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BOUNDED INDEX AND FOUR DIMENSIONAL SUMMABILITY METHODS

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Abstract. In this paper, we present a necessary and sufficient conditions on four dimensional matrix transformations that preserve entireness, bounded index and absolute convergence of double sequences. We begin this analysis with the following observation: The four dimensional Taylor matrix T transforms the set of double sequences of bounded index into itself. After this observation, we present general characterizations for four dimensional RH-regular matrix transformations for the space of entire, bounded index, and absolutely summable double sequences.

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1. Introduction

Lepson [15] presented the following notion of bounded index of an entire function. An entire function $f(z)$ is of bounded index if there exists an integer N independent of z, such that

$$\max_{\{l:0\leq l\leq N\}}\left\{\frac{|f^{(l)}(z)|}{l!}\right\}\geq\frac{|f^{(n)}(z)|}{n!}\text{ for all }n.$$

The least such integer N is called the index of $f(z)$. The reader may refer to the papers [5, 3, 4, 6, 7, 8, 9, 10, 13, 14, 18, 22] for the notion of bounded index of an entire function. The main goal of this paper is to extend this notion to two-dimensional space. To accomplish this we begin with the presentation of the following notion. If $f(z_1, z_2)$ is a holomorphic function in the bicylinder

$$\{|z_1-a|<r_1, |z_2-b|<r_2\}$$

then at all point of the bicylinder

$$f(z_1,z_2)=\sum_{k,l=0,0}^{\infty,\infty}c_{k,l}(z_1-a)^k(z_2-b)^l$$

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where

$$c_{k,l} = \frac{1}{k!l!}\left[\frac{\partial^{k+l} f(z_1,z_2)}{\partial z_1^k \partial z_2^l}\right]_{z_1=a;z_2=b} = \frac{1}{k!l!} f^{(k,l)}(a,b).$$

Using this notion we present in [16, 17, 21] the following notion of a bounded index for a holomorphic bivariate function. A holomorphic bivariate function $f(z_1, z_2)$ is of bounded index if there exist integers M and N independent of z_1 and z_2, respectively, such that

$$\max_{\{(k,l):0,0\le k,l\le M,N\}}\left\{\frac{|f^{(k,l)}(z_1,z_2)|}{k!l!}\right\} \geq \frac{|f^{(m,n)}(z_1,z_2)|}{m!n!} \text{ for all } m \text{ and } n.$$

We shall say the bivariate holomorphic function f is of bounded index (M, N), if M and N are the smallest integers such that the above inequality holds. But it leads to the following question: What is the index of the function of f if the corresponding inequality holds for (N, M) and (M, N)? Thus, the index of the function is not uniquely defined. It is better to define an index of a bivariate function as height of the vector (N, M) i.e. $N + M$. The least such integer $N + M$ is called the index of the function f and is denoted by $N(f)$.

2. Definitions, Notations, and Preliminary Results

Definition 2.1. [23] A double sequence $x = (x_{kl})$ has a **Pringsheim limit** L (denoted by P-lim $x = L$) provided that, given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > N$. Such an x is described more briefly as "P-convergent".

Definition 2.2. [19] A double sequence $\{y\}$ is a **double subsequence** of $\{x\}$ provided that there exist increasing index sequences (n_j) and (k_j) such that, if $x_j = x_{n_j k_j}$, then y is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

Robison [24] presented the following notion of a conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of this notion.

Definition 2.3. The four-dimensional matrix A is said to be **RH-regular** if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The four-dimensional matrix A is said to be **RH-conservative** if it maps every bounded P-convergent sequence into a P-convergent sequence.

Boundedness assumption was made because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [11] and [24].

Theorem 2.4. *The four-dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{mnkl} = 0$ *for each k and l;*
RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} = 1$;
RH_3: $P\text{-}\lim_{m,n} \sum_{k=0}^{\infty} |a_{mnkl}| = 0$ *for each l;*
RH_4: $P\text{-}\lim_{m,n} \sum_{l=0}^{\infty} |a_{mnkl}| = 0$ *for each k;*
RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{mnkl}|$ *is P-convergent;*
RH_6: *there exist finite positive integers Δ and Γ such that* $\sum_{k,l>\Gamma} |a_{mnkl}| < \Delta$.

