

$$\beta(hx, hy, t) \leq 1 \Rightarrow \varphi(M(fx, fy, t)) \geq \psi(\varphi(N(x, y))) \tag{2.1}$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(hx, hy, t), M(hx, fx, t), M(hy, fy, t)\},$$

$\varphi \in \Phi$ and $\psi \in \Psi$.

Remark 2.5. If $\beta(hx,hy,t)=1$ for all $x,y\in X$ and any $t>0$ and $\varphi(t)=t$ and $N(x,y)=M(x,y,t)$, then Definition 2.4 reduces to the Definition 3.1 given by Mihet (see [16]).

Theorem 2.6. *Let $(X,M,*)$ be a complete non-Archimedean fuzzy metric space and f be a fuzzy (h,β)-contractive mapping. Suppose that $fX\subset hX$, hX is a closed subset of X and the following conditions hold:*
(a) there exists $x_0\in X$ such that $\beta(hx_0,fx_0,t)\le 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n,x_{n+1},t)\le 1$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to\infty$, then $\beta(x_n,x,t)\le 1$,
(c) $\beta(hx,hy,t)\le 1$ for all $x,y\in Coin(f,h)$ and $t>0$.
Then f and h have a unique point of coincidence in X. Moreover, if f and h are weakly compatible, then f and h have a unique common fixed point.

Proof. Let $x_0\in X$ such that $\beta(hx_0,fx_0,t)\le 1$. Define sequences $\{x_n\}$ and $\{y_n\}$ by

$$y_n=fx_n=hx_{n+1} \text{ for all } n\in\mathbb{N}. \tag{2.2}$$

If $y_n=y_{n+1}$ then y_{n+1} is a point of coincidence of f and h. Suppose that $y_n\ne y_{n+1}$ for all $n\in\mathbb{N}$. By virtue of Lemma 2.3, we get

$$\beta(y_m,y_n,t)\le 1 \text{ for all } n\in\mathbb{N}\cup\{0\}.$$

Therefore by (2.1) and using (2.2), we get

$$\varphi(M(y_n,y_{n+1},t))=\varphi(M(fx_n,fx_{n+1},t))\ge\psi(\varphi(N(x_n,x_{n+1}))) \tag{2.3}$$

and since property of φ, we get

$$M(y_n,y_{n+1},t)\ge\varphi(M(fx_n,fx_{n+1},t))\ge\psi(\varphi(N(x_n,x_{n+1}))), \tag{2.4}$$

where

$$\begin{aligned} N(x_n,x_{n+1}) &= \min\{M(hx_n,hx_{n+1},t),M(hx_n,fx_n,t),M(hx_{n+1},fx_{n+1},t)\} \\ &= \min\{M(y_{n-1},y_n,t),M(y_{n-1},y_n,t),M(y_n,y_{n+1},t)\}. \end{aligned} \tag{2.5}$$

Thus from (2.4) and (2.5), we obtain

$$M(y_n,y_{n+1},t)\ge\psi(\min\{M(y_{n-1},y_n,t),M(y_n,y_{n+1},t)). \tag{2.6}$$

If $M(y_{n-1},y_n,t)>M(y_n,y_{n+1},t)$ for some $n\in\mathbb{N}$ and property of ψ, we get

$$M(y_n,y_{n+1},t)\ge\psi(M(y_n,y_{n+1},t))>M(y_n,y_{n+1},t)$$

which is a contradiction. So,

$$M(y_n,y_{n+1},t)\ge\psi(M(y_{n-1},y_n,t)).$$

Hence, repeating this inequality n times we obtain,

$$M(y_n,y_{n+1},t)\ge\psi^n(M(y_0,y_1,t))$$

Letting $n \to \infty$, from Lemma 1.6, we get

$$\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1. \tag{2.7}$$

Now, we want to show that $\{y_n\}$ is a Cauchy sequence. Suppose to the contrary, that $\{y_n\}$ is not a Cauchy sequence. Then there are $\varepsilon \in (0,1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $n(k) > m(k) > k$ and

$$M(y_{n(k)}, y_{m(k)}, t_0) \leq 1 - \varepsilon\ .$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality. Then

$$M(y_{m(k)-1}, y_{n(k)}, t_0) > 1 - \varepsilon$$

and so, for all $k \in \mathbb{N}$, we get

$$\begin{aligned} 1-\varepsilon &\geq M(y_{n(k)}, y_{m(k)}, t_0) \\ &\geq M(y_{m(k)-1}, y_{m(k)}, t_0) * M(y_{m(k)-1}, y_{n(k)}, t_0) \\ &\geq M(y_{m(k)-1}, y_{m(k)}, t_0) * (1-\varepsilon\ ). \end{aligned}$$

