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DEGENERATED TOPOLOGICAL SPACES

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Abstract. By using the compositions of interior and closure operators on a topological space we can get at most seven different operators. Topological spaces in which the number of different operators obtained in this way is strictly smaller than seven are called degenerated spaces. This paper provides insight about properties of the degenerated spaces, and introduces some new characterizations of them.

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1. Introduction

In [5] Kuratowski proved the following result.

Theorem 1.1. *(Closure-Complement Theorem) If $(X, \mathcal{T})$ is a topological space and $A \subseteq X$, then at most 14 sets can be obtained from A by taking closures and complements. Furthermore, there is a space in which this bound is attained.*

In this article we will use an equivalent formulation of the Kuratowski closure-complement theorem, i.e., the statement that at most 7 distinct sets can be obtained from a subset of a topological space by applying closures and interiors. This formulation implies that, in any topological space X, at most 7 distinct operators can be derived by using of the compositions of interiors and closures: Id, Int, Cl, Cl(Int), Int(Cl), Int(Cl(Int)) and Cl(Int(Cl)), where Id is the identity operator. The set $\mathcal{O}_X$, which consists of all these operators, can be ordered by inclusion (Figure 1).

If some of the given operators are coinciding, then the diagram shown in the previous figure is considered to be a degenerated one. The topological space for which this is the case is called degenerated space.

The goal of this article is to provide a survey about degenerated spaces with an accent on their structural characterizations. Most of these characterizations are well known and can be found in [2] and [7]. In this paper, some new characterizations of P-spaces, connected ED-spaces, and finite degenerated spaces are provided. Moreover, for all finite non-homeomorphic spaces with two, three and four elements, their corresponding k-number and K-number are calculated.

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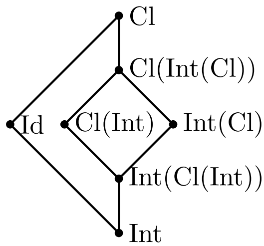

Figure 1: Partial order in $\mathcal{O}_X$

2. Preliminaries

Let us introduce the notation that will be used in this paper. By $(X, \mathcal{T})$ we denote a topological space on which no separation axiom is assumed. If $(X, \mathcal{T})$ is a topological space and $A \subseteq X$, then the interior and the closure of A in the space X will be denoted by $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$, respectively. Also, the subspace topology on A will be denoted by $\mathcal{T}_A$, and for $B \subseteq A$, the interior and the closure of B in the subspace $(A, \mathcal{T}_A)$ will be denoted by $\mathrm{Int}_A(B)$ and $\mathrm{Cl}_A(B)$, respectively. A subset D of the space X is dense if $\mathrm{Cl}(D) = X$. A subset C of the space X is co-dense if the set $X \setminus C$ is dense, or equivalently, if $\mathrm{Int}(C) = \emptyset$. A subset N of the space X is nowhere dense if $\mathrm{Int}(\mathrm{Cl}(N)) = \emptyset$. A point $x \in X$ is isolated if $\mathrm{Int}(\{x\})$ is an open set.

Basic properties of interior and closure operators can be found in most textbooks on general topology. The following proposition provides a list of the properties that will be considered in further discussion.

Proposition 2.1. *For every topological space $(X, \mathcal{T})$ and $A, B \subseteq X$ the following statements hold*

1. $\mathrm{Int}(A) \subseteq A \subseteq \mathrm{Cl}(A)$,
2. $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B),\ \mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$,
3. $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A),\ \mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$,
4. *If* $A \subseteq B$, *then* $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$ *and* $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$,
5. $\mathrm{Int}(A) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{A}))),\ \mathrm{Cl}(A) \supseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{A})))$,
6. $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{A})))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{A})),\ \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{A})))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{A}))$,
7. $\mathrm{Int}(A) = X \setminus \mathrm{Cl}(X \setminus A),\ \mathrm{Cl}(A) = X \setminus \mathrm{Int}(X \setminus A)$,
8. $\mathrm{Int}(A \setminus B) = \mathrm{Int} A \setminus \mathrm{Cl}(B)$,
9. $\mathrm{Int}(A \cup B) \subseteq \mathrm{Int}(A) \cup \mathrm{Cl}(B),\ \mathrm{Cl}(A \cap B) \supseteq \mathrm{Int}(A) \cap \mathrm{Cl}(B)$,
10. *If* A *is dense and* B *is not co-dense, then* $A \cap \mathrm{Int}(B) \neq \emptyset$,
11. *If* A *is dense and* B *is open, then* $\mathrm{Cl}(A \cap B) = \mathrm{Cl}(B)$,
12. *If* $B \subseteq A$, *then* $\mathrm{Int}_A(B) = A \setminus \mathrm{Cl}(A \setminus B)$ *and* $\mathrm{Cl}_A(B) = A \cap \mathrm{Cl}(B)$.