A double sequence $x = (x_{kl})$ of complex numbers is an entire sequence if

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} |x_{kl}| p^k q^l$$

converges for all positive integers p and q. If we denote the set of entire double sequences by $\mathcal{E}_2$, then we see that $\mathcal{E}_2$ can be identified with the class of entire functions. An entire double sequence $x = (x_{kl})$ is of bounded index if

$$f(z_1, z_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x_{kl} z_1^k z_2^l$$

is an entire function of bounded index. We will denote the set of double sequences of bounded index by $\mathcal{B}_2$. Furthermore, let $\mathcal{L}_u$ be the set of all absolutely summable double sequences, as usual, that is

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} |x_{kl}| < \infty.$$

A four dimensional matrix $A = (a_{mnkl})$ of complex entries which transforms $\mathcal{E}_2$ to $\mathcal{E}_2$ ($\mathcal{B}_2$ to $\mathcal{B}_2$, $\mathcal{L}_u$ to $\mathcal{L}_u$) will be called an $\mathcal{E}_2 - \mathcal{E}_2$ method ($\mathcal{B}_2 - \mathcal{B}_2$ method, $\mathcal{L}_u$-$\mathcal{L}_u$ method). In [21], Patterson has shown

Theorem 2.5. *A four dimensional matrix $A = (a_{mnkl})$ is an $\mathcal{E}_2$-$\mathcal{E}_2$ method if and only if for integers $p > 0$ and $q > 0$ there exist integers $\alpha = \alpha(p) > 0$ and $\beta = \beta(q) > 0$ and constants $M = M(p)$ and $N = N(q) > 0$ such that*

$$|a_{mnkl}| p^m q^n \le \alpha^k \beta^l M N$$

for all $m, n, k, l = 1, 2, 3, \ldots$.

The four dimensional Taylor matrix $T(r_1, r_2) = (a_{mnkl})$ is defined by

$$a_{mnkl} = \begin{cases} \binom{k}{m}\binom{l}{n}(1-r_1)^{m+1} r_1^{k-m}(1-r_2)^{n+1} r_2^{l-n}, & \text{if } k \ge m, l \ge n \\ 0, & \text{otherwise} \end{cases}$$

where r_1 and r_2 are complex numbers.
The reader may refer to the textbook [1] containing the chapter titled 'Double Sequences', [11, 12, 21] and the recent papers [2, 25, 26] for the domain of four dimensional triangle matrices in the double sequence spaces and the characterization of the classes of matrix transformations.

3. Main results

Theorem 3.1. *The four dimensional Taylor matrix $T(r_1, r_2) = (a_{mnkl})$ is a $\mathcal{B}_2$-$\mathcal{B}_2$ method for any complex numbers r_1 and r_2.*

Proof. Let $x = (x_{kl}) \in \mathcal{B}_2$, that is,

$$f(z_1, z_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x_{kl} z_1^k z_2^l$$

is a function of bounded index. Thus, for

$$y = (y_{mn}) = Ax \text{ where } y_{mn} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{mnkl} x_{kl},$$

$$\begin{aligned}
&g(z_1, z_2)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} y_{mn} z_1^m z_2^n\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{mnkl} x_{kl} z_1^m z_2^n\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=m}^{\infty}\sum_{l=n}^{\infty} \binom{k}{m}\binom{l}{n}(1-r_1)^{m+1} r_1^{k-m}(1-r_2)^{n+1} r_2^{l-n} x_{kl} z_1^m z_2^n\\
&= (1-r_1)(1-r_2)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{l}\binom{k}{m}(1-r_1)^m r_1^{k-m}(1-r_2)^n r_2^{l-n} x_{kl} z_1^m z_2^n\\
&= (1-r_1)(1-r_2)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x_{k,l}[(1-r_1)z_1 + r_1]^k[(1-r_2)z_2 + r_2]^l\\
&= (1-r_1)(1-r_2) f([1-r_1]z_1 + r_1, [1-r_2]z_2 + r_2).
\end{aligned}$$

From [21], we have that the class of functions of bounded index is closed under translation. Hence $g(z_1, z_2)$ is of bounded index, that is, $Ax = y = (y_{mn}) \in \mathcal{B}_2$. □

We now prove a result on functions of bounded index which we will require later on.

