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A CLASS OF SPIRALLIKE FUNCTIONS DEFINED BY RUSCHEWEYH-TYPE q-DIFFERENCE OPERATOR

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Abstract. In this paper, we define a new class of analytic functions $\mathcal{M}_q^m(\eta,\gamma,\lambda)$ involving Ruscheweyh-type q difference operator $\mathcal{D}_q(R_q^m f)$. Subordination results and Fekete-Szegö problem for this generalized function class are investigated. Sufficient conditions for a function to be in the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$ are also provided.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$.

A function $f\in\mathcal{A}$ is said to be in the class of γ-spirallike functions of order λ in $\mathbb{U}$, denoted by $\mathcal{S}^*(\gamma,\lambda)$ if

$$\Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)>\lambda\cos\gamma,\quad z\in\mathbb{U} \tag{1.2}$$

for $0\le\lambda<1$ and some real γ with $|\gamma|<\frac{\pi}{2}$. The class $\mathcal{S}^*(\gamma,\lambda)$ was studied by Libera [5] and Keogh and Merkes [4]. Note that $\mathcal{S}^*(\gamma,0)$ is the class of spirallike functions introduced by Špaček [14], $\mathcal{S}^*(0,\lambda)=\mathcal{S}^*(\lambda)$ is the class of starlike functions of order λ and $\mathcal{S}^*(0,0)=\mathcal{S}^*$ is the familiar class of starlike functions.

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For the constants λ, γ with $0 \leq \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$ denote

$$p_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma}(e^{-i\gamma} - 2\lambda\cos\gamma)z}{1-z}, \quad z \in \mathbb{U}. \tag{1.3}$$

The function $p_{\lambda,\gamma}(z)$ maps the open unit disk onto the half-plane

$$H_{\lambda,\gamma} = \left\{ z \in \mathbb{C} : \Re(e^{i\gamma}z) > \lambda\cos\gamma \right\}.$$

If

$$p_{\lambda,\gamma}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

then it is easy to check that

$$p_n = 2e^{-i\gamma}(1-\lambda)\cos\gamma, \quad \text{for all} \;\; n \geq 1. \tag{1.4}$$

The convolution or Hadamard product of two functions $f, g \in \mathcal{A}$, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U},$$

where f is given by (1.1) and g is given by

$$g(z) = \sum_{n=2}^{\infty} b_n z^n.$$

Denote by $\mathcal{B}$ the family of all analytic functions $w(z)$ that satisfy the conditions $w(0) = 0$ and $|w(z)| < 1, \; z \in \mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be subordinate to a function $g \in \mathcal{A}$, written $f \prec g$, if there exists a function $w \in \mathcal{B}$ such that $f(z) = g(w(z)), \; z \in \mathbb{U}$.

We briefly recall here the notion of *q-operators* i.e. *q-difference operators* that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [2] (also see [1, 3, 11]). Kanas and Răducanu [3] have used the fractional *q-calculus operators* to investigate certain classes of functions which are analytic in $\mathbb{U}$.

Consider $0 < q < 1$ and a non-negative integer n. The q-integer number or basic number n is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \; [0]_q = 0.$$

For a non-integer number t we will denote $[t]_q = \dfrac{1-q^t}{1-q}$.

The q-shifted factorial is defined as follows

$$[0]_q! = 1, \;\; [n]_q! = [1]_q[2]_q \ldots [n]_q.$$

Note that $\lim_{q\to 1^-}[n]_q = n$ and $\lim_{q\to 1^-}[n]_q! = n!$.