Definition 2.2. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

(i) $k(A)$ denotes the number of distinct sets obtainable from A by taking interiors and closures.

(ii) $k((X,\mathcal{T}))$ denotes $\max\{k(A) : A \subseteq X\}$.
(iii) $K((X,\mathcal{T}))$ denotes the number of distinct operators in $\mathcal{O}_X$.

Clearly, $k((X,\mathcal{T})) \le K((X,\mathcal{T})) \le 7$ holds for every topological space $(X,\mathcal{T})$.

Example 2.3. If $\mathcal{T}$ is the usual topology on the set $\mathbb{R}$ of real numbers, and $A = \{-1\} \cup (0,1) \cup (1,2) \cup (\mathbb{Q} \cap (2,3))$, where $\mathbb{Q}$ is the set of rational numbers, then it is easy to show that

$$\begin{aligned}
&\mathrm{Int}(A) = (0,1) \cup (1,2),\\
&\mathrm{Cl}(A) = \{-1\} \cup [0,3],\\
&\mathrm{Cl}(\mathrm{Int}(A)) = [0,2],\\
&\mathrm{Int}(\mathrm{Cl}(A)) = (0,3),\\
&\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = (0,2),\\
&\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = [0,3].
\end{aligned}$$

Thus, $k(A) = k((\mathbb{R},\mathcal{T})) = K((\mathbb{R},\mathcal{T})) = 7$.

If $(X,\mathcal{T})$ is a topological space with $K((X,\mathcal{T})) < 7$, then the diagram in Figure 1 is degenerated one. This gives motivation to the following definition.

Definition 2.4. Let $(X,\mathcal{T})$ be a topological space.
(i) $(X,\mathcal{T})$ is a degenerated space if $K((X,\mathcal{T})) < 7$.
(ii) $(X,\mathcal{T})$ is a non-degenerated or a K-space if $K((X,\mathcal{T})) = 7$.
(iii) $(X,\mathcal{T})$ is a full space if $k((X,\mathcal{T})) = K((X,\mathcal{T}))$.

3. Types of degenerated spaces

In this section we describe all possible types of degenerated spaces.

Definition 3.1. (i) A topological space is resolvable if it contains two disjoint dense subsets.
(ii) A space is irresolvable if it is not resolvable.
(iii) A space is open irresolvable, or OI-space, if any open subspace of this space is irresolvable.

Several characterizations of the OI-space are given in the following theorem.

Theorem 3.2. ([2],[7]) *For every topological space $(X,\mathcal{T})$, the following conditions are equivalent*
(1) *$(X,\mathcal{T})$ is an OI-space.*
(2) *Interior of every dense subset of X is dense.*
(3) *Every co-dense subset of X is nowhere dense.*
(4) *Every subset of X is a union of an open set and a nowhere dense set.*
(5) *For every $A \subseteq X$, it holds that*

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Cl}(\mathrm{Int}(A)),\ \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(\mathrm{Cl}(A)).$$

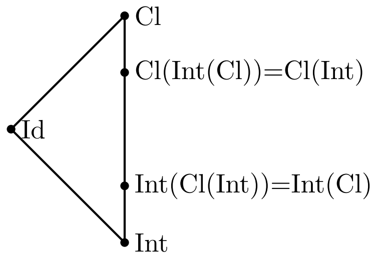

Figure 2: Degeneration in an OI-space

Corollary 3.3. *If $(X, \mathcal{T})$ is an OI-space, then $(X, \mathcal{T})$ is a degenerated space and $K(X, \mathcal{T}) \leqslant 5$ (Figure 2).*

A subspace of an OI-space does not need to be an OI-space, and the same applies for the products of OI-spaces. These facts are illustrated by the following example.

Example 3.4. (i) Let $\mathcal{T}$ be the cofinite topology on the set $\mathbb{N}$ of natural numbers. In the topological space $(\mathbb{N}, \mathcal{T})$ every infinite subset is dense, and since $\{1, 3, 5, \dots\}$ and $\{2, 4, 6, \dots\}$ are disjoint dense subsets, this topological space is resolvable.