Theorem 3.2. *Let $f(z_1, z_2)$ be a function of bounded index. If*

$$P - \lim_{m,n\to\infty} f^{(k,l)}(a_m, b_n) = 0 \quad \textit{for all positive integers } k, l;$$

then for all $r_1 > 0$ and $r_2 > 0$,

$$P - \lim_{m,n\to\infty} \max_{|z_1 - a_m| = r_1; |z_2 - b_n| = r_2} \{|f^{(k,l)}(z_1, z_2)|\} = 0 \textit{ for all positive integers } k, l.$$

where (a_m) and (b_n) are complex sequences.

Proof. Let f be of index $M+N$. It is sufficient to prove Theorem 3.2 for $r_1 = \frac{1}{2M+2}$ and $r_2 = \frac{1}{2N+2}$. Since f is of index $M+N$ we have for all a_m and b_n that there exist integers $u = u(a_m)$ with $0 \le u \le M$ and $v = v(b_n)$ with $0 \le v \le N$, such that for all $i, j = 1, 2, 3, \ldots$

$$\begin{aligned}
&\max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(u,v)}(z_1,z_2)|}{u!v!} \right\} \\
\ge\ & \max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(i,j)}(z_1,z_2)|}{i!j!} \right\} \\
\ge\ & \max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(u,v)}(z_1,z_2)|}{u!v!} \right\} \\
\ge\ & \max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(u+1,v+1)}(z_1,z_2)|}{(u+1)!(v+1)!} \right\} \\
\ge\ & \frac{1}{(u+1)!(v+1)!} \max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(u,v)}(z_1,z_2)| - |f^{(u,v)}(a_m,b_n)|}{r_1 r_2} \right\} \\
\ge\ & 2 \left(\max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(u,v)}(z_1,z_2)|}{u!v!} \right\} - \frac{|f^{(u,v)}(a_m,b_n)|}{u!v!} \right).
\end{aligned}$$

Thus

$$\max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(u,v)}(z_1,z_2)|}{u!v!} \right\} \le 2 \frac{|f^{(u,v)}(a_m,b_n)|}{u!v!}.$$

Therefore for $m, n, i, j = 0, 1, 2, 3, \ldots$,

$$\max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ \frac{|f^{(i,j)}(z_1,z_2)|}{i!j!} \right\} \le 2 \max_{0 \le s \le M; 0 \le t \le N} \frac{|f^{(s,t)}(a_m,b_n)|}{s!t!}.$$

Thus, using the hypothesis,

$$P - \lim_{m,n\to} \max_{|z_1-a_m|=r_1;|z_2-b_n|=r_2} \left\{ |f^{(i,j)}(z_1,z_2)| \right\} = 0.$$

□

4. The Matrix $A(f, w_{ij})$

For an entire function $f(z_1, z_2)$ and double sequence $(w_{ij}) = (\alpha_i \beta_j)$ of complex numbers define the four dimensional matrix transformation $A(f, w_{ij}) = (a_{mnkl})$ by

$$f(z_1,z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{mnkl} (z_1 - \alpha_m)^k (z_2 - \beta_n)^l$$

for $m, n = 0, 1, 2, \ldots$. For this matrix transformation we can express the Silverman-Toeplitz conditions for regularity as follows.

Theorem 4.1. *The four-dimensional matrix $A(f, w_{ij}) = (a_{mnkl})$ is RH-regular if and only if*

(*i*) *P*-$\lim_{m,n} f^{(k,l)}(\alpha_m, \beta_n) = 0$ *for each* k *and* l*;*
(*ii*) *P*-$\lim_{m,n} f(\alpha_m + 1, \beta_n + 1) = 1$*;*
(*iii*) *P*-$\lim_{m,n} \sum_{k=0}^{\infty} |a_{mnkl}| = 0$ *for each* l*;*
(*iv*) *P*-$\lim_{m,n} \sum_{l=0}^{\infty} |a_{mnkl}| = 0$ *for each* k*;*
(*v*) $\sum_{k,l=0,0}^{\infty,\infty} |a_{mnkl}|$ *is P-convergent;*
(*vi*) *there exist finite positive integers* Δ *and* Γ *such that* $\sum_{k,l>\Gamma} |a_{mnkl}| < \Delta$.