By taking $k \to \infty$ in the above inequality and using (2.7), we obtain

$$\lim_{k\to\infty} M(y_{n(k)}, y_{m(k)}, t_0) = 1 - \varepsilon. \tag{2.8}$$

From (FM-4), we have

$$\begin{aligned} &M(y_{m(k)}, y_{n(k)}, t_0) \\ &\quad\geq M(y_{m(k)}, y_{m(k)-1}, t_0) * M(y_{m(k)-1}, y_{n(k)-1}, t_0) * M(y_{n(k)-1}, y_{n(k)}, t_0) \end{aligned}$$

and we get

$$\lim_{k\to\infty} M(y_{n(k)-1}, y_{m(k)-1}, t_0) = 1 - \varepsilon\ . \tag{2.9}$$

In view of Lemma 2.3, we have $\beta(y_{n(k)}, y_{m(k)}, t) \leq 1$. By applying (2.1), we obtain

(2.10)

$$\varphi(M(y_{n(k)}, y_{m(k)}, t_0)) = \varphi(M(fx_{n(k)}, fx_{m(k)}, t_0)) \geq \psi(\varphi(N(x_{n(k)}, x_{m(k)}))),$$

where

$$\begin{aligned} N(x_{n(k)}, x_{m(k)}) &= \min\{M(hx_{n(k)}, hx_{m(k)}, t_0), M(hx_{n(k)}, fx_{n(k)}, t_0), \\ &\qquad M(hx_{m(k)}, fx_{m(k)}, t_0))\} \\ &= \min\{M(y_{n(k)-1}, y_{m(k)-1}, t_0), M(y_{n(k)-1}, y_{n(k)}, t_0), \\ &\qquad M(y_{m(k)-1}, y_{m(k)}, t_0))\} \end{aligned}$$

Now, from the properties of φ, ψ and from (2.7), (2.8), (2.9) and the above inequality, as $k \to \infty$ in (2.10), we have

$$\varphi(1-\varepsilon) \geq \psi(\varphi(1-\varepsilon)) > \varphi(1-\varepsilon)$$

which is a contradiction. So, this implies that $\varepsilon = 0$. Thus $\{y_n\}$ is a Cauchy sequence in X. From the completeness of $(X, M, *)$ there exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z. \tag{2.11}$$

From (2.2) and (2.11), we have

$$fx_n \to z \text{ and } hx_{n+1} \to z. \tag{2.12}$$

Since hX is closed, by (2.12), $z \in hX$. Therefore, there exists $u \in X$ such that $hu = z$. As $y_n \to z$ and from Definition 2.1 with condition (b), we get

$$\beta(y_n, z, t) \le 1 \text{ for all } n \in \mathbb{N} \text{ and for all } t > 0.$$

Now, applying inequality (2.1), we get

$$\begin{aligned} &\varphi(M(fx_n, fu, t)) \\ &\quad \ge \quad \psi(\varphi(N(x_n, u))) \\ &\quad = \quad \psi(\varphi(\min\{M(hx_n, hu, t), M(hx_n, fx_n, t), M(hu, fu, t)\})). \end{aligned} \tag{2.13}$$

Taking $n \to \infty$ in (2.13) and using the properties of φ, ψ and the above inequality we have

$$\varphi(M(z, fu, t)) \ge \psi(\varphi(\min\{1, 1, M(z, fu, t)\})) > \varphi(M(z, fu, t))$$

which implies $M(z, fu, t) = 1$, that is $fu = z$. Thus, we deduce

$$z = hu = fu, \tag{2.14}$$

and so z is a point of coincidence for f and h. The uniqueness of the point of coincidence is a consequence of inequality (2.1) and condition (c), and so we omit the details.

