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Sharp Wirtinger's type inequalities for double integrals with applications

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Abstract. In this work, sharp Wirtinger type inequalities for double integrals are established. As applications, two sharp Čebyšev type inequalities for absolutely continuous functions whose second partial derivatives belong to L^2 space are proved.

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1. Introduction

The theory of Fourier series has a significant role in almost all branches of mathematical and numerical analysis. A very interesting connection between inequalities and Fourier series has been made more than a hundred years ago. The celebrated Bessel's integral inequality

$$2a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx, \tag{1.1}$$

was named after Bessel's death and considered from that time as the first link in this connection and starting point for other related works after the end of 18-th century.

In 1916, Wirtinger [8] credibly proved his inequality regarding square integrable periodic functions, which reads:

Theorem 1.1. *Let f be a real valued function with period 2π and $\int_0^{2\pi} f(x)\, dx = 0$. If $f' \in L^2[0, 2\pi]$, then*

$$\int_0^{2\pi} f^2(x)\, dx \leq \int_0^{2\pi} f'^2(x)\, dx, \tag{1.2}$$

with equality if and only if $f(x) = A\cos x + B\sin x$, $A, B \in \mathbb{R}$.

Many authors have considered a main attention for Wirtinger's inequality and therefore, several generalizations, counterparts and refinements was collected in a chapter of the book [16].

In 1967, Diaz and Metcalf [9] have extended and generalized Wirtinger inequality and they proved the following result:

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Theorem 1.2. *Let f be continuously differentiable on (a,b). Suppose $f(t_1)=f(t_2)$ for $a\le t_1\le t_2\le b$, then the inequality*

$$\int_a^b [f(x)-f(t_1)]^2\,dx \le \frac{4}{\pi^2}\max\left\{(t_1-a)^2,(b-t_2)^2,\left(\frac{t_2-t_1}{2}\right)^2\right\}\int_a^b f'^2(x)\,dx, \tag{1.3}$$

holds. In particular, if $t_1=t_2=t$, then

$$\int_a^b [f(x)-f(t)]^2\,dx \le \frac{4}{\pi^2}\left[\frac{b-a}{2}+\left|t-\frac{a+b}{2}\right|\right]^2\int_a^b f'^2(x)\,dx, \tag{1.4}$$

For other related results see [7], [6] and [15].

One of the most directly applicable usages of (1.3) is in several works regarding the famous *Čebyšev functional*

$$\mathcal{T}(f,g)=\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.5}$$

which compare or measure the difference between the integral of the product of two functions with the product of their integrals.

In 1970, Ostrowski [17] proved that if $f',g'\in L^2[a,b]$, then there exists a constant C, $0\le C\le \frac{b-a}{8}$, such that

$$|\mathcal{T}(f,g)|\le C\,\|f'\|_2\,\|g'\|_2\,. \tag{1.6}$$

After that in 1973, A. Lupaş [14] has improved the result of Ostrowski's (1.6) and proved that

$$|\mathcal{T}(f,g)|\le \frac{b-a}{\pi^2}\,\|f'\|_2\,\|g'\|_2\,, \tag{1.7}$$

where the constant $\frac{1}{\pi^2}$ is the best possible.

In this work we deal with the problem: what is the best possible constant C such that the inequality

$$\int_c^d\int_a^b f^2(x,y)\,dxdy \le C\int_c^d\int_a^b\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dxdy \tag{1.8}$$

holds whenever $f,g\in\mathfrak{L}^2(I)$. This question is a natural extension of Diaz-Metcalf inequality (1.3), as well as the complementary works of Beesack and Milovanović in one variable, see [7], [6] and [15].

Accordingly, for the *Čebyšev functional*

$$\mathcal{T}(f,g):=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,g(t,s)\,dsdt - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(t,s)\,dtds,$$

what is the best possible constant C' such that the inequality

$$|\mathcal{T}(f,g)| \le C' \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_2 \left\| \frac{\partial^2 g}{\partial x \partial y} \right\|_2$$

holds, and this is an extension of the Lupaş inequality (1.7).