The Jackson's *q-derivative operator* or *q-difference operator* for a function $f \in \mathcal{A}$ is defined by

$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(qz)-f(z)}{z(q-1)} & , z \neq 0 \\ f'(0) & , z = 0. \end{cases} \tag{1.5}$$

Note that for $n \in \mathbb{N} = \{1, 2, \dots\}$ and $z \in \mathbb{U}$

$$\mathcal{D}_q z^n = [n]_q z^{n-1}.$$

Further, we define the operator $\mathcal{D}_q^m f(z), m \in \mathbb{N}$ as follows

$$\mathcal{D}_q^0 f(z) = f(z) \text{ and } \mathcal{D}_q^m f(z) = \mathcal{D}_q(\mathcal{D}_q^{m-1} f(z)).$$

For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, the *q-generalized Pochhammer* symbol is defined by

$$[t]_n = [t]_q[t+1]_q[t+2]_q \dots [t+n-1]_q.$$

Moreover, for $t > 0$ the q-Gamma function is given by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \text{ and } \quad \Gamma_q(1) = 1.$$

Using the definition of Ruscheweyh differential operator [12], in [3] Kanas and Răducanu introduced the Ruscheweyh *q-differential operator* defined by

$$\mathcal{R}_q^m f(z) = f(z) * F_{q,m+1}(z) \quad z \in \mathbb{U},\ m > -1, \tag{1.6}$$

where $f \in \mathcal{A}$ and

$$F_{q,m+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+m)}{[n-1]_q! \Gamma_q(1+m)} z^n. \tag{1.7}$$

From (1.6) we have

$$\mathcal{R}_q^0 f(z) = f(z), \qquad \mathcal{R}_q^1 f(z) = z\mathcal{D}_q f(z)$$

and

$$\mathcal{R}_q^n f(z) = \frac{z\mathcal{D}_q^n \left(z^{n-1} f(z)\right)}{[n]_q!} \quad n \in \mathbb{N}.$$

For $f \in \mathcal{A}$ given by (1.1), in view of (1.6) and (1.7), we obtain

$$\mathcal{R}_q^m f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+m)}{[n-1]_q! \Gamma_q(1+m)} a_n z^n, \quad z \in \mathbb{U}. \tag{1.8}$$

Note that

$$\lim_{q\to 1^-} F_{q,m+1}(z) = \frac{z}{(1-z)^{m+1}}$$

and

$$\lim_{q\to 1^-} \mathcal{R}_q^m f(z) = f(z) * \frac{z}{(1-z)^{m+1}}.$$

Moreover,

$$\mathcal{D}_q(\mathcal{R}_q^m f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q \Phi_q(n,m) a_n z^{n-1}, \tag{1.9}$$

where

$$\Phi_q(n,m) = \frac{\Gamma_q(n+m)}{[n-1]_q! \Gamma_q(1+m)}. \tag{1.10}$$

Using Ruscheweyh differential operator various new classes of convex and starlike functions have been defined. Therefore it seems natural to use Ruscheweyh q-differential operator to introduce the following class of functions.

Definition 1.1. For $0 \le \eta < 1, 0 \le \lambda < 1, |\gamma| < \dfrac{\pi}{2}$ denote by $\mathcal{M}_q^m(\eta,\gamma,\lambda)$ the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(e^{i\gamma} \frac{z\mathcal{D}_q(\mathcal{R}_q^m f(z))}{(1-\eta)\mathcal{R}_q^m f(z) + \eta z \mathcal{D}_q(\mathcal{R}_q^m f(z))}\right) > \lambda \cos\gamma, \quad z \in \mathbb{U}. \tag{1.11}$$

When $q \to 1$ the class $\mathcal{M}_q^0(\eta,\gamma,\lambda) \equiv \mathcal{S}(\eta,\gamma,\lambda)$ consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\Re\left(e^{i\gamma} \frac{zf'(z)}{(1-\eta)f(z) + \eta z f'(z)}\right) > \lambda \cos\gamma, \quad z \in \mathbb{U}$$

which have been studied by Murugusundaramoorthy [7] and Orhan et al., [10] defined by various integral and differential operators.

The main object of this paper is to obtain sharp upper-bounds for the Fekete-Szegö problem and subordination results for the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$. We also find sufficient conditions for a function to be in this class.