By (2.14) and using the weak compatibility of f and h, we obtain

$$fz = fhu = hfu = hz$$

and so $fz = hz$. Uniqueness of the point of coincidence implies $z = fz = hz$. Consequently, z is a unique common fixed point of f and h. □

Corollary 2.7. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space and f be a triangular (h, β)-admissible mapping. Suppose that $fX \subset hX$, hX is a closed subset of X such that*

$$\beta(hx, hy, t)\varphi(M(fx, fy, t)) \ge \psi(\varphi(N(x, y)))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(hx, hy, t), M(hx, fx, t), M(hy, fy, t)\}$$

$\varphi \in \Phi$, $\psi \in \Psi$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1})) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) $\beta(hx, hy, t) \leq 1$ for all $x, y \in Coin(f, h)$ and $t > 0$.
Then f and h have a unique point of coincidence in X. Moreover, if f and h are weakly compatible, then f and h have a unique common fixed point.

If we choose $h = I_X$ in Theorem 2.6, we have the following corollary.

Corollary 2.8. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space and let f be a triangular β-admissible mapping such that*

$$\beta(x, y, t) \leq 1 \Rightarrow \varphi(M(fx, fy, t)) \geq \psi(\varphi(N(x, y)))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(x, y, t), M(x, fx, t), M(y, fy, t)\}$$

$\varphi \in \Phi$ and $\psi \in \Psi$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1})) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) $\beta(x, y, t) \leq 1$, whenever $x = fx$, $y = fy$ and for $t > 0$.
Then f has a unique fixed point.

If we take $\varphi(t) = t$ in Corollary 2.8, we have the following corollary.

Corollary 2.9. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space and let f be a triangular β-admissible mapping such that*

$$\beta(x, y, t) \leq 1 \Rightarrow M(fx, fy, t) \geq \psi(N(x, y))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(x, y, t), M(x, fx, t), M(y, fy, t)\}$$

$\psi \in \Psi$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1})) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) $\beta(x, y, t) \leq 1$, whenever $x = fx$, $y = fy$ and for $t > 0$.
Then f has a unique fixed point.

Example 2.10. Let $X = [0, +\infty)$, $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $t > 0$. Clearly, $(X, M, *)$ is a complete non-Archimedean fuzzy metric space. Let $f, h : X \to X$ considered in Example 2.2. Suppose that $\beta : X \times X \times [0, +\infty) \to \mathbb{R}^+$ is given by

$$\beta(x, y, t) = \begin{cases} e^{-(x+y)} & , \quad x, y \in [0, 1] \\ 3 & , \quad \textit{otherwise}. \end{cases}$$

f is a triangular (h,β)-admissible mapping. Indeed, if $\beta(hx,hy,t) \leq 1$, then $e^{-h(x+y)} \leq 1$. Then $hx, hy \in [0,1]$ and so $x, y \in [0,1]$. Thus $\beta(fx,fy,t) = e^{-f(x+y)} \leq 1$. Now assume that $\beta(x,z,t) \leq 1$ and $\beta(z,y,t) \leq 1$, so $x, z \in [0,1]$ and $z, y \in [0,1]$. Then, $x, y \in [0,1]$ and so $\beta(x,y,t) \leq 1$. Also, it is easy to see that f is a fuzzy (h,β)-contractive mapping.
Let, $x, y \in [0,1]$ and $x < y$. Then $\beta(hx,hy,t) \leq 1$ and

$$\begin{aligned}
1 &\geq \varphi(1) = \varphi(M(fx,fy,t)) \\
&\geq \psi(\varphi(\min\{M(hx,hy,t), M(hx,fx,t), M(hy,fy,t)\})) \\
&= \psi(1) = 1.
\end{aligned}$$

Otherwise, $\beta(hx,hy,t) = 3$ and

$$\begin{aligned}
1 &\geq \varphi(1) = \varphi(M(fx,fy,t)) \\
&\geq \psi(\varphi(\min\{M(hx,hy,t), M(hx,fx,t), M(hy,fy,t)\})) \\
&= \psi(\frac{1}{6}) > \frac{1}{6}.
\end{aligned}$$

Further, there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$. Indeed for $x_0 = 0$, we have $\beta(h0, f0, t) = 1 \leq 1$. Finally, $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. By the definition of the function β, it follows that $x_n \in [0,1]$ for all $n \in \mathbb{N}$, so $x \in [0,1]$. Therefore $\beta(x_n, x, t) \leq 1$. Thus, all the required hypotheses of Theorem 2.6 are satisfied and hence f and h have a unique common fixed point.