2. Wirtinger's type inequalities

Let I be a two dimensional interval and denote by I° its interior. For $a,b,c,d \in \mathbb{R}$, we consider the subset $\mathbb{D} := \{(x,y) : a \le x \le b, c \le y \le d\} \subseteq \mathbb{R}^2$ such that $\mathbb{D} \subset I^{\circ}$. Also, define the subsets I_- and I^- of I as follows:

$$I^- := I - \{b,d\} = [a,b) \times [c,d)\,, \quad \text{and} \quad I_- := I - \{a,c\} = (a,b] \times (c,d]$$

In the sequel, throughout this work, we assume that $f : I \to \mathbb{R}$ satisfies the boundary conditions: $f(a,\cdot) = f(\cdot,c) = 0$, $f_x(a,\cdot) = f_x(\cdot,c) = 0$, $f_y(a,\cdot) = f_y(\cdot,c) = 0$ on I^-. Also, we assume $f(b,\cdot) = f(\cdot,d) = 0$, $f_x(b,\cdot) = f_x(\cdot,d) = 0$, $f_y(b,\cdot) = f_y(\cdot,d) = 0$ on I_-, and both conditions on I°.

Let $\mathfrak{L}^2(I)$ be the space of all functions f which are absolutely continuous on I, with $\int_c^d \int_a^b \left| \frac{\partial^2 f}{\partial x \partial y} \right|^2 dxdy < \infty$ and f satisfies the above boundary conditions.

Theorem 2.1. *Let $f \in \mathfrak{L}^2(I^-)$. Then the inequality*

$$\int_c^d \int_a^b f^2(x,y)\,dxdy \le \frac{16}{\pi^4}(b-a)^2(d-c)^2 \int_c^d \int_a^b \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 dxdy \tag{2.1}$$

is valid. The constant $\frac{16}{\pi^2}$ is the best possible, in the sense that it cannot be replaced by a smaller one.

Proof. Let $a \le x < b$ and $c \le y < d$. Since f is absolutely continuous then we can write $f(x,y) = \int_c^y \int_a^x f_{ts}(t,s)\,dtds$. If a and c are real numbers this is equivalent to saying that $f(a,c) = 0$ and f is absolutely continuous on $[a,b) \times [c,d)$. Setting

$$f(x,y) = g_1(x) g_2(y) h(x,y),$$

where

$$g_1(x) = \sin \omega_1 (x-a)\,, \forall x \in [a,b),$$

with $\omega_1 = \lambda_1^{1/2}$ and $\lambda_1 = \frac{\pi^2}{4(b-a)^2}$, and

$$g_2(y) = \sin \omega_2 (y-c)\,, \forall y \in [c,d),$$

with $\omega_2 = \lambda_2^{1/2}$ and $\lambda_2 = \frac{\pi^2}{4(d-c)^2}$.

Firstly, let us observe that since $g_1'(x) = \omega_1 \cos \omega_1 (x-a)$, so that $g_1''(x) = -\omega_1^2 g_1(x)$. Similarly, we have $g_2''(y) = -\omega_2^2 g_2(y)$.

For simplicity, since

$$\frac{\partial f}{\partial x} = g_1 g_2 \frac{\partial h}{\partial x} + g_2 h g_1',$$

then

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= g_1 g_2 \frac{\partial^2 h}{\partial x \partial y} + g_1 g_2' \frac{\partial h}{\partial x} + g_1' g_2' h + g_1' g_2 \frac{\partial h}{\partial y} \\
&= \frac{d}{dy}\left(g_1 g_2 \frac{\partial h}{\partial x} + g_1' g_2 h \right) \\
&= g_2' \left(g_1 \frac{\partial h}{\partial x} + g_1' h \right) + g_2 \frac{d}{dy}\left(g_1 \frac{\partial h}{\partial x} + g_1' h \right).
\end{aligned}$$

Setting

$$\Phi := \Phi(x,y) = g_1 \frac{\partial h}{\partial x} + g_1' h = \frac{f_x}{g_2} \Rightarrow g_2 \Phi = f_x,$$

therefore

$$\frac{d}{dy}\left(g_1 g_2 \frac{\partial h}{\partial x} + g_1' g_2 h \right) = g_2' \left(g_1 \frac{\partial h}{\partial x} + g_1' h \right) + g_2 \left(g_1 \frac{\partial h}{\partial x} + g_1' h \right)' = \Phi g_2' + g_2 \Phi_y'.$$