Let Y be an infinite set disjoint with $\mathbb{N}$ and $\mathcal{U}$ be a non-principal ultrafilter on Y. The set $X := \mathbb{N} \cup Y$ can be equipped with the topology $\mathcal{T}_1 = \{U \cup V : U \in \mathcal{T}, V \in \mathcal{U} \cup \{\emptyset\}\}$. Let $D \subseteq X$ be a dense set. If for some $V \in \mathcal{U}$ it holds that $D \cap V = \emptyset$, then $D \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, this implies $Y \setminus D \in \mathcal{U}$, i.e. $Y \setminus D \in \mathcal{T}_1$, which is impossible because D is a dense set. Hence, for all $V \in \mathcal{U}$ it holds that $D \cap V \neq \emptyset$, and, since $\mathcal{U}$ is an ultrafilter, this implies that $D \in \mathcal{U}$. Thus, $\mathrm{Int}(D) = D$ is also a dense set, and space $(X, \mathcal{T}_1)$ is an OI-space. It is easy to show that the subspace topology on $\mathbb{N}$ is equal to $\mathcal{T}$, which implies that the subspace $\mathbb{N}$ of the space $(X, \mathcal{T}_1)$ is not an OI-space.

(ii) Let $\mathcal{U}$ be a non-principal ultrafilter on the set $\mathbb{N}$ of natural numbers. Family $\mathcal{T} = \mathcal{U} \cup \{\emptyset\}$ is the topology. Every dense set in the space $(\mathbb{N}, \mathcal{T})$ is also open, implying that this space is an OI-space. Let $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ have the product topology. We will show that $D = \{(n, n) : n \in \mathbb{N}\}$ is a dense and co-dense set in $\mathbb{N}^2$, which makes this space resolvable. To see that D is a dense set, note that every nonempty open set $W \subseteq \mathbb{N}^2$ contains a nonempty basic open set $U \times V$, where $U, V \in \mathcal{T}$. Since U and V are nonempty sets, we have $U, V \in \mathcal{U}$. So, $U \cap V \in \mathcal{U}$, which implies that U and V are not disjoint. If $n \in U \cap V$, then $(n, n) \in (U \times V) \cap D$, which implies $W \cap D \neq \emptyset$, concluding that D is a dense set. Also, since $\mathcal{U}$ is an ultrafilter, every set in $\mathcal{U}$ is infinite, which implies that if $U, V \in \mathcal{U}$ and $m \in U$, there exists $n \in V$, with $m \neq n$. Hence $(m, n) \in (U \times V) \setminus D$, implying $(U \times V) \cap (\mathbb{N}^2 \setminus D) \neq \emptyset$. Thus, $\mathbb{N}^2 \setminus D$ is also a dense set.

Definition 3.5. A topological space is extremally disconnected, or ED-space, if the closure of every open set in this space is an open set.

Remark 3.6. Some topologists require that an extremally disconnected space satisfies the additional condition of being a Hausdorff space. In this paper this condition will not be included. We note that this approach can lead to a situation that an extremally disconnected space can be a connected space.

Several characterizations of the ED-space are given by the following theorem.

Theorem 3.7. *([2],[3]) For any topological space $(X, \mathcal{T})$ the following conditions are equivalent*

(1) *$(X, \mathcal{T})$ is an ED-space.*
(2) *Every pair of disjoint open subsets of X have disjoint closures.*
(3) *For every $A \subseteq X$ it holds that*

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Int}(\mathrm{Cl}(A)), \; \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Cl}(\mathrm{Int}(A)).$$

Corollary 3.8. *If $(X, \mathcal{T})$ is an ED-space, then $(X, \mathcal{T})$ is a degenerated space and $K(X, \mathcal{T}) \leqslant 5$ (Figure 3).*

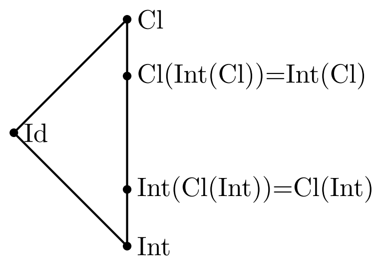

Figure 3: Degeneration in an ED-space

The next characterization of the connected ED-space is a novel result.

Theorem 3.9. *If $(X, \mathcal{T})$ is an ED-space, then the following conditions are equivalent*

(1) $(X, \mathcal{T})$ is a connected space.
(2) Every nonempty open set in X is dense.

Proof. (1) $\Rightarrow$ (2). Let $U \subseteq X$ be a nonempty open set. If U is not a dense set, then $\mathrm{Cl}(U)$ is a non-trivial clopen set in X, which is a contradiction by the connectedness of the space $(X, \mathcal{T})$.

(2) $\Rightarrow$ (1). Let $U \subseteq X$ be an open set. If $U = \emptyset$, then $\mathrm{Cl}(U) = \emptyset$, and if U is nonempty, then $\mathrm{Cl}(U) = X$. In both cases $\mathrm{Cl}(U)$ is an open set, hence $(X, \mathcal{T})$ is an ED-space. $\square$

It is relatively easy to show that open or dense subspaces of an ED-space are also ED-spaces. However, closed subspace of an ED-space does not need to be an ED-space, and same applies for products of ED-spaces. These facts are illustrated by the following example.