2. Membership characterizations

In this section we obtain several sufficient conditions for a function $f \in \mathcal{A}$ to be in the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ and let δ be a real number with $0 \le \delta < 1$. If*

$$\left| \frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\eta)R_q^m f(z) + \eta z \mathcal{D}_q(R_q^m f(z))} - 1 \right| \le 1 - \delta, \quad z \in \mathbb{U} \tag{2.1}$$

then $f \in \mathcal{M}_q^m(\eta,\gamma,\lambda)$ provided that

$$|\gamma| \le \cos^{-1}\left(\frac{1-\delta}{1-\lambda}\right).$$

Proof. From (2.1) it follows that

$$\frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\eta)R_q^m f(z)+\eta z\mathcal{D}_q(R_q^m f(z))}=1+(1-\delta)w(z),$$

where $w(z)\in\mathcal{B}$. We have

$$\Re\left(e^{i\gamma}\frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\eta)R_q^m f(z)+\eta z\mathcal{D}_q(R_q^m f(z))}\right)=\Re[e^{i\gamma}(1+(1-\delta)w(z))]$$

$$=\cos\gamma+(1-\delta)\Re(e^{i\gamma}w(z))\geq\cos\gamma-(1-\delta)|e^{i\gamma}w(z)|>\cos\gamma-(1-\delta)\geq\lambda\cos\gamma,$$

provided that $|\gamma|\leq\cos^{-1}\left(\frac{1-\delta}{1-\lambda}\right)$. Thus, the proof is completed. □

If in Theorem 2.1 we take $\delta=1-(1-\lambda)\cos\gamma$ we obtain the following result.

Corollary 2.2. *Let $f\in\mathcal{A}$. If*

$$\left|\frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\eta)R_q^m f(z)+\eta z\mathcal{D}_q(R_q^m f(z))}-1\right|\leq(1-\lambda)\cos\gamma,\quad z\in\mathbb{U}\tag{2.2}$$

then $f\in\mathcal{M}_q^m(\eta,\gamma,\lambda)$.

A sufficient condition for a function $f\in\mathcal{A}$ to be in the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$, in terms of coefficients inequality is obtained in the next theorem.

Theorem 2.3. *If a function $f\in\mathcal{A}$ given by (1.1) satisfies the inequality*

$$\sum_{n=2}^{\infty}[(1-\eta)([n]_q-1)\sec\gamma+(1-\lambda)(1+\eta([n]_q-1))]\Phi_q(n,m)|a_n|\leq 1-\lambda,$$

where $0\leq\eta<1, 0\leq\lambda<1, |\gamma|<\frac{\pi}{2}$, and $\Phi_q(n,m)$ is defined by (1.6), then it belongs to the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$.

Proof. By the virtue of Corollary 2.2, it suffices to show that the condition (2.2) is satisfied. We have

$$\left|\frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\eta)R_q^m f(z)+\eta z\mathcal{D}_q(R_q^m f(z))}-1\right|$$

$$=(1-\eta)\left|\frac{\sum_{n=2}^{\infty}([n]_q-1)\Phi_q(n,m)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}(1-\eta+[n]_q\eta)\Phi_q(n,m)a_nz^{n-1}}\right|$$

$$<(1-\eta)\frac{\sum_{n=2}^{\infty}(n-1)\Phi_q(n,m)|a_n|}{1-\sum_{n=2}^{\infty}(1-\eta+\eta[n]_q)\Phi_q(n,m)|a_n|}.$$

The last expression is bounded above by $(1-\lambda)\cos\gamma$, if

$$\sum_{n=2}^{\infty}(1-\eta)([n]_q-1)\Phi_q(n,m)|a_n|$$

$$\leq (1-\lambda)\cos\gamma\left(1-\sum_{n=2}^{\infty}(1-\eta+\eta[n]_q)\Phi_q(n,m)|a_n|\right),$$

which is equivalent to

$$\sum_{n=2}^{\infty}[(1-\eta)([n]_q-1)\sec\gamma+(1-\lambda)(1+\eta([n]_q-1))]\Phi_q(n,m)|a_n|\leq 1-\lambda.$$

□

3. Subordination Result

In this section, we obtain subordination results for the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$. To prove our results we need the following definition and lemmas.

Definition 3.1. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)=\sum\limits_{n=1}^{\infty}a_nz^n$, $a_1=1$ is regular, univalent and convex in $\mathbb{U}$, we have

$$\sum_{n=1}^{\infty}b_na_nz^n\prec f(z),\quad z\in\mathbb{U}. \tag{3.1}$$

In 1961, Wilf [15] proved the following subordinating factor sequence.