Now, if $a < \alpha < \beta < b$, and $c < \gamma < \delta < d$, we have

$$\begin{aligned}
\int_\alpha^\beta \int_\gamma^\delta \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 dy dx &= \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' + g_2 \Phi_y' \right)^2 dy dx \\
&= \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' \right)^2 \left(1 + \frac{g_2 \Phi_y'}{\Phi g_2'} \right)^2 dy dx \\
&\geq \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' \right)^2 \left(1 + 2 \frac{g_2 \Phi_y'}{\Phi g_2'} \right) dy dx \\
&= \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' \right)^2 dy dx + 2 \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' \right) g_2 \Phi_y' dy dx \\
&= g_2 \left(g_2' \right) \Phi^2 \Big|_\gamma^\delta \Big|_\alpha^\beta - \int_\alpha^\beta \int_\gamma^\delta \left(g_2 \left(g_2'' \right) + \left(g_2' \right)^2 \right) \Phi^2 dy dx \\
&\quad + \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' \right)^2 dy dx \\
&= g_2 \left(g_2' \right) \Phi^2 \Big|_\gamma^\delta \Big|_\alpha^\beta - \int_\alpha^\beta \int_\gamma^\delta \left(-\omega_2^2 g_2^2 + \left(g_2' \right)^2 \right) \Phi^2 dy dx \qquad (2.2) \\
&\quad + \int_\alpha^\beta \int_\gamma^\delta \left(\Phi g_2' \right)^2 dy dx
\end{aligned}$$

$$
\begin{aligned}
&= g_2\left(g_2'\right)\Phi^2\Big|_\gamma^\delta\Big|_\alpha^\beta + \omega_2^2\int_\alpha^\beta\int_\gamma^\delta g_2^2\Phi^2 dydx \\
&\quad - \int_\alpha^\beta\int_\gamma^\delta \left(g_2'\right)^2\Phi^2 dydx + \int_\alpha^\beta\int_\gamma^\delta \left(\Phi g_2'\right)^2 dydx \\
&= g_2\left(g_2'\right)\Phi^2\Big|_\gamma^\delta\Big|_\alpha^\beta + \omega_2^2\int_\alpha^\beta\int_\gamma^\delta g_2^2\Phi^2 dydx
\end{aligned}
$$

$$
= g_2\left(g_2'\right)\Phi^2\Big|_\gamma^\delta\Big|_\alpha^\beta + \omega_2^2\int_\alpha^\beta\int_\gamma^\delta \left(\frac{\partial f}{\partial x}\right)^2 dydx \tag{2.3}
$$

where, in (2.2) we integrate by parts, assuming that $u = g_2\left(g_2'\right)$ and $dv = 2\Phi_y'\Phi$. Now, we also have

(2.4)

$$
\begin{aligned}
&\int_\alpha^\beta int_\gamma^\delta \left(\frac{\partial f}{\partial x}\right)^2 dydx \\
&= \int_\alpha^\beta\int_\gamma^\delta \left\{\left(g_1 g_2\frac{\partial h}{\partial x} + g_1' g_2 h\right)\right\}^2 dydx \\
&= \int_\alpha^\beta\int_\gamma^\delta \left\{\left(g_1' g_2 h\right)^2\left(1 + \frac{g_1 g_2\frac{\partial h}{\partial x}}{g_1' g_2 h}\right)^2\right\} dydx \\
&\geq \int_\alpha^\beta\int_\gamma^\delta \left\{\left(g_1' g_2 h\right)^2\left(1 + 2\frac{g_1 g_2\frac{\partial h}{\partial x}}{g_1' g_2 h}\right)\right\} dydx \\
&= \int_\alpha^\beta\int_\gamma^\delta \left(g_1' g_2 h\right)^2 dydx + 2\int_\alpha^\beta\int_\gamma^\delta \left(g_1' g_2 h\right)\left(g_1 g_2\frac{\partial h}{\partial x}\right) dydx \\
&= g_2^2 g_1\left(g_1'\right)h^2\Big|_\gamma^\delta\Big|_\alpha^\beta - \int_\alpha^\beta\int_\gamma^\delta \left(g_1 g_1'' + \left(g_1'\right)^2\right)g_2^2 h^2 dydx \\
&\quad + \int_\alpha^\beta\int_\gamma^\delta \left(g_1' g_2 h\right)^2 dydx \\
&= g_2^2 g_1\left(g_1'\right)h^2\Big|_\gamma^\delta\Big|_\alpha^\beta - \int_\alpha^\beta\int_\gamma^\delta \left(-\omega_1^2 g_1^2 + \left(g_1'\right)^2\right)g_2^2 h^2 dydx \\
&\quad + \int_\alpha^\beta\int_\gamma^\delta \left(g_1' g_2 h\right)^2 dydx \\
&= g_2^2 g_1\left(g_1'\right)h^2\Big|_\gamma^\delta\Big|_\alpha^\beta + \omega_1^2\int_\alpha^\beta\int_\gamma^\delta g_1^2 g_2^2 h^2 dydx \\
&\quad - \int_\alpha^\beta\int_\gamma^\delta \left(g_1'\right)^2 g_2^2 h^2 dydx + \int_\alpha^\beta\int_\gamma^\delta \left(g_1' g_2 h\right)^2 dydx
\end{aligned}
$$

$$
= g_2^2 g_1\left(g_1'\right)h^2\Big|_\gamma^\delta\Big|_\alpha^\beta + \omega_1^2\int_\alpha^\beta\int_\gamma^\delta g_1^2 g_2^2 h^2 dydx. \tag{2.5}
$$