3. Cylic Results

In this section, we give some fixed point results involving cyclic mappings which can be regarded as consequences of the theorems presented in the previous section.

Theorem 3.1. *Let A and B be two closed subsets of complete non-Archimedean fuzzy metric space $(X,M,*)$ such that $A \cap B \neq 0$ and $f,h : A \cup B \to A \cup B$ be mappings such that $fA \subset hB$, $fB \subset hA$ and f is a triangular (h,β)-admissible mapping. Assume that $h(A \cup B)$ is a closed subset of X such that*

$$\varphi(M(fx,fy,t)) \geq \psi(\varphi(N(x,y))) \tag{3.1}$$

for all $x \in A$, $y \in B$, $t > 0$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$.
Then,
(i) If h is one to one then there exists $z \in A \cap B$ such that $fz = hz$.

(ii) If f and h are weakly compatible, then f and h have a unique common fixed point $z \in A \cap B$.

Proof. Define $\beta : X \times X \times (0, \infty) \to \mathbb{R}^+$

$$\beta(x, y, t) = \begin{cases} 1 & x \in hA, y \in hB \text{ or } x \in hB, y \in hA \\ 0 & \text{otherwise.} \end{cases}$$

Let $\beta(hx, hy, t) \leq 1$. Then $hx \in hA$ and $hy \in hB$. Since h is one to one, we have $x \in A$ and $y \in B$. So, $fx \in hB$ and $fy \in hA$. Hence, $\beta(fx, fy, t) \leq 1$. Therefore, f is a triangular (h, β)-admissible mapping.

Since $A \cap B \neq \emptyset$, there exists $x_0 \in A \cap B$. This implies that $hx_0 \in hA$ and $fx_0 \in hB$. So, $\beta(hx_0, fx_0, t) \leq 1$.

Let $\{x_n\}$ be a sequence in X such that $\beta(x_n, x_{n+1)}) \leq 1$ for all n and $x_n \to x$ as $n \to \infty$. Then $x_n \in hA$ and $x_{n+1} \in hB$. This implies that $x \in hA \cap hB$. So, we get $\beta(x_n, x, t) \leq 1$. Then the conditions (a) and (b) of Theorem 2.6 hold. So there exist $u, z \in A \cup B$ such that $u = fz = hz$. On the other hand, since h is one to one, there exist $z_1 \in A, z_2 \in B$ such that $hz_1 = hz_2 = u$ implies $z_1 = z_2 = z$. Therefore, $u = hz$ for $z \in A \cap B$. If f and h are weakly compatible, following the proof of Theorem 2.6, we have $u = fu = hu$. The uniqueness of the common fixed point follows from (3.1). □

4. Some Results in Fuzzy Metric Spaces

Theorem 4.1. *Let $(X, M, *)$ be a G-complete fuzzy metric space and f be a fuzzy (h, β)-contractive mapping. Suppose that $fX \subset hX$, hX is a closed subset of X and the following conditions hold:*
(a) there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) $\beta(hx, hy, t) \leq 1$ for all $x, y \in Coin(f, h)$ and $t > 0$.
Then f and h have a unique point of coincidence in X. Moreover, if f and h are weakly compatible, then f and h have a unique common fixed point.