Lemma 3.2. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\}>0,\quad z\in\mathbb{U}. \tag{3.2}$$

Theorem 3.3. *Let $f\in\mathcal{M}_q^m(\eta,\gamma,\lambda)$ and $g(z)$ be any function in the usual class of convex functions $\mathcal{C}$, then*

$$\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}(f*g)(z)\prec g(z), \tag{3.3}$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\gamma<1$; $0\leq\lambda<1$, with

$$\Phi_q(2,m)=\frac{\Gamma_q(2+m)}{\Gamma_q(1+m)} \tag{3.4}$$

and

(3.5)

$$\Re\{f(z)\} > -\frac{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}, \quad z\in\mathbb{U}.$$

The constant factor $\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}$ *in (3.3) cannot be replaced by a larger number.*

Proof. Let $f\in\mathcal{M}_q^m(\eta,\gamma,\lambda)$ and suppose that $g(z)=z+\sum_{n=2}^{\infty}c_nz^n\in\mathcal{C}$. Then

$$\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}(f*g)(z) \tag{3.6}$$

$$=\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}\left(z+\sum_{n=2}^{\infty}c_na_nz^n\right).$$

Thus, by Definition 3.1, the subordination result holds true if

$$\left\{\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 3.2, this is equivalent to the following inequality

(3.7)

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}a_nz^n\right\}>0.$$

By noting the fact that $\frac{[(1-\eta)([n]_q-1)\sec\gamma+(1-\lambda)(1+\eta([n]_q-1))]\Phi_q(n,m)}{1-\lambda}$ is an increasing function for $n\geq 2$ and in particular

$$\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{(1-\lambda)}\leq \frac{[(1-\eta)([n]_q-1)\sec\gamma+(1-\lambda)(1+\eta([n]_q-1))]\Phi_q(n,m)}{1-\lambda}, \tag{3.8}$$

for $n \geq 2$, $|\eta| < \frac{\pi}{2}$. Therefore, for $|z| = r < 1$, we have

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}\sum_{n=1}^{\infty}a_nz^n\right\}$$
$$=\Re\left\{1+\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}z\right.$$
$$\left.+\frac{\sum\limits_{n=2}^{\infty}((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)a_nz^n}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}\right\}$$
$$\geq 1-\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}r$$
$$-\frac{\sum\limits_{n=2}^{\infty}((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta)|a_n|r^n}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}$$
$$\geq 1-\frac{(((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}r$$
$$-\frac{1-\gamma}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}r^2$$
$$>0,\quad |z|=r<1,$$

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (3.7) and hence also the subordination result (3.3) asserted by Theorem 3.3. The inequality (3.5) follows from (3.3) by taking

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in\mathcal{C}.$$

Next we consider the function

$$F(z):=z-\frac{1-\gamma}{[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}z^2$$

where $|\eta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$, $0 \leq \lambda < 1$ and $\Phi_q(2,m)]$ is given by (3.4). Clearly $F \in \mathcal{M}_q^m(\eta,\gamma,\lambda)$. For this function (3.3) becomes

$$\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}F(z)\prec\frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\Re\left(\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}F(z)\right)\right\}=-\frac{1}{2},z\in\mathbb{U}.$$

This shows that the constant $\frac{((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)}{2[1-\lambda+((1-\eta)q\sec\gamma+(1-\lambda)(1+q\eta))\Phi_q(2,m)]}$ cannot be replaced by any larger one. □

4. The Fekete-Szegö problem

The problem of finding sharp upper-bounds for the functional $|a_3-\mu a_2^2|$ for different subclasses of $\mathcal{A}$ is known as the Fekete-Szegö problem. Over the years this problem has been investigated by many authors including [6], [9], [13] etc. In order to obtain sharp upper-bounds for the Fekete-Szegö functional for the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$ the following lemma is required (see, e.g., [8], p.108).