Proof. For the matrix $A(f, w_{ij}) = (a_{mnkl})$, we have $a_{mnkl} = \frac{f^{(k,l)}(\alpha_m,\beta_n)}{k!l!}$ for $m, n, k, l = 0, 1, 2, 3, \ldots$ and

$$f(\alpha_m + 1, \beta_n + 1) = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_{mnkl}.$$

Hence, conditions $(i)-(ii)$ are identical to the RH-regularity conditions $RH_1 - RH_2$ and the other conditions are same the RH-regularity conditions $RH_3 - RH_6$. □

Theorem 4.2. *If* $f(z_1, z_2)$ *is an entire function of bounded index then* $A(f, w_{ij}) = (a_{mnkl})$ *is not RH-regular for any double sequence* (w_{ij})*.*

Proof. By Theorem 3.2, P-$\lim_{m,n} f^{(k,l)}(\alpha_m, \beta_n) = 0$ for each k and l implies P-$\lim_{m,n} f(\alpha_m + 1, \beta_n + 1) = 0$ since f is bounded index. Thus, conditions (i) and (ii) of Theorem 4.1 can not be satisfied simultaneously. Therefore, $A(f, w_{ij}) = (a_{mnkl})$ is not RH-regular. □

It is worth noting however, that functions of bounded index can give rise to RH-conservative matrix. For example, let $f(z_1, z_2) = e^{z_1+z_2}$ (f is bounded index with index 0) and choose $w_{mn} = (2\pi im, 2\pi in)$. Thus

$f^{(k,l)}(\alpha_m, \beta_n) = 1$ for all $m, n = 0, 1, 2, 3, \ldots,$

$f(\alpha_m + 1, \beta_n + 1) = e^2$ for all $m, n, k, l = 0, 1, 2, 3, \ldots,$

$$P - \lim_{m,n} \sum_{k=0}^{\infty} a_{mnkl} \text{ exists for each l,}$$

$$P - \lim_{m,n} \sum_{l=0}^{\infty} a_{mnkl} \text{ exists for each k,}$$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{mnkl} = e^2 \text{ for all } m, n = 0, 1, 2, 3, \ldots, \text{ and}$$

there exists a finite positive integer Γ such that $\sum_{k,l>\Gamma} |a_{mnkl}| < e^2$.

We now examine the matrix $A^{\top}(f, w_{ij}) = (b_{mnkl})$ which is defined by

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mnkl}(z_1 - \alpha_k)^m (z_2 - \beta_l)^n$$

for $k, l = 0, 1, 2, \ldots$. The matrix $A^\top(f, w_{ij}) = (b_{mnkl})$ is the transpose of $A(f, w_{ij}) = (a_{mnkl})$, that is, $a_{mnkl} = b_{nmlk}$ for $m, n, k, l = 0, 1, 2, 3, \ldots$.

Theorem 4.3. *If $f(z_1, z_2)$ is an entire function of bounded index then for any sequence (w_{ij}), $A^\top(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{L}_u - \mathcal{L}_u$ method if and only if*

$$\sup_{m,n}\{|f^{(k,l)}(\alpha_m, \beta_n)|\} < \infty \text{ for } k, l = 0, 1, 2, 3, \ldots.$$

Proof. □

In [20], Patterson showed that a necessary and sufficient condition for a matrix $A = (a_{mnkl})$ to be an $\mathcal{L}_u - \mathcal{L}_u$ method is

$$\sup_{k,l}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mnkl}|\right\} < \infty.$$

Let $A^\top(f, w_{ij}) = (b_{mnkl})$ be an $\mathcal{L}_u - \mathcal{L}_u$ method. Thus, there exists a constant Δ such that

$$\sup_{k,l}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|b_{mnkl}|\right\} \leq \Delta \text{ for } k, l = 0, 1, 2, 3, \ldots.$$