Substitute (2.5) in (2.3), we get

$$\begin{aligned}\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dxdy \geq\ & g_2\left(g_2'\right)\Phi^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}+\omega_2^2 g_2^2 g_1\left(g_1'\right)h^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}\\
&+\omega_1^2\omega_2^2\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g_1^2 g_2^2 h^2 dydx\\
=\ & g_2\left(g_2'\right)\frac{f_x^2}{g_2^2}\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}+\omega_2^2 g_2^2 g_1\left(g_1'\right)\frac{f^2}{g_1^2 g_2^2}\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}\\
&+\omega_1^2\omega_2^2\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g_1^2 g_2^2\frac{f^2}{g_1^2 g_2^2}dydx\\
=\ & \left(\frac{g_2'}{g_2}\right)f_x^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}+\omega_2^2\left(\frac{g_1'}{g_1}\right)f^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}\\
&+\omega_1^2\omega_2^2\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} f^2 dydx\end{aligned}$$

Hence,

$$\begin{aligned}(2.6)\quad &\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} f^2 dydx\\
&\leq \frac{1}{\omega_1^2\omega_2^2}\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dxdy-\frac{1}{\omega_1^2\omega_2^2}\left(\frac{g_2'}{g_2}\right)f_x^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}-\frac{1}{\omega_1^2}\left(\frac{g_1'}{g_1}\right)f^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta}\end{aligned}$$

Now, since

$$0\leq f^2\left(\alpha,\gamma\right)=\left(\int_c^{\gamma}\int_a^{\alpha} f_{ts}\left(t,s\right)dtds\right)^2\leq\left(\alpha-a\right)\left(\gamma-c\right)\int_c^{\gamma}\int_a^{\alpha} f_{ts}^2\left(t,s\right)dtds$$

then

$$0\leq f^2\left(\alpha,\gamma\right)\leq\left(\alpha-a\right)\left(\gamma-c\right)\int_c^{\gamma}\int_a^{\alpha} f_{ts}^2\left(t,s\right)dtds\longrightarrow 0,$$

as $\alpha\longrightarrow a^+$ and $\gamma\longrightarrow c^+$, i.e., $f^2(\alpha,\gamma)=0$ and therefore

$$\left\{\frac{1}{\omega_1^2}\left(\frac{g_1'}{g_1}\right)\right\}\cdot f^2\left(\alpha,\gamma\right)\longrightarrow 0 \ \text{ as }\ \alpha\longrightarrow a^+ \text{ and } \gamma\longrightarrow c^+.$$

Similarly,

$$\begin{aligned}0\leq f_x^2\left(\alpha,\gamma\right)&=\left(\int_c^{\gamma} f_{xy}\left(\alpha,y\right)-f_{xy}\left(a,y\right)dy\right)^2\\
&\leq\left(\gamma-c\right)\int_c^{\gamma}\left(f_{xy}\left(\alpha,y\right)-f_{xy}\left(a,y\right)\right)^2 dy\end{aligned}$$

then

$$0 \le f_x^2(\alpha,\gamma) \le (\gamma - c)\int_c^{\gamma} \left(f_{xy}(\alpha,y) - f_{xy}(a,y)\right)^2 dy \longrightarrow 0,$$

as $\gamma \longrightarrow c^+$, i.e., $f^2(\alpha,\gamma) = 0$ and therefore

$$\left\{\frac{1}{\omega_1^2\omega_2^2}\left(\frac{g_2'}{g_2}\right)\right\} \cdot f_x^2(\alpha,\gamma) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow a^+ \text{ and } \gamma \longrightarrow c^+.$$

Then, from (2.6)) it follows

$$\begin{aligned}
&\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} f^2 dydx \\
&\le \frac{1}{\omega_1^2\omega_2^2}\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dxdy - \frac{1}{\omega_1^2\omega_2^2}\left(\frac{g_2'}{g_2}\right) f_x^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta} - \frac{1}{\omega_1^2}\left(\frac{g_1'}{g_1}\right) f^2\Big|_{\gamma}^{\delta}\Big|_{\alpha}^{\beta} \\
&\le \frac{1}{\omega_1^2\omega_2^2}\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dxdy,
\end{aligned}$$

where $a < \beta < b$ and $c < \delta < d$. Now let $\alpha \longrightarrow a^+$, $\beta \longrightarrow b^-$ and $\gamma \longrightarrow c^+$, $\delta \longrightarrow d^-$ to obtain the inequality (2.1).