Lemma 4.1. *Let the function $w \in \mathcal{B}$ be given by $w(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathbb{U}$. Then*

$$|w_1| \le 1 \ \ \text{and} \ \ |w_2| \le 1 - |w_1|^2 \tag{4.1}$$

and

$$|w_2 - s w_1^2| \le \max\{1, |s|\} \ \ \text{for any complex number } s. \tag{4.2}$$

The functions $w(z) = z$ and $w(z) = z^2$, or one of their rotations show that both inequalities (4.1) and (4.2) are sharp.

First we obtain sharp upper-bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$, with μ a real parameter.

Theorem 4.2. *Let $f \in \mathcal{M}_q^m(\eta,\gamma,\lambda)$ be given by (1.1) and let μ be a real number. Then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{2(1-\lambda)\cos\gamma}{q^2(1+q)(1-\eta)^2\Phi_q(3,m)} \times \\ \times \left[2 + q(1+\eta) - 2\lambda(1+q\eta) - \mu(1+q)\dfrac{2(1-\lambda)\Phi_q(3,m)}{\Phi_q^2(2,m)}\right], \\ \text{if } \mu \le \sigma_1 \\ \dfrac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)}, \\ \text{if } \sigma_1 \le \mu \le \sigma_2 \\ \dfrac{2(1-\lambda)\cos\gamma}{q^2(1+q)(1-\eta)^2\Phi_q(3,m)} \times \\ \times \left[\mu(1+q)\dfrac{2(1-\lambda)\Phi_q(3,m)}{\Phi_q^2(2,m)} + 2\lambda(1+q\eta) - q(1+\eta) - 2\right], \\ \text{if } \mu \ge \sigma_2 \end{cases}, \tag{4.3}$$

where

$$\sigma_1 = \frac{1+q\eta}{1+q}\,\frac{\Phi_q^2(2,m)}{\Phi_q(3,m)} \ \ \text{and} \tag{4.4}$$

$$\sigma_2 = \frac{1+q-\lambda(1+q\eta)}{(1+q)(1-\lambda)}\,\frac{\Phi_q^2(2,m)}{\Phi_q(3,m)}, \tag{4.5}$$

with $\Phi_q(2,m)$, *is given by* (3.4) *and* $\Phi_q(3,m)=\dfrac{\Gamma_q(3+m)}{(1+q)\Gamma_q(1+m)}$.

All estimates are sharp.

Proof. Let $f\in\mathcal{M}_q^m(\eta,\gamma,\lambda)$ be given by (1.1). From the definition of the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$, there exists $w\in\mathcal{B}, w(z)=w_1z+w_2z^2+w_3z^3+\dots$ such that

$$\frac{z\mathcal{D}_q(R_q^m f(z))}{(1-\eta)R_q^m f(z)+\eta z\mathcal{D}_q(R_q^m f(z))}=p_{\lambda,\gamma}(w(z)),\quad z\in\mathbb{U}. \tag{4.6}$$

Set $p_{\lambda,\gamma}(z)=1+p_1z+p_2z^2+p_3z^3+\dots$. Equating the coefficients of z and z^2 on both sides of (4.6) we obtain

$$a_2=\frac{p_1w_1}{q(1-\eta)\Phi_2(2,m)}$$

and

$$a_3=\frac{1}{q(1+q)(1-\eta)\Phi_q(3,m)}\left[\left(\frac{1+q\eta}{q(1-\eta)}p_1^2+p_2\right)w_1^2+p_1w_2\right].$$

From (1.4) we have $p_1=p_2=2e^{-i\gamma}(1-\lambda)\cos\gamma$ and thus we obtain

$$a_2=\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{q(1-\eta)\Phi_q(2,m)}w_1 \tag{4.7}$$

and

$$a_3=\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)}\left[\left(2e^{-i\gamma}(1-\lambda)\cos\gamma\frac{1+q\eta}{q(1-\eta)}+1\right)w_1^2+w_2\right]. \tag{4.8}$$

Then, it follows

$$\left|a_3-\mu a_2^2\right|\le\frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)}\times$$
$$\left\{\left|\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{q(1-\eta)}\left(1+q\eta-\mu(1+q)\frac{\Phi_q(3,m)}{\Phi_2^2(2,m)}\right)+1\right||w_1|^2+|w_2|\right\}.$$