Example 3.10. The discrete space on the set $\mathbb{N}$ of natural numbers is obviously an ED-space. The Čech-Stone compactification $\beta\mathbb{N}$ is also an ED-space. However, in the subspace $\beta\mathbb{N}\setminus\mathbb{N}$, the closure of the union of a strictly increasing sequence of clopen sets is never open. Thus, the closed subspace $\beta\mathbb{N}\setminus\mathbb{N}$ of the space $\beta\mathbb{N}$ is not an ED-space. Similarly, the space $\beta\mathbb{N}\times\beta\mathbb{N}$ equipped with the product topology is not an ED-space, because the closure of the open set $D=\{(x,x) : x\in\beta\mathbb{N}\}$ is not open.

Definition 3.11. A topological space is Q-space, if this space is an OI-space which is also an ED-space.

Theorem 3.12. *[2] For every topological space $(X,\mathcal{T})$ the following conditions are equivalent*

(1) $(X,\mathcal{T})$ *is a Q-space.*

(2) *For every $A\subseteq X$, it holds that*

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))).$$

Corollary 3.13. *If $(X,\mathcal{T})$ is a Q-space, then $(X,\mathcal{T})$ is a degenerated space and $K(X,\mathcal{T})\leqslant 4$ (Figure 4).*

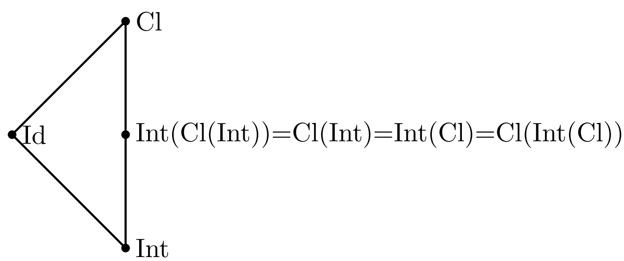

Figure 4: Degeneration in a Q-space

Definition 3.14. A topological space is a partition space, or P-space, if the topology of this space is induced by the base of topology whose elements are pairwise disjoint sets.

In the following theorem, several characterizations of P-spaces are given. We remark that third characterization is a novel result.

Theorem 3.15. ([2],[8]) *For any topological space $(X,\mathcal{T})$ the following conditions are equivalent*

(1) $(X, \mathcal{T})$ *is a* P*-space.*
(2) *Every open set in* X *is closed.*
(3) *The only nowhere dense set in* X *is the empty set.*
(4) *For every* $A \subseteq X$*, it holds*

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Cl}(A),\ \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(A).$$

Proof. (1) ⇒ (2). Let $U \subseteq X$ be an open set, and let $\mathcal{B}$ be the family of pairwise disjoint sets which form the basis for the topology $\mathcal{T}$. Since U is an open set, there is a family $\mathcal{B}' \subseteq \mathcal{B}$ such that $U = \bigcup_{V \in \mathcal{B}'} V$. This implies $X \setminus U = \bigcup_{V \in \mathcal{B} \setminus \mathcal{B}'} V$, leading to the conclusion that $X \setminus U$ is an open set, i.e. that U is a closed set.

(2) ⇒ (1). Consider the family

$$\mathcal{F} = \{\mathcal{A} \subseteq \mathcal{T} \setminus \{\emptyset\} : \text{ all distinct sets in } \mathcal{A} \text{ are pairwise disjoint }\}.$$

This family is nonempty, and every chain in a partially ordered set $(\mathcal{F}, \subseteq)$ has an upper bound in $\mathcal{F}$. By virtue of Zorn's lemma, there is a maximal element $\mathcal{B}$ in the family $\mathcal{F}$. Since the family $\mathcal{B} \cup \{X \setminus \bigcup \mathcal{B}\}$ is not maximal in $(\mathcal{F}, \subseteq)$, it follows that $X = \bigcup \mathcal{B}$. Let $U \subseteq X$ be a nontrivial open set. Then U is a clopen set, and the family $\mathcal{B}' = \{B \in \mathcal{B} : B \cap U \neq \emptyset\}$ is nonempty. Clearly, $U \subseteq \bigcup \mathcal{B}'$. Assume that $U \subsetneq \bigcup \mathcal{B}'$ holds, and let $V \in \mathcal{B}'$ be a set such that $(X \setminus U) \cap V \neq \emptyset$. This means that $\{U \cap V, (X \setminus U) \cap V\} \in \mathcal{F}$, so, using the maximality of $\mathcal{B}$, we can find sets $W_1, W_2 \in \mathcal{B}$ such that $U \cap V = W_1$ and $(X \setminus U) \cap V = W_2$. But, since $V, W_1, W_2 \in \mathcal{B}$, this leads to $W_1 = V = W_2$, which is a contradiction. Thus, $U = \bigcup \mathcal{B}'$, proving that $\mathcal{B}$ is a base for the topology $\mathcal{T}$, and subsequently, that $(X, \mathcal{T})$ is a P-space.