To obtain the sharpness, assume that (2.1) holds with another constant $K > 0$,

$$\int_c^d\int_a^b f^2(x,y)\,dxdy \le K(b-a)^2(d-c)^2\int_c^d\int_a^b\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dxdy. \tag{2.7}$$

Define the function $f : [a,b) \times [c,d) \to \mathbb{R}$, given by

$$f(x,y) = C\sin\left(\frac{\pi}{2}\cdot\frac{x-a}{b-a}\right)\sin\left(\frac{\pi}{2}\cdot\frac{y-c}{d-c}\right).$$

Therefore, we have

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\pi^2}{4(b-a)(d-c)}\cos\left(\frac{\pi}{2}\cdot\frac{x-a}{b-a}\right)\cos\left(\frac{\pi}{2}\cdot\frac{y-c}{d-c}\right),$$

$\int_a^b\int_c^d f^2(x,y)\,dydx = \frac{(b-a)(d-c)}{4}$, and $\int_a^b\int_c^d\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dydx = \frac{\pi^4}{64(b-a)(d-c)}$. If we substitute in (2.7)

$$\frac{(b-a)(d-c)}{4} \le K(b-a)^2(d-c)^2\frac{\pi^4}{64(b-a)(d-c)},$$

which means that $K \ge \frac{16}{\pi^4}$, thus the constant $\frac{16}{\pi^4}$ is the best possible and the inequality (2.1) is sharp. □

Corollary 2.2. *If $f \in \mathfrak{L}^2(I_-)$, then the inequality* (2.1) *still holds, and the inequality is sharp.*

Proof. The proof goes similarly to the proof of Theorem 2.1, with a few changes in the auxiliary function 'sin' in both variables x and y defined on the bidimensional interval I_-. To obtain the sharpness, define the function $f : (a, b] \times (c, d] \to \mathbb{R}$, given by

$$f(x, y) = C \sin\left(\frac{\pi}{2} \cdot \frac{b - x}{b - a}\right) \sin\left(\frac{\pi}{2} \cdot \frac{d - y}{d - c}\right),$$

where C is constant. □

Corollary 2.3. *Let $f \in \mathfrak{L}^2(I)$. Under the assumptions of Theorem 2.1 and Corollary 2.2 together, the inequality*

$$\int_c^d \int_a^b |f(x, y) - f(\xi, \eta)|^2 \, dxdy \tag{2.8}$$
$$\leq \frac{16}{\pi^4} \left[\frac{b - a}{2} + \left|\xi - \frac{a + b}{2}\right|\right]^2 \left[\frac{d - c}{2} + \left|\eta - \frac{c + d}{2}\right|\right]^2 \int_c^d \int_a^b \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dxdy$$

is valid for all $(\xi, \eta) \in \mathbb{D}^\circ$. The constant $\frac{1}{\pi^4}$ is the best possible.

Proof. Apply Theorem 2.1 and Corollary 2.3 on the right hand side of the equation

$$\begin{aligned}
&\int_c^d \int_a^b |f(x, y) - f(\xi, \eta)|^2 \, dxdy \\
&\quad = \int_c^\eta \int_a^\xi |f(x, y) - f(\xi, \eta)|^2 \, dxdy + \int_\eta^d \int_a^\xi |f(x, y) - f(\xi, \eta)|^2 \, dxdy \\
&\qquad + \int_c^\eta \int_\xi^b |f(x, y) - f(\xi, \eta)|^2 \, dxdy + \int_\eta^d \int_\xi^b |f(x, y) - f(\xi, \eta)|^2 \, dxdy
\end{aligned}$$

and the make the substitution $h(x, y) = |f(x, y) - f(\xi, \eta)|^2$. To obtain the sharpness define $f : [a,b] \times [c,d] \to \mathbb{R}$, given by

$$f(x,y) = K_0 + \begin{cases} K_1 \sin\left(\frac{\pi}{2}\cdot\frac{a+\xi-2x}{\xi-a}\right)\sin\left(\frac{\pi}{2}\cdot\frac{c+\eta-2y}{\eta-c}\right)u_\xi(\tau)\,u_\eta(\psi), \\ \qquad a \le x \le \xi, c \le y \le \eta \\ K_2 \sin\left(\frac{\pi}{2}\cdot\frac{2x-\xi-b}{b-\xi}\right)\sin\left(\frac{\pi}{2}\cdot\frac{c+\eta-2y}{\eta-c}\right)u_\xi(-\tau)\,u_\eta(\psi), \\ \qquad \xi \le x \le b, c \le y \le \eta \\ K_3 \sin\left(\frac{\pi}{2}\cdot\frac{a+\xi-2x}{\xi-a}\right)\sin\left(\frac{\pi}{2}\cdot\frac{2y-\eta-d}{d-\xi}\right)u_\xi(\tau)\,u_\eta(-\psi), \\ \qquad a \le x \le \xi, \eta \le x \le d \\ K_4 \sin\left(\frac{\pi}{2}\cdot\frac{2x-\xi-b}{b-\xi}\right)\sin\left(\frac{\pi}{2}\cdot\frac{2y-\eta-d}{d-\eta}\right)u_\xi(-\tau)\,u_\eta(-\psi), \\ \qquad \xi \le x \le b, \eta \le x \le d \end{cases}$$