In view of Lemma 4.1(4.1) we have

$$\left|a_3-\mu a_2^2\right|\le\frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)}\times$$
$$\left\{1+\left[\left|\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{q(1-\eta)}\left(1+q\eta-\mu(1+q)\frac{\Phi_q(3,m)}{\Phi_2^2(2,m)}\right)+1\right|-1\right]|w_1|^2\right\}$$

or

$$\left|a_3-\mu a_2^2\right|\le \tag{4.9}$$
$$\frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)}\left[1+\left(\sqrt{1+M(2+M)\cos^2\gamma}-1\right)|w_1|^2\right],$$

where

$$M = \frac{2(1-\lambda)}{q(1-\eta)}\left(1+q\eta-\mu(1+q)\frac{\Phi_q(3,m)}{\Phi_q^2(2,m)}\right). \tag{4.10}$$

Denote by

$$F(x,y) = 1 + \left(\sqrt{1+M(2+M)x^2}-1\right)y^2,$$

where $x = \cos\gamma$, $y = |w_1|$ and $(x,y) \in [0,1]\times[0,1]$.

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)\times(0,1)$. Thus, the maximum must be attained at a boundary point. Since $F(x,0)=1, F(0,y)=1$ and $F(1,1) = |1+M|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0)=1$ or $F(1,1)=|1+M|$.

Therefore, from (4.9) we obtain

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)}\max\{1,|1+M|\}, \tag{4.11}$$

where M is given by (4.10).

Consider first the case $|1+M|\geq 1$. If $\mu \leq \sigma_1$, where σ_1 is given by (4.4), then $M\geq 0$ and from (4.11) we obtain

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)^2\Phi_q(3,m)}\times \left[2+q(1+\eta)-2\lambda(1+q\eta)-\mu(1+q)\frac{2(1-\lambda)\Phi_q(3,m)}{\Phi_q^2(2,m)}\right],$$

which is the first part of the inequality (4.3). If $\mu\geq\sigma_2$, where σ_2 is given by (4.5), then $M\leq -2$ and it follows from (4.11) that

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)^2\Phi_q(3,m)}\times \left[\mu(1+q)\frac{2(1-\lambda)\Phi_q(3,m)}{\Phi_q^2(2,m)}+2\lambda(1+q\eta)-q(1+\eta)-2\right]$$

and this is the third part of (4.3).

Now, suppose $\sigma_1\leq\mu\leq\sigma_2$. Then $|1+M|\leq 1$ and thus, from (4.11) we obtain

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)},$$

which is the second part of the inequality (4.3).

In view of Lemma 4.1, the results are sharp for $w(z)=z$ and $w(z)=z^2$ or one of their rotations. □

Next, we consider the Fekete-Szegö problem for the class $\mathcal{M}_q^m(\eta,\gamma,\lambda)$ with complex parameter μ.

Theorem 4.3. *Let $f \in \mathcal{M}_q^m(\eta,\gamma,\lambda)$ be given by (1.1) and let μ be a complex number. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)} \times \max\left\{1, \left|\frac{2(1-\lambda)\cos\gamma}{q(1-\eta)}\left(\mu(1+q)\frac{\Phi_q(3,m)}{\Phi_q^2(2,m)} - 1 - q\eta\right) - e^{i\gamma}\right|\right\}. \tag{4.12}$$

Proof. Assume that $f \in \mathcal{M}_q^m(\eta,\gamma,\lambda)$. Making use of (4.7) and (4.8) we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\lambda)\cos\gamma}{q(1+q)(1-\eta)\Phi_q(3,m)} \times \left|w_2 - \left[\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{q(1-\eta)}\left(\mu(1+q)\frac{\Phi_q(3,m)}{\Phi_q^2(2,m)} - 1 - q\eta\right) - 1\right] w_1^2\right|.$$

The inequality (4.12) follows as an application of Lemma 4.1(4.2) with

$$s = \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{q(1-\eta)}\left(\mu(1+q)\frac{\Phi_q(3,m)}{\Phi_q^2(2,m)} - 1 - q\eta\right) - 1.$$

□

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