Hence

$$|b_{mnkl}| = \frac{|f^{(m,n)}(\alpha_k, \beta_l)|}{m!n!} \leq \Delta \text{ for } m, n, k, l = 0, 1, 2, 3, \ldots.$$

Therefore

$$\sup_{k,l}\left\{|f^{(m,n)}(\alpha_k, \beta_l)|\right\} \leq m!n!\Delta < \infty \text{ for } k, l = 0, 1, 2, 3, \ldots.$$

Now let f be an entire function is of bounded index and (w_{ij}) be a double sequence such that

$$\sup_{m,n}\{|f^{(k,l)}(\alpha_m, \beta_n)|\} < \infty \text{ for } k, l = 0, 1, 2, 3, \ldots.$$

Since $f(z_1, z_2)$ is of bounded index we have that $f(2z_1, 2z_2)$ is of bounded index (see [21]). Let $M + N$ be the index of $f(2z_1, 2z_2)$. Thus

$$\max_{0\leq i\leq M; 0\leq j\leq N}\left\{\frac{2^{i+j}|f^{(i,j)}(z_1, z_2)|}{i!j!}\right\} \leq \frac{2^{m+n}|f^{(m,n}(z_1, z_2)|}{m!n!}$$

for all (z_1, z_2) and all m, n. Hence

$$\max_{0\leq i\leq M; 0\leq j\leq N}\left\{\frac{|f^{(i,j)}(z_1, z_2)|}{i!j!}\right\} \leq \frac{2^{m-M+n-N}|f^{(m,n}(z_1, z_2)|}{m!n!}$$

for all (z_1, z_2) and all m, n. Therefore

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|b_{mnkl}| &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{|f^{(m,n)}(\alpha_k,\beta_l)|}{m!n!}\\ &\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{m-M+n-N}\max_{0\leq i\leq M;0\leq j\leq N}\left\{\frac{|f^{(i,j)}(\alpha_k,\beta_l)|}{i!j!}\right\}\\ &\leq 2^{M+N+2}M!N!\max_{0\leq i\leq M;0\leq j\leq N}\left\{|f^{(i,j)}(\alpha_k,\beta_l)|\right\}.\end{aligned}$$

Now since

$$\sup_{k,l}\left\{\max_{0\leq i\leq M;0\leq j\leq N}|f^{(i,j)}(\alpha_k,\beta_l)|\right\}<\infty,$$

we have

$$\sup_{k,l}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|b_{mnkl}|<\infty\right\},$$

which shows that $A^{\top}(f, w_{ij})$ is an $\mathcal{L}_u - \mathcal{L}_u$ method.

Theorem 4.4. *If $f(z_1, z_2)$ is an entire function of bounded index, then for any double sequence (w_{ij}), $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{E}_2 - \mathcal{E}_2$ method if and only if for all integers $p > 0$, $q > 0$ and there exist constants M and N such that*

$$|f^{(m,n)}(\alpha_k,\beta_l)|\leq p^k q^l MN \text{ for } k,l,m,n=0,1,2,3,\ldots.$$

Proof. If $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{E}_2 - \mathcal{E}_2$ method then by Theorem 2.5, for all integers $p > 0$, $q > 0$ there exist integers $s > 0$, $t > 0$ and constants $S > 0$, $T > 0$ such that

$$|b_{mnkl}|p^m q^n < s^k t^l ST \text{ for some } m,n,k,l=0,1,2,3,\ldots.$$

Thus, for $p = 1$, $q = 1$ there exist integers s, t and constants $S > 0$, $T > 0$ such that

$$|b_{mnkl}|=\frac{|f^{(m,n)}(\alpha_k,\beta_l)|}{m!n!}\leq s^k t^l ST \text{ for some } m,n,k,l=0,1,2,3,\ldots.$$

Hence, for all integers m and n there exist integers $p = s, q = t$ and constants $M = m!S$ and $N = n!T$ such that

$$|f^{(m,n)}(\alpha_k,\beta_l)|\leq s^k t^l STm!n! = p^k q^l MN \text{ for some } k,l=0,1,2,3,\ldots.$$

