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PELL NUMBERS IDENTITIES FROM TOEPLITZ-HESSENBERG DETERMINANTS AND PERMANENTS

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Abstract. In this paper, we investigate some families of Toeplitz-Hessenberg determinants and permanents the entries of which are Pell numbers with consecutive, even, and odd subscripts. As a consequence, we obtain for these numbers new identities involving multinomial coefficients.

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1. Introduction and preliminaries

The well-known *Fibonacci sequence* $(F_n)_{n\geq 0}$ is defined by the recurrence: for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2},$$

where $F_0 = 0$, $F_1 = 1$. Furthermore, similar to the Fibonacci sequence, the *Pell sequence* $(P_n)_{n\geq 0}$ is defined by the recurrence: for $n \geq 2$,

$$P_n = 2P_{n-1} + P_{n-2}, \tag{1}$$

where $P_0 = 0$, $P_1 = 1$.

The Pell sequence has a rich history and many remarkable properties [6, 7]. As well as being used to approximate the square root of 2, the Pell numbers can be used to find square triangular numbers, to construct integer approximations to the right isosceles triangle, and to solve certain combinatorial enumeration problems [1, 2, 9, 12]. Some examples of recent papers involving Pell numbers and their generalizations include [3, 4, 8, 10, 13, 14, 15].

The purpose of the present paper is to investigate the determinants and permanents of some families of Toeplitz-Hessenberg matrices whose entries are Pell numbers with successive, odd or even subscripts. As a result, we obtain for these numbers new identities involving multinomial coefficients. Also, we establish a connection between Pell numbers and Fibonacci numbers using Toeplitz-Hessenberg determinants.

Some results of this paper were announced without proofs in [5].

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2. Toeplitz-Hessenberg determinants and permanents

A lower *Toeplitz-Hessenberg matrix* is a square matrix of the order n in the form

$$M_n(a_0; a_1, \dots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}, \tag{2}$$

where $a_i \neq 0$ for at least one $i > 0$ and $a_0 \neq 0$.

Expanding Toeplitz-Hessenberg determinant and permanent, which we will denote by $\det(M_n)$ and $\operatorname{perm}(M_n)$, repeatedly along the last row, we obtain the following recurrences:

$$\begin{aligned} \det(M_n) &= \sum_{i=1}^{n} (-a_0)^{i-1} a_i \det(M_{n-i}), \\ \operatorname{perm}(M_n) &= \sum_{i=1}^{n} a_0^{i-1} a_i \operatorname{perm}(M_{n-i}), \end{aligned} \tag{3}$$

where, by definition, $\det(M_0) = 1$ and $\operatorname{perm}(M_0) = 1$.

It can also easily be verified that

$$\det(M_n(a_0; a_1, \dots, a_n)) = \operatorname{perm}(M_n(-a_0; a_1, \dots, a_n)).$$

We investigate a particular case of Toeplitz-Hessenberg matrix, in which all subdiagonal elements are 1.

To simplify notation, we denote $\det(a_1, \dots, a_n) = \det\left(M_n(1; a_1, a_2, \dots, a_n)\right)$ and $\operatorname{perm}(a_1, \dots, a_n) = \operatorname{perm}\left(M_n(1; a_1, a_2, \dots, a_n)\right)$.

In the next two sections, we evaluate Toeplitz-Hessenberg determinants and permanents with special Pell numbers entries.

3. Fibonacci numbers via Toeplitz-Hessenberg determinants with Pell numbers entries

The next theorem gives a connection between Fibonacci numbers and Pell numbers using Toeplitz-Hessenberg determinants.

Theorem 3.1. *For all $n \geq 1$, the following formulas hold:*

$$F_n = (-1)^{n-1} \det(P_1, P_2, \dots, P_n), \tag{4}$$

$$F_{2n+3} = \det(P_3, P_4, \dots, P_{n+2}). \tag{5}$$

Proof. We will prove formula (4) using the principle of mathematical induction on n. The proof of (5) follow similarly, so we omit it for brevity.

Let $D_n = \det(P_1, P_2, \dots, P_n)$. The formula (4) clearly holds, when $n = 1$ and $n = 2$. Suppose it is true for all $k \le n - 1$, where $n \ge 3$. Using recurrence (1) and (3), we have

$$\begin{aligned}
D_n &= \sum_{k=1}^{n} (-1)^{k-1} P_k D_{n-k} \\
&= P_1 D_{n-1} + \sum_{k=2}^{n} (-1)^{k-1} \left(2P_{k-1} + P_{k-2}\right) D_{n-k} \\
&= D_{n-1} + 2\sum_{k=1}^{n-1} (-1)^{k} P_k D_{n-k-1} + \sum_{k=0}^{n-2} (-1)^{k+1} P_k D_{n-k-2} \\
&= D_{n-1} - 2D_{n-1} + D_{n-2} = -D_{n-1} + D_{n-2}.
\end{aligned}$$

Using the induction hypothesis and the Fibonacci recurrence, we obtain

$$\begin{aligned}
D_n &= -(-1)^{n-2} F_{n-1} + (-1)^{n-3} F_{n-2} \\
&= (-1)^{n-1} F_n.
\end{aligned}$$

Consequently, formula (4) is true in the n case and thus, by induction, it holds for all positive integers.