where K_0, K_1, K_2, K_3 and K_2 are arbitrary constants, $\tau = 2\xi - a - b$, $\psi = 2\eta - c - d$ and $u_s(t)$ is the unit step function. □

3. Sharp bounds for the Čebyšev functional

The *Čebyšev functional*

$$\mathcal{T}(f,g) := \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,g(t,s)\,dsdt$$
$$- \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(t,s)\,dtds \tag{3.1}$$

has interesting applications in the approximation of the integral of a product, as pointed out in the references below.

In order to represent the remainder of the Taylor formula in an integral form which provides a better estimation using the Grüss type inequalities, Hanna et al. [20], generalized the Korkine identity for double integrals and therefore Grüss type inequalities were proved.

In 2002, Pachpatte [18] has established two inequalities of Grüss type involving continuous functions of two independent variables whose first and second partial derivatives exist, are continuous and belong to $L_\infty(\mathbb{D})$; for details see [18]. For more results about multivariate and multidimensional Grüss type inequalities the reader may refer to [2, 4, 3, 5, 11, 12, 20, 13, 19].

Recently, the author of this paper [1] established various inequalities of Grüss type for functions of two variables under various assumptions on the functions involved.

In view of Corollary 2.3, we may state the following result.

Theorem 3.1. *If $f, g \in \mathfrak{L}^2(\mathbb{D})$, then*

$$|\mathcal{T}(f,g)| \le \frac{1}{\pi^4}(b-a)^2(d-c)^2 \left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_2 \left\|\frac{\partial^2 g}{\partial x \partial y}\right\|_2, \tag{3.2}$$

$\frac{1}{\pi^4}$ *is the best possible.*

Proof. By the triangle inequality and then using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&|\mathcal{T}(f,f)|^2 \\
&= \frac{1}{\Delta^2}\left|\int_c^d \int_a^b \left[f(x,y) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]\right. \\
&\quad \left.\left[f(x,y) - \frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right] dxdy\right|^2. \\
&\le \frac{1}{\Delta}\int_c^d \int_a^b \left[f(x,y) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]^2 dxdy \\
&\quad \times \frac{1}{\Delta}\int_c^d \int_a^b \left[f(x,y) - \frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right]^2 dxdy.
\end{aligned} \tag{3.3}$$

Now, since

$$\begin{aligned}
\mathcal{T}(f,f) &= \frac{1}{\Delta}\int_c^d \int_a^b \left(f(x,y) - \frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right)^2 dxdy \\
&= \frac{1}{\Delta}\int_c^d \int_a^b \left[f^2(x,y) - 2f(x,y)\frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right. \\
&\quad \left. + \left(\frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right)^2\right] dxdy \\
&= \frac{1}{\Delta}\int_c^d \int_a^b f^2(x,y)dxdy - \left(\frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right)^2,
\end{aligned}$$

where $\Delta := (b-a)(d-c)$.

Therefore, from (3.3)

$$\begin{aligned}
|\mathcal{T}(f,f)|^2 &\le \frac{1}{\Delta}\int_c^d \int_a^b \left[f(x,y) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]^2 dxdy \\
&\quad \times \frac{1}{\Delta}\int_c^d \int_a^b \left[f(x,y) - \frac{1}{\Delta}\int_c^d \int_a^b f(t,s)\,dtds\right]^2 dxdy \\
&= \frac{1}{\Delta}\int_c^d \int_a^b \left[f(x,y) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]^2 dxdy \times \mathcal{T}(f,f).
\end{aligned}$$

Therefore,

$$\mathcal{T}(f,f) \le \frac{1}{\Delta}\int_c^d\int_a^b \left[f(x,y) - f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]^2 dxdy.$$

Applying (2.8), we get

$$\int_c^d\int_a^b \left[f(x,y) - f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]^2 dxdy \le \frac{1}{\pi^4}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_2^2.$$