(2) ⇒ (3). Let $N \subseteq X$ be a nowhere dense set. Since $X \setminus \mathrm{Cl}(N)$ is open, then, by assumption, this set is also closed, whence $\mathrm{Cl}(X \setminus \mathrm{Cl}(N)) = X \setminus \mathrm{Cl}(N)$. Thus, $N \subseteq \mathrm{Cl}(N) = \mathrm{Int}(\mathrm{Cl}(N)) = \emptyset$, which implies $N = \emptyset$.

(3) ⇒ (4). Let $A \subseteq X$. The set $N := \mathrm{Cl}(A) \setminus \mathrm{Int}(\mathrm{Cl}(A))$ is closed, and $\mathrm{Int}(N) = \mathrm{Int}(\mathrm{Cl}(A) \setminus \mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Int}(\mathrm{Cl}(A)) \setminus \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \emptyset$, which implies that N is a nowhere dense set. Thus, $\mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$, and also $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. In a similar way we can prove that $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

(4) ⇒ (2). Let $U \subseteq X$ be an open set. Then $\mathrm{Cl}(U) = \mathrm{Cl}(\mathrm{Int}(U)) = \mathrm{Int}(U) = U$, so U is also a closed set. □

Corollary 3.16. *If* $(X, \mathcal{T})$ *is a* P*-space, then* $(X, \mathcal{T})$ *is a degenerated space and* $K(X, \mathcal{T}) \leqslant 3$ *(Figure 5).*

Theorem 3.17. *[2] Every* P*-space is an* ED*-space.*

Theorem 3.18. *Let* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *be* P*-spaces.*
1. *Every subspace of* X *is a* P*-space.*
2. *The topological product* $X \times Y$ *is a* P*-space.*

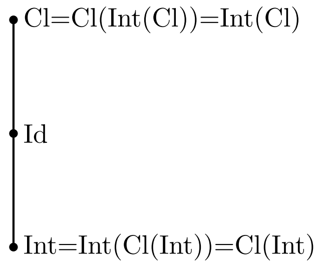

Figure 5: Degeneration in a P-space

Proof. 1. Let $A \subseteq X$ be a nonempty set. If $\mathcal{B}$ is a partition of X and a base for the topology $\mathcal{T}$ as well, then $\mathcal{B}_A := \{B \cap A : B \in \mathcal{B}\}$ is a partition of A and also a base for the topology $\mathcal{T}_A$.

2. Let $\mathcal{B}_X$ be a partition of X and also a base for the topology $\mathcal{T}_X$. Similarly, let $\mathcal{B}_Y$ be a partition of Y and also a base for the topology $\mathcal{T}_Y$. Then the family $\mathcal{B}_X \times \mathcal{B}_Y$ is a partition of $X \times Y$, and also a base for the topology $\mathcal{T}_X \times \mathcal{T}_Y$. □

Every discrete space $(X, \mathcal{T})$ is a degenerated space with $K(X, \mathcal{T}) = 1$, because for any $A \subseteq X$, it holds that $\text{Int}(A) = A = \text{Cl}(A)$. Moreover, it is not difficult to check that every discrete space is also a P-space, Q-space, ED-space and OI-space.

Relations between considered types of degenerated spaces are illustrated in Figure 6. Their corresponding K-numbers are determined by the following proposition, which can be easily proven.

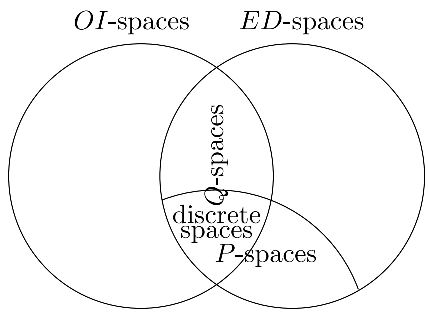

Figure 6: Relations between different types of degenerated spaces

Proposition 3.19. *For every topological space $(X, \mathcal{T})$ the following statements hold*

1. *If $(X, \mathcal{T})$ is a non-discrete P-space, then $K((X, \mathcal{T})) = 3$.*
2. *If $(X, \mathcal{T})$ is a non-discrete Q-space, then $K((X, \mathcal{T})) = 4$.*

3. *If* $(X, \mathcal{T})$ *is an* OI*-space, but not* ED*-space, or is an* ED*-space, but not* OI*-space, then* $K((X, \mathcal{T})) = 5$.