Proof. Let $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$. Define a sequences $\{x_n\}$ and $\{y_n\}$ by

$$y_n = fx_n = hx_{n+1} \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

If $y_n = y_{n+1}$ then y_{n+1} is a point of coincidence of f and h. Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. By virtue of Lemma 2.3, we get

$$\beta(y_m, y_n, t) \leq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore by (2.1) and using (4.1), we get

$$\varphi(M(y_n, y_{n+1}, t)) = \varphi(M(fx_n, fx_{n+1}, t)) \geq \psi(\varphi(N(x_n, x_{n+1}))) \tag{4.2}$$

and since property of φ, we get

$$M(y_n, y_{n+1}, t) \geq \varphi(M(fx_n, fx_{n+1}, t)) \geq \psi(\varphi(N(x_n, x_{n+1}))) \tag{4.3}$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \min\{M(hx_n, hx_{n+1}, t), M(hx_n, fx_n, t), M(hx_{n+1}, fx_{n+1}, t)\} \\ (4.4) \qquad &= \min\{M(y_{n-1}, y_n, t), M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)\} \end{aligned}$$

Thus from (4.3) and (4.4), we obtain

$$M(y_n, y_{n+1}, t) \geq \psi(\min\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)).$$

If $M(y_{n-1}, y_n, t) > M(y_n, y_{n+1}, t)$ for some $n \in \mathbb{N}$ and property of ψ, we get

$$M(y_n, y_{n+1}, t) \geq \psi(M(y_n, y_{n+1}, t)) > M(y_n, y_{n+1}, t)$$

which is a contradiction. So,

$$M(y_n, y_{n+1}, t) \geq \psi(M(y_{n-1}, y_n, t)).$$

Hence, repeating this inequality n times we obtain,

$$M(y_n, y_{n+1}, t) \geq \psi^n(M(y_0, y_1, t)).$$

Letting $n \to \infty$, from Lemma 1.6 we get,

$$\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1.$$

Thus, for any $p > 0$,we have

$$\begin{aligned} &M(y_n, y_{n+p}, t) \\ &\quad \geq \ast(M(y_n, y_{n+1}, \frac{t}{p}), M(y_{n+1}, y_{n+2}, \frac{t}{p}), \ldots, M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\quad \geq \ast(\psi^n(\varphi(M(y_0, y_1, \frac{t}{p}))), \psi^{n+1}(\varphi(M(y_0, y_1, \frac{t}{p}))), \ldots \\ &\qquad , \psi^{n+p-1}(\varphi(M(y_0, y_1, \frac{t}{p}))) \\ &\quad = \ast_{i=0}^{p-1} \psi^{n+i}(M(y_0, y_1, \frac{t}{p})). \end{aligned}$$

By Lemma 1.6, for every $i \in \{0, 1, ..., p-1\}$, we obtain that

$$\lim_{n\to\infty} \psi^{n+i}(M(x_0, x_1, \frac{t}{p})) = 1.$$

According to the continuitiy of t-norm $\ast$, it can easily be verified that $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$. Thus $\{y_n\}$ is a Cauchy sequence in X. From the completeness of $(X, M, \ast)$ there exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z. \tag{4.5}$$

From (4.1) and (4.5), we have

$$fx_n \to z \text{ and } hx_{n+1} \to z. \tag{4.6}$$

Since hX is closed, by (4.6), $z \in hX$. Therefore, there exists $u \in X$ such that $hu = z$. As $y_n \to z$ and from Lemma 2.3 with condition (b), we get

$$\beta(y_n, z, t) \leq 1 \text{ for all for all } n \in \mathbb{N} \text{ and for all } t > 0.$$

Now, applying inequality (2.1), we get

$$\begin{aligned} \varphi(M(fx_n, fu, t)) &\geq \psi(\varphi(N(x_n, u))) \\ &= \psi(\varphi(\min\{M(hx_n, hu, t), M(hx_n, fx_n, t), M(hu, fu, t)\}. \end{aligned} \tag{4.7}$$

Taking $n \to \infty$ in (4.7) and using the properties of φ, ψ and the above inequality we have

$$\varphi(M(z, fu, t)) \geq \psi(\varphi(\min\{1, 1, M(z, fu, t)\})) > \varphi(M(z, fu, t))$$

which implies $M(z, fu, t) = 1$, that is $fu = z$. Thus, we deduce

$$z = hu = fu \tag{4.8}$$

and so z is a point of coincidence for f and h. The uniqueness of the point of coincidence is a consequence of the conditions (a) and (c), and so we omit the details.