Thus,

$$\mathcal{T}(f,f) \le \frac{1}{\pi^4}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_2^2. \tag{3.4}$$

Using a similar argument we can observe that

$$\mathcal{T}(g,g) \le \frac{1}{\pi^4}(b-a)^2(d-c)^2\left\|\frac{\partial^2 g}{\partial x\partial y}\right\|_2^2. \tag{3.5}$$

Finally, since

$$|\mathcal{T}(f,g)| \le \mathcal{T}^{1/2}(f,f)\,\mathcal{T}^{1/2}(g,g) \le \frac{1}{\pi^4}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_2\left\|\frac{\partial^2 g}{\partial x\partial y}\right\|_2.$$

which proves (3.2). To obtain the sharpness, assume that (3.2) holds with another constant $K>0$,

$$|\mathcal{T}(f,g)| \le K(b-a)^2(d-c)^2\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_2\left\|\frac{\partial^2 g}{\partial x\partial y}\right\|_2. \tag{3.6}$$

Define the functions $f,g:\mathbb{D}\to\mathbb{R}$, given by

$$f(x,y) = \sin\left(\frac{\pi}{2}\cdot\frac{a+b-2x}{b-a}\right)\sin\left(\frac{\pi}{2}\cdot\frac{c+d-2y}{d-c}\right) = g(x,y),$$

therefore, we have

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\pi^2}{(b-a)(d-c)}\cos\left(\frac{\pi}{2}\cdot\frac{a+b-2x}{b-a}\right)\cos\left(\frac{\pi}{2}\cdot\frac{c+d-2y}{d-c}\right) = \frac{\partial^2 g}{\partial x\partial y},$$

$$\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx = \frac{(b-a)(d-c)}{4},$$

and

$$\int_a^b\int_c^d \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 dydx = \frac{(b-a)(d-c)}{4} = \int_a^b\int_c^d \left(\frac{\partial^2 g}{\partial x\partial y}\right)^2 dydx.$$

Substituting in (3.6)

$$\frac{(b-a)(d-c)}{4} \le K\pi^4 \frac{(b-a)(d-c)}{4},$$

which means that $K \ge \frac{1}{\pi^4}$, thus the constant $\frac{1}{\pi^4}$ is the best possible and the inequality (3.2) is sharp. □

Theorem 3.2. *Let $f \in \mathfrak{L}^2(\mathbb{D})$ and let $g : \mathbb{D} \to \mathbb{R}$ satisfy that there exist real numbers γ, Γ such that $\gamma \le g(x,y) \le \Gamma$ for all $(x,y) \in \mathbb{D}$, then*

$$|\mathcal{T}(f,g)| \le \frac{4}{\pi^2}(b-a)^{1/2}(d-c)^{1/2}(\Gamma-\gamma)\left\|\frac{\partial^2 f}{\partial x \partial y}\right\|_2. \tag{3.7}$$

The constant $\frac{4}{\pi^2}$ is the best possible.

Proof. Since

$$T(f,g) = \frac{1}{\Delta}\int_a^b \int_c^d \left[f(x,y) - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right] \times \left[g(x,y) - \frac{1}{\Delta}\int_a^b \int_c^d g(t,s)\right] dydx$$

Taking the absolute value of both sides and making use of the triangle inequality, we get

$$\begin{aligned}
&|T(f,g)| \\
&\le \frac{1}{\Delta}\int_a^b \int_c^d \left|f(x,y) - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right| \\
&\qquad \times \left|g(x,y) - \frac{1}{\Delta}\int_a^b \int_c^d g(t,s)\right| dydx \\
&\le \frac{1}{\Delta}\left(\int_a^b \int_c^d \left|f(x,y) - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right|^2 dydx\right)^{1/2} \\
&\qquad \times \left(\int_a^b \int_c^d \left|g(x,y) - \frac{1}{\Delta}\int_a^b \int_c^d g(t,s)\right|^2 dydx\right)^{1/2}
\end{aligned} \tag{3.8}$$

As in Theorem 1 in [1], we have observed that since there exist $\gamma, \Gamma \ge 0$ such that $\gamma \le g(x,y) \le \Gamma$ for all $(x,y) \in \mathbb{D}$, then

$$\int_c^d \int_a^b \left|g(x,y) - \frac{1}{\Delta}\int_c^d \int_a^b g(t,s)\,dtds\right|^2 dxdy \le \frac{1}{4}(\Gamma-\gamma)^2 \Delta \tag{3.9}$$