□

4. Some Toeplitz-Hessenberg determinants and permanents with Pell numbers entries

Next, we investigate several Toeplitz-Hessenberg determinants whose entries are Pell numbers with consecutive, even and odd subscripts.

Theorem 4.1. *Let $n \ge 1$, except when noted otherwise. Then*

$$\begin{aligned}
\det(P_0, P_1, \dots, P_{n-1}) &= (-1)^{n-1} \left\lfloor 2^{n-2} \right\rfloor, \\
\det(P_0, P_2, \dots, P_{2n-2}) &= \frac{(-3-\sqrt{6})^{n-1} - (-3+\sqrt{6})^{n-1}}{\sqrt{6}}, \\
\det(P_1, P_3, \dots, P_{2n-1}) &= (-1)^{n-1} 4 \cdot 5^{n-2}, \qquad n \ge 2, \\
\det(P_2, P_3, \dots, P_{n+1}) &= (-1)^{n} \left\lfloor \frac{2}{n} \right\rfloor, \\
\det(P_2, P_4, \dots, P_{2n}) &= \frac{(-2+\sqrt{3})^{n} - (-2-\sqrt{3})^{n}}{\sqrt{3}}, \\
\det(P_3, P_5, \dots, P_{2n+1}) &= (-1)^{n-1} 4, \qquad n \ge 2, \\
\det(P_4, P_6, \dots, P_{2n+2}) &= \frac{(3+\sqrt{10})^{n} - (3-\sqrt{10})^{n}}{\sqrt{10}}, \qquad n \ge 2,
\end{aligned} \tag{6}$$

where $\lfloor \alpha \rfloor$ is the floor of α.

Proof. We will prove formula (6) using induction on n; the others can be proved in the same way. Let $D_n = \det(P_1, P_3, \ldots, P_{2n-1})$.

When $n = 1$ and $n = 2$, the formula holds. Assuming (6) to hold for $n-1$, we proved it for $n \geq 3$. Using (1), (3), and the well-known formula [7, p. 193]

$$P_{2k-2} = 2\sum_{i=1}^{k-1} P_{2i-1},$$

we then obtain

$$\begin{aligned}
D_n &= \sum_{k=1}^{n}(-1)^{k-1}P_{2k-1}D_{n-k} \\
&= P_1D_{n-1} + \sum_{k=2}^{n}(-1)^{k-1}\left(2P_{2k-2} + P_{2k-3}\right)D_{n-k} \\
&= D_{n-1} + 2\sum_{k=2}^{n}(-1)^{k-1}P_{2k-2}D_{n-k} + \sum_{k=1}^{n-1}(-1)^{k}P_{2k-1}D_{n-k-1} \\
&= D_{n-1} + 2\sum_{k=2}^{n}(-1)^{k-1}P_{2k-2}D_{n-k} - D_{n-1} \\
&= 2\sum_{k=2}^{n}(-1)^{k-1}P_{2k-2}D_{n-k} \\
&= 4\sum_{k=2}^{n}\sum_{i=1}^{k-1}(-1)^{k-1}P_{2i-1}D_{n-k} \\
&= 4\sum_{i=1}^{n-1}(-1)^{i}\sum_{k=1}^{n-i}(-1)^{k-1}P_{2k-1}D_{n-k-i} \\
&= 4\sum_{i=1}^{n-2}(-1)^{i}D_{n-i} + 4(-1)^{n-1}D_1.
\end{aligned}$$

Using the induction hypothesis, we have

$$\begin{aligned}
D_n &= 4\sum_{i=1}^{n-2}(-1)^{i}\cdot\frac{4(-5)^{n-i-1}}{5} + 4(-1)^{n-1} \\
&= 4(-1)^{n-1}(5^{n-2}-1) + 4(-1)^{n-1} \\
&= \frac{4(-5)^{n-1}}{5}.
\end{aligned}$$

Since the formula holds for n, it follows by induction that it is true for all positive integers. $\square$

Similar formulas hold true for Toeplitz-Hessenberg permanents with Pell numbers entries.