Now let $f(z_1, z_2)$ be an entire function of bounded index and (w_{ij}) be a double sequence such that for all positive integers m, n there exist the positive integers p, q and a constant Δ such that

$$|f^{(m,n)}(\alpha_k,\beta_l)|\leq p^k q^l\Delta \text{ for } k,l=0,1,2,3,\ldots.$$

Let $M+N$ be the index of $f(z_1,z_2)$. Thus there exists integers $r_1>0$, $r_2>0$ and constants S,T such that for $m\leq M$, $n\leq N$,

$$|f^{(m,n)}(\alpha_k,\beta_l)|\leq r_1^k r_2^l ST \text{ for } k,l=0,1,2,3,\ldots.$$

For integers $p>0$ and $q>0$, we have (by [21]) $h(z_1,z_2)=f(pz_1,qz_2)$ is of bounded index. Thus $p>0$ and $q>0$, let M_p+N_q be the index of

$h(z_1,z_2)$ that is, for any $(z_1,z_2)\in\mathbb{C}^2$ and $i,j=0,1,2,\ldots,$

$$\begin{aligned}\frac{|h^{(i,j)}(z_1,z_2)|}{i!j!} &= p^iq^j\frac{|f^{(i,j)}(z_1,z_2)|}{i!j!}\\ &\leq \max_{0\leq i\leq M_p;0\leq j\leq N_q}\left\{p^iq^j\frac{|f^{(i,j)}(z_1,z_2)|}{i!j!}\right\}\\ &= \max_{0\leq i\leq M_p;0\leq j\leq N_q}\left\{\frac{|h^{(i,j)}(z_1,z_2)|}{i!j!}\right\}.\end{aligned}$$

Hence, for $i,j=0,1,2,\ldots,$

$$\begin{aligned}p^iq^j\frac{|f^{(i,j)}(\alpha_k,\beta_l)|}{i!j!} &\leq \max_{0\leq i\leq M_p;0\leq j\leq N_q}\left\{p^iq^j\frac{|f^{(i,j)}(\alpha_k,\beta_l)|}{i!j!}\right\}\\ &\leq p^{M_p}q^{N_q}\max_{0\leq i\leq M;0\leq j\leq N}\frac{|f^{(i,j)}(\alpha_k,\beta_l)|}{i!j!}\\ &\leq p^{M_p}q^{N_q}\max_{0\leq i\leq M;0\leq j\leq N}\left\{|f^{(i,j)}(\alpha_k,\beta_l)|\right\}\\ &\leq p^{M_p}q^{N_q}r_1^kr_2^lST \qquad k,l=0,1,2,\ldots.\end{aligned}$$

Therefore, for integers $p>0$ and $q>0$, there exist integers $u=r_1$, $v=r_2$ and constants $\lambda=p^{M_p}S$ and $\mu=q^{N_q}T$ such that

$$|b_{m,n,k,l}|p^mq^n=\frac{|f^{(m,n)}(\alpha_k,\beta_l)|}{m!n!}p^mq^n\leq p^{M_p}r_1^kSq^{N_q}r_2^lT=u^kv^l\lambda\mu$$

for $m,n,k,l=0,1,2,\ldots.$ Thus, by Theorem 2.5, $A^\top(f,w_{ij})=(b_{mnkl})$ is an $\mathcal{E}_2-\mathcal{E}_2$ method. □

We now show that the condition that $f(z_1,z_2)$ is of bounded index can not be omitted Theorem 4.4.

Theorem 4.5. *There exists an entire function $f(z_1,z_2)$ of exponential type and of unbounded index and there exist integers p and q and constant Δ with*

$$|f^{(m,n)}(\alpha_k,\beta_l)|\leq p^mq^n\Delta$$

for $k,l=0,1,2,\ldots.$ but $A^\top(f,w_{ij})=(b_{mnkl})$ is not an $\mathcal{E}_2-\mathcal{E}_2$ method.