By (4.8) and using the weak compatibility of f and h, we obtain

$$fz = fhu = hfu = hz$$

and so $fz = hz$. Uniqueness of the point of coincidence implies $z = fz = hz$. Consequently, z is a unique common fixed point of f and h. □

Corollary 4.2. *Let $(X, M, *)$ be a G-complete fuzzy metric space and f be a triangular (h, β)-admissible mapping. Suppose that $fX \subset hX$, is a closed subset of X such that*

$$\beta(hx, hy, t)\varphi(M(fx, fy, t)) \geq \psi(\varphi(N(x, y)))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(hx, hy, t), M(hx, fx, t), M(hy, fy, t)\}$$

$\varphi \in \Phi$ and $\psi \in \Psi$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(hx_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1)}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) $\beta(hx, hy, t) \leq 1$ for all $x, y \in Coin(f, h)$ and $t > 0$.
Then f and h have a unique point of coincidence in X. Moreover, if f and h are weakly compatible, then f and h have a unique common fixed point.

If we choose $h = I_X$ in Theorem 4.1, we have the following corollary.

Corollary 4.3. *Let $(X, M, *)$ be a G-complete fuzzy metric space and let f be a triangular β-admissible mapping such that*

$$\beta(x, y, t) \leq 1 \Rightarrow \varphi(M(fx, fy, t)) \geq \psi(\varphi(N(x, y)))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(x, y, t), M(x, fx, t), M(y, fy, t)\}$$

$\varphi \in \Phi$ and $\psi \in \Psi$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1)}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) $\beta(x, y, t) \leq 1$, whenever $x = fx$, $y = fy$ and for $t > 0$.
Then f has a unique fixed point.

If we take $\varphi(t) = t$ in Corollary 4.3, we have the following corollary.

Corollary 4.4. *Let $(X, M, *)$ be a G-complete fuzzy metric space and let f be a triangular β-admissible mapping such that*

$$\beta(x, y, t) \leq 1 \Rightarrow M(fx, fy, t) \geq \psi(N(x, y))$$

for all $x, y \in X$ and $t > 0$, where

$$N(x, y) = \min\{M(x, y, t), M(x, fx, t), M(y, fy, t)\}$$

$\psi \in \Psi$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1)}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$,
(c) if $\beta(x, y, t) \leq 1$, whenever $x = fx$, $y = fy$ and for $t > 0$.
Then f has a unique fixed point.

If we choose $h = I_X$ in Theorem 4.1, following similar arguments as those given in the proof of Theorem 3.1, we have the following theorem.

Theorem 4.5. *Let A and B be two closed subsets of G-complete fuzzy metric space $(X, M, *)$ such that $A \cap B \neq \emptyset$ and $f : A \cup B \to A \cup B$ be a mapping s such that $fA \subset B$, $fB \subset A$. Assume that $A \cup B$ is a closed subset of X such that*

$$\varphi(M(fx, fy, t)) \geq \psi(\varphi(N(x, y)))$$

for all $x \in A$, $y \in B$ and $t > 0$ and the following conditions hold:
(a) there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$,
(b) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$.
Then, f has a unique fixed point $z \in A \cap B$.

Remark 4.6. If $\beta(hx, hy, t) = 1$ for all $x, y \in X$ and any $t > 0$ and $\varphi(t) = t$ and $N(x, y) = M(x, y, t)$, then Theorem 4.1 reduces to Theorem 3.1 given by Shen et al. (see [25]).

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References

[1] Altun, I., and Miheţ, D. Ordered non-Archimedean fuzzy metric spaces and some fixed point results. *Fixed Point Theory Appl.* (2010), Art. ID 782680, 11.

[2] Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math. 3* (1922), 133–181.

[3] Chauhan, S., Bhatnagar, S., and Radenović, S. Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces. *Matematiche (Catania) 68*, 1 (2013), 87–98.

[4] Chauhan, S., Pant, B. D., and Imdad, M. Coincidence and common fixed point theorems in non-Archimedean Menger PM-spaces. *Cubo 15*, 3 (2013), 31–44.