Theorem 4.2. *For all $n \geq 1$, the following formulas hold:*

$$\operatorname{perm}(P_0, P_1, \ldots, P_{n-1}) = \frac{(1+\sqrt{3})^{n-1} - (1-\sqrt{3})^{n-1}}{2\sqrt{3}},$$

$$\operatorname{perm}(P_0, P_2, \ldots, P_{2n-2}) = \frac{(3+\sqrt{10})^{n-1} - (3-\sqrt{10})^{n-1}}{\sqrt{10}},$$

$$\operatorname{perm}(P_1, P_2, \ldots, P_n) = \frac{1}{\sqrt{13}}\left(\left(\frac{3+\sqrt{13}}{2}\right)^n - \left(\frac{3-\sqrt{13}}{2}\right)^n\right),$$

$$\operatorname{perm}(P_1, P_3, \ldots, P_{2n-1}) = \frac{\sqrt{41}}{82}\left((5+\sqrt{41})A^{n-1} - (5-\sqrt{41})\left(\frac{2}{A}\right)^{n-1}\right),$$

$$\operatorname{perm}(P_2, P_3, \ldots, P_{n+1}) = \frac{(3+\sqrt{6})(2+\sqrt{6})^n + (3-\sqrt{6})(2-\sqrt{6})^n}{12},$$

$$\operatorname{perm}(P_2, P_4, \ldots, P_{2n}) = \frac{(4+\sqrt{15})^n - (4-\sqrt{15})^n}{\sqrt{15}}, \tag{7}$$

where $A = (7+\sqrt{41})/2$.

Proof. We will prove formula (7) using induction on n; the others can be proved in the same way. Let

$$D_n = \operatorname{perm}(P_2, P_4, \ldots, P_{2n}).$$

When $n = 1$ and $n = 2$, the formula holds. Assuming (7) to hold for $n-1$, we proved it for $n \geq 2$. Using (3) and well-known formula [7, p. 193]

$$P_{2k-1} = 2\sum_{i=1}^{k-1} P_{2i} + 1,$$

we then obtain

$$\begin{aligned}
D_n &= \sum_{k=1}^{n} P_{2k} D_{n-k} \\
&= \sum_{k=1}^{n} \left(2P_{2k-1} + P_{2k-2}\right) D_{n-k} \\
&= 2\sum_{k=1}^{n}\left(2\sum_{i=1}^{k-1} P_{2i} + 1\right) D_{n-k} + \sum_{k=2}^{n} P_{2k-2} D_{n-k} \\
&= 4\sum_{k=1}^{n}\sum_{i=1}^{k-1} P_{2i} D_{n-k} + 2\sum_{k=1}^{n} D_{n-k} + \sum_{k=1}^{n-1} P_{2k} D_{n-k-1} \\
&= 4\sum_{i=1}^{n-1}\sum_{k=1}^{n-i} P_{2i} D_{n-i-k} + 2\sum_{k=1}^{n} D_{n-k} + D_{n-1} \\
&= 4\sum_{i=1}^{n-1} D_{n-i} + 2\sum_{k=1}^{n} D_{n-k} + D_{n-1}
\end{aligned}$$

$$= 4\left(\sum_{i=1}^{n} D_{n-i} - D_0\right) + 2\sum_{k=1}^{n} D_{n-k} + D_{n-1}$$
$$= 6\left(\sum_{i=1}^{n-2} D_{n-i} + D_0 + D_1\right) - 4 + D_{n-1}$$
$$= 6\sum_{i=1}^{n-2} D_{n-i} + D_{n-1} + 14.$$

Thus,

$$D_n = 6\sum_{i=1}^{n-2}\frac{(4+\sqrt{15})^{n-i} - (4-\sqrt{15})^{n-i}}{\sqrt{15}} + \frac{(4+\sqrt{15})^{n-1} - (4-\sqrt{15})^{n-1}}{\sqrt{15}} + 14$$
$$= \frac{(4+\sqrt{15})^{n} - (4-\sqrt{15})^{n}}{\sqrt{15}}.$$

Since the formula holds for n, it follows that it is true for all positive integers. $\square$

5. Multinomial extensions

In this section, we focus on multinomial extension of Theorems 3.1, 4.1, and 4.2.

It is known that the determinant and permanent of M_n can be evaluated using *Trudi's formulas* [11, Ch. 7] as follows:

$$\det(M_n) = \sum_{\substack{t=(t_1,t_2,\ldots,t_n)\\ t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-a_0)^{n-|t|} s_n(t)\, a_1^{t_1} a_2^{t_2}\cdots a_n^{t_n}, \tag{8}$$

$$\operatorname{perm}(M_n) = \sum_{\substack{t=(t_1,t_2,\ldots,t_n)\\ t_1,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} a_0^{n-|t|} s_n(t)\, a_1^{t_1} a_2^{t_2}\cdots a_n^{t_n}, \tag{9}$$

where $|t| = t_1 + \cdots + t_n$ and $s_n(t) = \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ is the multinomial coefficient.