We conclude this section by proving that no other type of degenerated spaces is possible.

Theorem 3.20. *[2] If* $(X, \mathcal{T})$ *is a degenerated space, then* $(X, \mathcal{T})$ *is an* OI*-space or an* ED*-space or a* Q*-space or a* P*-space.*

Proof. Since $K((X, \mathcal{T})) < 7$, some of the operators in the set $\mathcal{O}_X$ coincide. All of 21 possibilities are given in Table 1. For the most of these cases it is easy to prove the claim, but for some of them the additional work is required. E.g., if $(X, \mathcal{T})$ is a space in which Int=Int(Cl(Int)) holds, then, by symmetry, Cl=Cl(Int(Cl)) also holds. Furthermore, in this space, for any set $A \subseteq X$, the set $B := \mathrm{Cl}(A) \setminus \mathrm{Int}(\mathrm{Cl}(A))$ is nowhere dense, which implies $B \subseteq \mathrm{Cl}(B) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))) = \mathrm{Cl}(\emptyset) = \emptyset$. Therefore, $\mathrm{Cl}(A) = \mathrm{Int}(\mathrm{Cl}(A))$ $=\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, and $\mathrm{Int}(A)$=$\mathrm{Cl}(\mathrm{Int}(A))$=$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ similarly. Since these equalities hold for every $A \subseteq X$, by Theorem 3.15, $(X, \mathcal{T})$ is a P-space. □

Table 1: The list of possibilities for a degenerated space $(X, \mathcal{T})$

For degenerated space $(X, \mathcal{T})$ holds	Type of the space
Int=Id or Cl=Id	discrete space
Int=Int(Cl(Int)) or Cl=Cl(Int(Cl))	P-space
Int=Int(Cl) or Cl=Cl(Int)	discrete space
Int=Cl(Int) or Cl=Int(Cl)	P-space
Int=Cl(Int(Cl)) or Cl=Int(Cl(Int))	discrete space
Int=Cl	discrete space
Int(Cl(Int))=Cl(Int) or Cl(Int(Cl))=Int(Cl)	ED-space
Int(Cl(Int))=Int(Cl) or Cl(Int(Cl))=Cl(Int)	OI-space
Int(Cl(Int))=Cl(Int(Cl))	Q-space
Int(Cl(Int))=Id or Cl(Int(Cl))=Id	discrete space
Int(Cl)=Cl(Int)	Q-space
Int(Cl)=Id or Cl(Int)=Id	discrete space

Corollary 3.21. $(X, \mathcal{T})$ *is a degenerated space if and only if* $(X, \mathcal{T})$ *is an* OI*-space or an* ED*-space.*

4. Classification of spaces by their k-number

We start this section with observation that a topological space $(X, \mathcal{T})$ is discrete if and only if $k((X, \mathcal{T})) = 1$. Therefore, every discrete space is a full space. The same characterization holds for P-spaces.

Theorem 4.1. *Every* P*-space is a full space.*

Proof. Let $(X, \mathcal{T})$ be a non-discrete P-space, and let $\mathcal{B}$ be a partition of X, and a base for the topology $\mathcal{T}$ as well. The space $(X, \mathcal{T})$ is non-discrete, so it must contain a non-isolated point $x \in X$. Then $\mathrm{Int}(\{x\}) = \mathrm{Cl}(\mathrm{Int}(\{x\})) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\{x\}))) = \emptyset$ and $\mathrm{Cl}(\{x\}) = \mathrm{Int}(\mathrm{Cl}(\{x\})) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\{x\}))) = B$, where $B \in \mathcal{B}$ is the set that contains x. Clearly, $\emptyset \neq B \neq \{x\}$, whence $k(\{x\}) = 3$. Thus, $k((X, \mathcal{T})) = K((X, \mathcal{T})) = 3$, so $(X, \mathcal{T})$ is a full space. □

Definition 4.2. A topological space is door space if every subset of this space is either open or closed.

Theorem 4.3. *[1] Every door space is an OI-space.*

Theorem 4.4. *[6] If $(X, \mathcal{T})$ is a door space, then exactly one of the following conditions holds*

1. *$(X, \mathcal{T})$ is a discrete space.*
2. *$(X, \mathcal{T})$ has exactly one non-isolated point.*
3. *There is a set $B \subseteq X$, whose complement $X \setminus B$ has cardinality at least two, and there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{T} = \mathcal{U} \cup \{C \subseteq X : C \subseteq B\}$.*

Theorem 4.5. *[2] $k((X, \mathcal{T})) = 2$ if and only if $(X, \mathcal{T})$ is a non-discrete door space.*

Theorem 4.6. *[2] If $(X, \mathcal{T})$ is a Q-space, then $k((X, \mathcal{T})) = 3$ if and only if $(X, \mathcal{T})$ is not a door space and every neutral set has clopen interior or closure.*

Theorem 4.7. *[2] Let $(X, \mathcal{T})$ be a space with two disjoint open sets U, V each containing nowhere dense singletons. Then $k((X, \mathcal{T})) \geqslant 4$.*

5. Finite degenerated spaces

For finite OI-spaces we have the following simple characterization.