On the other hand, using the elementary inequality

$$(A+B+C+D)^2 \leq 4\left(A^2+B^2+C^2+D^2\right),$$

for all $A, B, C, D \geq 0$, we also have

$$\begin{aligned}
&\int_a^b \int_c^d \left| f(x,y) - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right|^2 dydx \\
&\leq \int_a^b \int_c^d |f(x,y)-f(a,c)|^2 dydx + \int_a^b \int_c^d |f(x,y)-f(a,d)|^2 dydx \\
&\quad + \int_a^b \int_c^d |f(x,y)-f(b,c)|^2 dydx + \int_a^b \int_c^d |f(x,y)-f(b,d)|^2 dydx
\end{aligned}$$

Applying (2.8) for each integral above and simplifying we get

$$\begin{aligned}
\text{(3.10)}\quad &\int_a^b \int_c^d \left| f(x,y) - \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right|^2 dydx \\
&\qquad \leq \frac{64}{\pi^4}(b-a)^2(d-c)^2 \int_c^d \int_a^b \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dxdy.
\end{aligned}$$

Combining the inequalities (3.9) and (3.10) with (3.8) we get the desired result (3.7).

To prove the sharpness of (3.7) assume that it holds with constant $C>0$, i.e.,

$$\text{(3.11)}\qquad |\mathcal{T}(f,g)| \leq C(b-a)^{1/2}(d-c)^{1/2}(\Gamma-\gamma)\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_2,$$

and consider the functions $f, g : \mathbb{D} \to \mathbb{R}$ be defined as

$$\begin{aligned}
f(x,y) &= \sin\left(\frac{\pi}{2}\cdot\frac{a+b-2x}{b-a}\right)\sin\left(\frac{\pi}{2}\cdot\frac{c+d-2y}{d-c}\right), \\
g(x,y) &= \operatorname{sgn}\left(x-\frac{a+b}{2}\right)\cdot \operatorname{sgn}\left(y-\frac{c+d}{2}\right).
\end{aligned}$$

As in the proof of Theorem 3.1, $f \in \mathcal{L}^2(\mathbb{D})$, and $\Gamma-\gamma=2$, $\int_a^b \int_c^d g(t,s)\,dsdt = 0$,

$$\int_a^b \int_c^d f(x,y)\,g(x,y)\,dydx = \frac{4}{\pi^2}(b-a)(d-c),$$

and

$$\int_a^b \int_c^d \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dydx = \frac{(b-a)(d-c)}{4}.$$

Making use of (3.11) we get $\frac{4}{\pi^2} \leq C$, which proves that $\frac{4}{\pi^2}$ is the best possible and thus the proof is completely finished. □

3.1. An inequality of Ostrowski's type

The mean value theorem for double integrals reads that: If f is continuous on $[a,b]\times[c,d]$, then there exists $(\eta,\xi)\in[a,b]\times[c,d]$ such that

$$f(\eta,\xi)=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds. \tag{3.12}$$

What about if one needs to measure the difference between the image of an arbitrary point $(x,y)\in[a,b]\times[c,d]$ and the average value

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds?$$

In this way Ostrowski introduced his famous inequality regarding differentiable functions and its average values. In [10, 11, 12, 20, 13] and other related works many authors have studied the Ostrowski type inequalities for various types of functions of several variables.

In the following, we present a which bound belongs to L_2 norm for the Ostrowski inequality.

Theorem 3.3. *Let $f\in\mathfrak{L}^2(\mathbb{D})$, then*

$$\begin{aligned}&\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dtds\right| \\ &\qquad\le\frac{4}{\pi^2\Delta^{1/2}}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_2\end{aligned} \tag{3.13}$$

for all $(x,y)\in[a,b]\times[c,d]$. In special case, choose $(x,y)=\left(\frac{a+b}{2},\frac{c+d}{2}\right)$

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dtds\right|\le\frac{1}{\pi^2}\Delta^{1/2}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_2$$

Proof. Since

$$\begin{aligned}&f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dtds \\ &\qquad=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d[f(x,y)-f(t,s)]\,dtds\end{aligned}$$

Taking the modulus, applying the triangle inequality and then using the Cauchy-

Schwarz inequality, we get

$$\begin{aligned}
&\left|f(x,y)-\frac{1}{\Delta}\int_a^b\int_c^d f(t,s)\,dtds\right| \\
&\le \frac{1}{\Delta}\int_a^b\int_c^d |f(x,y)-f(t,s)|\,dtds \\
&\le \frac{1}{\Delta^{1/2}}\left(\int_a^b\int_c^d |f(x,y)-f(t,s)|^2\,dtds\right)^{1/2} \\
&\le \frac{4}{\pi^2\Delta^{1/2}}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_2,
\end{aligned}$$

which follows by (2.8), and this proves (3.13). □

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