For example, from (8) and (9) we obtain

$$\det(M_4) = \binom{4}{4,0,0,0}a_1^4 - \binom{3}{2,1,0,0}a_1^2a_2 + \binom{2}{1,0,1,0}a_1a_3$$
$$+ \binom{2}{0,2,0,0}a_2^2 - \binom{1}{0,0,0,1}a_4$$
$$= a_1^4 - 3a_1^2a_2 + 2a_1a_3 + a_2^2 - a_4;$$
$$\operatorname{perm}(M_3) = \binom{3}{3,0,0}a_1^3 + \binom{2}{1,1,0}a_1a_2 + \binom{1}{0,0,1}a_3$$
$$= a_1^3 + 2a_1a_2 + a_3.$$

Trudi's formulas (8) and (9), coupled with Theorems 3.1, 4.1, 4.2 yield the following Pell identities with multinomial coefficients.

Corollary 5.1. *Let $n \geq 1$, except when noted otherwise, and let $a = \frac{7+\sqrt{41}}{2}$, $b = 5 + \sqrt{41}$, $c = 3 + \sqrt{6}$, $\tau_n = t_1 + 2t_2 + \cdots + nt_n$, $t_i \geq 0$. Then*

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_0^{t_1} P_1^{t_2} \cdots P_{n-1}^{t_n} = -2^{n-2}, \quad n \geq 2,$$

$$\sum_{\tau_n=n} s_n(t) P_0^{t_1} P_1^{t_2} \cdots P_{n-1}^{t_n} = \frac{(1+\sqrt{3})^{n-1} - (1-\sqrt{3})^{n-1}}{2\sqrt{3}},$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_0^{t_1} P_2^{t_2} \cdots P_{2n-2}^{t_n} = \frac{(3-\sqrt{6})^{n-1} - (3+\sqrt{6})^{n-1}}{\sqrt{6}},$$

$$\sum_{\tau_n=n} s_n(t) P_0^{t_1} P_2^{t_2} \cdots P_{2n-2}^{t_n} = \frac{(3+\sqrt{10})^{n-1} - (3-\sqrt{10})^{n-1}}{\sqrt{10}}, \tag{10}$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n} = -F_n, \tag{11}$$

$$\sum_{\tau_n=n} s_n(t) P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n} = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{2^n\sqrt{13}},$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_1^{t_1} P_3^{t_2} \cdots P_{2n-1}^{t_n} = -4 \cdot 5^{n-2}, \quad n \geq 2,$$

$$\sum_{\tau_n=n} s_n(t) P_1^{t_1} P_3^{t_2} \cdots P_{2n-1}^{t_n} = \frac{\sqrt{41}}{82}\left(ba^{n-1} + \frac{2^{n+3}}{ba^{n-1}}\right),$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_2^{t_1} P_3^{t_2} \cdots P_{n+1}^{t_n} = 0, \quad n \geq 3, \tag{12}$$

$$\sum_{\tau_n=n} s_n(t) P_2^{t_1} P_3^{t_2} \cdots P_{n+1}^{t_n} = \frac{c^2(2+\sqrt{6})^n + 3(2-\sqrt{6})^n}{12c},$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_2^{t_1} P_4^{t_2} \cdots P_{2n}^{t_n} = \frac{(2-\sqrt{3})^n - (2+\sqrt{3})^n}{\sqrt{3}},$$

$$\sum_{\tau_n=n} s_n(t) P_2^{t_1} P_4^{t_2} \cdots P_{2n}^{t_n} = \frac{(4+\sqrt{15})^n - (4-\sqrt{15})^n}{\sqrt{15}},$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_3^{t_1} P_4^{t_2} \cdots P_{n+2}^{t_n} = (-1)^n F_{2n+3},$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_3^{t_1} P_5^{t_2} \cdots P_{2n+1}^{t_n} = -4, \quad n \geq 2,$$

$$\sum_{\tau_n=n} (-1)^{|t|} s_n(t) P_4^{t_1} P_6^{t_1} \cdots P_{2n+2}^{t_n} = \frac{(-3-\sqrt{10})^n - (-3+\sqrt{10})^n}{\sqrt{10}}, \quad n \geq 2.$$

Example 5.2. It follows from (11), (12), and (10) that

$$P_1^3 - 2P_1P_2 + P_3 = F_3,$$
$$P_2^4 - 3P_2^2P_3 + 2P_2P_4 + P_3^2 - P_5 = 0,$$
$$P_0^5 + 4P_0^3P_2 + 3P_0^2P_4 + 3P_0P_2^2 + 2P_0P_6 + 2P_2P_4 + P_8 = 456,$$

respectively.

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