[5] Deng, Z. Fuzzy pseudometric spaces. *J. Math. Anal. Appl. 86*, 1 (1982), 74–95.

[6] Di Bari, C., and Vetro, C. Fixed points, attractors and weak fuzzy contractive mappings in a fuzzy metric space. *J. Fuzzy Math. 13*, 4 (2005), 973–982.

[7] Dinarvand, M. Some fixed point results for admissible Geraghty contraction type mappings in fuzzy metric spaces. *Iran. J. Fuzzy Syst. 14*, 3 (2017), 161–177, 191.

[8] George, A., and Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets and Systems 64*, 3 (1994), 395–399.

[9] Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets and Systems 27*, 3 (1988), 385–389.

[10] Gregori, V., and Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets and Systems 125*, 2 (2002), 245–252.

[11] Isik, H., Samet, B., and Vetro, C. Cyclic admissible contraction and applications to functional equations in dynamic programming. *Fixed Point Theory Appl.* (2015), 2015:163, 19.

[12] Istrăţescu, V. *An Introduction to Theory of Probabilistic Metric Spaces, with Applications*. Ed, Tehnică. Bucureşti, in Romanian, 1974.

[13] Jungck, G., and Rhoades, B. E. Fixed points for set valued functions without continuity. *Indian J. Pure Appl. Math. 29*, 3 (1998), 227–238.

[14] Kramosil, I., and Michálek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika (Prague) 11*, 5 (1975), 336–344.

[15] Miheţ, D. On fuzzy contractive mappings in fuzzy metric spaces. *Fuzzy Sets and Systems 158*, 8 (2007), 915–921.

[16] Miheţ, D. Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets and Systems 159*, 6 (2008), 739–744.

[17] Radenović, S. Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension. *Bull. Allahabad Math. Soc. 31*, 1 (2016), 39–55.

[18] Radenović, S. Some remarks on mappings satisfying cyclical contractive conditions. *Afr. Mat. 27*, 1-2 (2016), 291–295.

[19] Salimi, P., Vetro, C., and Vetro, P. Some new fixed point results in non-Archimedean fuzzy metric spaces. *Nonlinear Anal. Model. Control 18*, 3 (2013), 344–358.

[20] Samet, B., Vetro, C., and Vetro, P. Fixed point theorems for α-ψ-contractive type mappings. *Nonlinear Anal. 75*, 4 (2012), 2154–2165.

[21] Sangurlu, M., and Turkoglu, D. Fixed point theorems for $(\psi \circ \varphi)$—contractions in a fuzzy metric spaces. *J. Nonlinear Sci. Appl. 8*, 5 (2015), 687–694.

[22] Schweizer, B., and Sklar, A. Statistical metric spaces. *Pacific J. Math. 10* (1960), 313–334.

[23] Schweizer, B., and Sklar, A. *Probabilistic Metric Spaces.* North-Holland, Amsterdam, 1983.

[24] Sedghi, S., Shobkolaei, N., Došenović, T., and Radenović, S. Suzuki-type of common fixed point theorems in fuzzy metric spaces. *Math. Slovaca 68*, 2 (2018), 451–462.

[25] Shen, Y. H., Qiu, D., and Chen, W. Fixed point theory for cyclic ϕ-contractions in fuzzy metric spaces. *Iran. J. Fuzzy Syst. 10*, 4 (2013), 125–133, 153.

[26] Turkoglu, D., and Sangurlu, M. Fixed point theorems for fuzzy ψ-contractive mappings in fuzzy metric spaces. *J. Intell. Fuzzy Systems 26*, 1 (2014), 137–142.

[27] Vasuki, R., and Veeramani, P. Fixed point theorems and Cauchy sequences in fuzzy metric spaces. *Fuzzy Sets and Systems 135*, 3 (2003), 415–417.

[28] Vetro, C. Fixed points in weak non-Archimedean fuzzy metric spaces. *Fuzzy Sets and Systems 162* (2011), 84–90.

[29] Vetro, C., Gopal, D., and Imdad, M. Common fixed point theorems for (ϕ,ψ)-weak contractions in fuzzy metric spaces. *Indian J. Math. 52*, 3 (2010), 573–590.

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