Theorem 5.1. *[2] Let X be a finite set. $(X, \mathcal{T})$ is an OI-space if and only if every nonempty open set in this space contains an isolated point.*

Proof. ($\Rightarrow$) If $U = \{x\}$, then x is an isolated point of U. Suppose $U = \{x_1, \ldots, x_n\} \subseteq X$, with $n \geqslant 2$, is an open set in the OI-space $(X, \mathcal{T})$, and that every point of U is non-isolated. Since $(X, \mathcal{T})$ is an OI-space, this implies that every singleton in U is a nowhere dense set. From $\mathrm{Cl}(U) = \mathrm{Cl}(\{x_1, \ldots, x_{n-1}\}) \cup \mathrm{Cl}(\{x_n\})$, it follows that $\mathrm{Int}(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(\{x_1, \ldots, x_{n-1}\}) \cup \mathrm{Int}(\mathrm{Cl}(\{x_n\})) = \mathrm{Cl}(\{x_1, \ldots, x_{n-1}\})$. Continuing with this process, we would eventually get $\mathrm{Int}(\mathrm{Cl}(U)) \subseteq \mathrm{Int}(\mathrm{Cl}(\{x_1\})) = \emptyset$. But, this would lead to $U = \mathrm{Int}(U) \subseteq \mathrm{Int}(\mathrm{Cl}(U)) = \emptyset$, which is a contradiction.

($\Leftarrow$) Let the space $(X, \mathcal{T})$ satisfy the given condition, and let $D \subseteq X$ be a dense set. For every nonempty open set $U \subseteq X$ there exists an isolated point $x \in U$ and, since $\{x\}$ is an open set, it follows that $D \cap \{x\} \neq \emptyset$, which means $x \in D$. Therefore, $x \in \mathrm{Int}(D)$, whence $\mathrm{Int}(D) \cap U \neq \emptyset$, so $\mathrm{Int}(D)$ is a dense set. □

Theorem 5.2. *[4] Every finite nonempty open set in a T_0-space contains an isolated point.*

Corollary 5.3. *Every finite T_0-space is an OI-space.*

It is well known fact that every finite T_1-space is discrete. From the previous corollary we conclude that every finite T_0-space is also a degenerated space. In fact, the majority of finite topological spaces are degenerated, which is illustrated by the following example.

Example 5.4. On the set $X = \{a\}$ there is only one topology $\mathcal{T}_1 = \{\emptyset, \{a\}\}$, and the space $(X, \mathcal{T}_1)$ is discrete. On the set $X = \{a, b\}$ there are 4 topologies and 3 non-homeomorphic topologies. All of these spaces are degenerated (Table 2).

Table 2: Non homeomorphic topologies on the set $X = \{a, b\}$

Topology	Type of the space	$k(X)$	$K(X)$
$\mathcal{T}_1 = \{\emptyset, X\}$	P-space	3	3
$\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$	Q-space	2	4
$\mathcal{T}_3 = P(X)$	discrete space	1	1

On the set $X = \{a, b, c\}$ there are 29 topologies and 9 non-homeomorphic topologies. All of these spaces are degenerated (Table 3).

Table 3: Non homeomorphic topologies on the set $X = \{a, b, c\}$

Topology	Type of the space	$k(X)$	$K(X)$
$\mathcal{T}_1 = \{\emptyset, X\}$	P-space	3	3
$\mathcal{T}_2 = \{\emptyset, \{c\}, X\}$	Q-space	3	4
$\mathcal{T}_3 = \{\emptyset, \{a, b\}, X\}$	ED-space	3	5
$\mathcal{T}_4 = \{\emptyset, \{c\}, \{a, b\}, X\}$	P-space	3	3
$\mathcal{T}_5 = \{\emptyset, \{c\}, \{b, c\}, X\}$	Q-space	3	4
$\mathcal{T}_6 = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$	Q-space	2	4
$\mathcal{T}_7 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$	OI-space	2	5
$\mathcal{T}_8 = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$	Q-space	2	4
$\mathcal{T}_9 = P(X)$	discrete space	1	1

On the set $X = \{a, b, c, d\}$ there are 355 topologies and 33 non-homeomorphic topologies. The only non-degenerated space is $(X, \mathcal{T}_{14})$. (Table 4).