Proof. Let (a_m) and (b_n) be the sequences positive numbers such that $a_1 = 0$ and $b_1 = 0$, $(a_{mn}) = (a_m b_n)$ also be a double sequence and

$$a_{k+1l+1} \geq \max_{k,l=1,2,\dots} \left\{ 3(k+1)(l+1)a_k b_l, a_k^{\frac{k+1}{k}} b_l^{\frac{l+1}{l}} \right\}.$$

Because of

$$f(z_1, z_2) = \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} \left(1 - \frac{z_1}{a_m}\right)^m \left(1 - \frac{z_2}{b_n}\right)^n,$$

it is a product of two entire functions of one variable, so $f(z_1, z_2)$ is an entire function of exponential type and of unbounded index(see [6]). Also

$$\lim_{m,n\to\infty} \frac{|f^{(m,n)}(a_m, b_n)|}{m!n!} = \infty.$$

Therefore, there exist sequences (m_k) and (n_l) such that

$$\frac{|f^{(m_k,n_l)}(a_m, b_n)|}{m_k!n_l!} \geq k!l!$$

Choose the sequence $(w_{kl}) = (\alpha_k \beta_l)$ by $\alpha_0 = 0$, $\beta_0 = 0$, $\alpha_k = a_{m_k}$ and $\beta_l = b_{n_l}$ for all positive k, l. Thus, for

$$f(z_1, z_2) = \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} \left(1 - \frac{z_1}{a_m}\right)^m \left(1 - \frac{z_2}{b_n}\right)^n$$

and the double sequence (w_{kl}) we have

$$f^{(m,n)}(\alpha_k, \beta_l) = 0 \text{ for } k > m, l > n.$$

Hence, for all integers m and n there exist integers $p = 1$ and $q = 1$ and constant $\Delta = \max_{k\leq m; l\leq n}\{|f^{(m,n)}(\alpha_k, \beta_l)|\}$ such that

$$|f^{(m,n)}(\alpha_k, \beta_l)| \leq p^k q^l \Delta$$

for $k, l = 1, 2, \dots$. Now, for $A^{\top}(f, w_{ij}) = (b_{mnkl})$ and $k, l = 1, 2, \dots$,

$$|b_{m,n,k,l}| = \frac{|f^{(m_k,n_l)}(\alpha_k, \beta_l)|}{m_k!n_l!} \geq k!l!.$$

Therefore, for any integers r_1, r_2 and any constants $S > 0$, $T > 0$, there exist k_0, l_0 such that

$$|b_{mnkl}| \geq k!l! > r_1^k r_2^l ST \text{ for } k > k_0, l > l_0.$$

Thus by Theorem 2.5, $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is not an $\mathcal{E}_2 - \mathcal{E}_2$ method. □

Theorem 4.6. *Let $f(z_1, z_2)$ is an entire function of bounded index and (w_{ij}) be a double sequence of complex numbers. If either $A(f, w_{ij}) = (a_{mnkl})$ or $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{L}_u - \mathcal{L}_u$ method then $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{E}_2 - \mathcal{E}_2$ method.*

Proof. If either $A(f, w_{ij}) = (a_{mnkl})$ or $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{L}_u - \mathcal{L}_u$ method then either

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mnkl}| \leq \Delta,$$

or

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|b_{mnkl}| \leq \Delta$$

Thus either

$$|a_{mnkl}| = \frac{|f^{(k,l)}(\alpha_m, \beta_n)|}{k!l!} \leq \Delta.$$

or

$$|b_{mnkl}| = \frac{|f^{(n,m)}(\alpha_k, \beta_l)|}{m!n!} \leq \Delta.$$

for $m, n, k, l = 0, 1, 2, \ldots$. Hence,

$$\frac{|f^{(n,m)}(\alpha_k, \beta_l)|}{m!n!} \leq \Delta.$$

for $m, n, k, l = 0, 1, 2, \ldots$. Therefore, for each m and n there exist $p = 1$, $q = 1$, $S = m!\Delta$ and $T = n!$ such that

$$|f^{(n,m)}(\alpha_k, \beta_l)| \leq m!n!\Delta = p^k q^l ST$$

for $k, l = 0, 1, 2, \ldots$. Thus, by Theorem 2.5, $A^{\top}(f, w_{ij}) = (b_{mnkl})$ is an $\mathcal{E}_2 - \mathcal{E}_2$ method. □

Conclusion

In this paper, we studied the four dimensional Taylor matrix transformations of double sequences of bounded index into itself and gave some related results. Almost all these results are the extension of the results in [8] from single to the double sequences.

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