Let us notice that all topological spaces from the previous example are either degenerated or non-full. In [4] it is shown that every finite, non-degenerated, full space must have at least 7 elements. In this paper an example of the full K-space $(X, \mathcal{T})$ is given, where $X = \{a, b, c, d, e, f, g\}$ and $\mathcal{T}$ is the topology

Table 4: Non homeomorphic topologies on the set $X = \{a, b, c, d\}$

Topology	Type of the space	$k(X)$	$K(X)$
$\mathcal{T}_1 = \{\emptyset, X\}$	P-space	3	3
$\mathcal{T}_2 = \{\emptyset, \{a,b,c\}, X\}$	ED-space	3	5
$\mathcal{T}_3 = \{\emptyset, \{a\}, X\}$	Q-space	3	4
$\mathcal{T}_4 = \{\emptyset, \{a\}, \{a,b,c\}, X\}$	Q-space	3	4
$\mathcal{T}_5 = \{\emptyset, \{a,b\}, X\}$	ED-space	3	5
$\mathcal{T}_6 = \{\emptyset, \{a,b\}, \{a,b,c\}, X\}$	ED-space	3	5
$\mathcal{T}_7 = \{\emptyset, \{a\}, \{a,b\}, X\}$	Q-space	3	4
$\mathcal{T}_8 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$	OI-space	3	5
$\mathcal{T}_9 = \{\emptyset, \{a,b,c\}, \{d\}, X\}$	P-space	3	3
$\mathcal{T}_{10} = \{\emptyset, \{a\}, \{a,b,c\}, \{a,d\}, X\}$	Q-space	3	4
$\mathcal{T}_{11} = \{\emptyset, \{a\}, \{a,b,c\}, \{d\}, \{a,d\}, X\}$	Q-space	3	4
$\mathcal{T}_{12} = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, \{a,d\}, X\}$	ED-space	4	5
$\mathcal{T}_{13} = \{\emptyset, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$	ED-space	3	5
$\mathcal{T}_{14} = \{\emptyset, \{a,b\}, \{c\}, \{a,b,c\}, X\}$	K-space	4	7
$\mathcal{T}_{15} = \{\emptyset, \{a,b\}, \{c\}, \{a,b,c\}, \{a,b,d\}, X\}$	ED-space	3	5
$\mathcal{T}_{16} = \{\emptyset, \{a,b\}, \{c\}, \{a,b,c\}, \{d\}, \{a,b,d\}, \{c,d\}, X\}$	P-space	3	3
$\mathcal{T}_{17} = \{\emptyset, \{b,c\}, \{a,d\}, \{a,b,c,d\}\}$	P-space	3	3
$\mathcal{T}_{18} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$	Q-space	3	4
$\mathcal{T}_{19} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}$	Q-space	3	4
$\mathcal{T}_{20} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}$	OI-space	4	5
$\mathcal{T}_{21} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X\}$	Q-space	3	4
$\mathcal{T}_{22} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X\}$	OI-space	3	5
$\mathcal{T}_{23} = \{\emptyset, \{a\}, \{a,b\}, \{c\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X\}$	Q-space	3	4
$\mathcal{T}_{24} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X\}$	Q-space	3	4
$\mathcal{T}_{25} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$	OI-space	3	5
$\mathcal{T}_{26} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X\}$	OI-space	3	5
$\mathcal{T}_{27} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,d\}, \{a,b,d\}, X\}$	Q-space	4	4
$\mathcal{T}_{28} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X\}$	Q-space	2	4
$\mathcal{T}_{29} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X\}$	Q-space	2	4
$\mathcal{T}_{30} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, X\}$	OI-space	2	5
$\mathcal{T}_{31} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{c\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{b,c\}, \{a,b,d\}, \{a,c,d\}, X\}$	Q-space	2	4
$\mathcal{T}_{32} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}$	OI-space	2	5
$\mathcal{T}_{33} = P(X)$	discrete space	1	1

generated by the base $\mathcal{B} = \{\{a\}, \{g\}, \{a, b\}, \{f, g\}, \{c, e\}, \{a, b, c, d, e, f, g\}\}$. It is easy to check that for the set $A = \{a, c, f\}$ it holds that

$$\begin{aligned}
&\mathrm{Int}(A) = \{a\},\\
&\mathrm{Cl}(A) = \{a, b, c, d, e, f\},\\
&\mathrm{Cl}(\mathrm{Int}(A)) = \{a, b, d\},\\
&\mathrm{Int}(\mathrm{Cl}(A)) = \{a, b, c, e\},\\
&\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \{a, b\},\\
&\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \{a, b, c, d, e\}.
\end{aligned}$$

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