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LYAPUNOV-TYPE INEQUALITY FOR NONLINEAR SYSTEMS WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. In this paper, we introduce a fractional nonlinear system of differential equations including the Riemann-Liouville derivatives. We present some new Lyapunov-type inequalities for this fractional nonlinear system and its special cases. A study of the boundedness and the behavior of oscillatory solutions is also given.

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1. Introduction

The well-known Lyapunov inequality

$$\int_a^b |q(u)|du > \frac{4}{b-a}, \tag{1.1}$$

was introduced by Aleksandr Mikhailovich Lyapunov [26] in 1893 for the following boundary value problem

$$y''(t) + q(t)y(t) = 0, \qquad a < t < b, \tag{1.2}$$

$$y(a) = y(b) = 0,$$

where $q(t)$ is a real and continuous function. In recent years, with developments in the theory of fractional calculus (see for example [7, 8, 12, 16, 13, 14, 15]), many authors have studied the associated inequalities of the fractional differential equations, for example see [2, 1, 19, 21, 27, 36, 37]. In this sense, for the differential equation (1.2), some authors generalized it as the fractional boundary value problems and obtained the associated Lyapunov inequalities with the

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fractional Riemann-Liouville, Caputo and Prabhakar derivatives [13, 17, 18], [23, 24].

In view of the nonlinear differential equations, several authors generalized also the classical Lyapunov inequality (1.1) for the half-linear and nonlinear differential equation [5, 6, 11, 20, 22, 29, 30, 32, 33, 34, 35]. For example, Lyapunov-type inequalities can be found in [28] for the Emden-Fowler type equations and were obtained for the first time by Elbert [10] for the half-linear equation. Also, the proof of its extension can be found in the book of Došlý and Řehák [9]. In [32], the authors obtained the Lyapunov inequality for the nonlinear system (a generalization of the Emden-Fowler-type and half-linear equations)

$$\begin{cases} x'(t) = \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), & \gamma > 1, \\ u'(t) = -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t), & \beta > 1, \end{cases} \tag{1.3}$$

with initial conditions

$$x(a) = x(b) = 0, \qquad a, b \in \mathbb{R},\ a < b.$$

Furthermore, in [15] authors study the nonlinear system (1.3) including the Prabhakar fractional derivative and present some new Lyapunov-type inequalities for this system and consider some special cases of the system.

In this work, as fractionalization of the nonlinear system (1.3), we study the following fractional nonlinear system with the Riemann-Liouville derivative of order μ $(0 < \mu < 1)$

$$\begin{cases} D^{\mu}x(t) = \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), & \gamma > 1, \\ D^{\mu}u(t) = -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t), & \beta > 1, \end{cases} \tag{1.4}$$

where the functions α_1, β_1 and β_2 are continuous functions such that $\beta_1(t) > 0$ for $t \in [t_0, \infty)$ and initial conditions are

$$x(a) = x(b) = 0, \qquad (I_{a^+}^{1-\mu}u)(b) = (I_{b^-}^{1-\mu}u)(a) = 0, \qquad a, b \in \mathbb{R}\ (a < b).$$

We intend to obtain some new Lyapunov-type inequalities for the fractional nonlinear system (1.4) and some special cases. For this purpose, we suppose that the nontrivial solution $(x(t), u(t))$ of the fractional nonlinear system (1.4) exists.

The plan of the paper is the following. In the next section, we recall some definitions and properties of the fractional calculus. In Section 3, we state fundamental theorems about the Lyapunov-type inequalities for fractional nonlinear system (1.4) and in some special cases we reduce the obtained inequalities. In Section 4, we discuss the boundedness of solution $(x(t), u(t))$ and behavior of zeros.

2. Definitions

Definition 2.1. For $m - 1 < \mu < m$, $m \in \mathbb{N}$ and $f \in L^1(a, b)$, the left-sided and right-sided Riemann-Liouville fractional integral and derivative are defined

as follows [31]
(2.1)
$$(I^{\mu}_{a+}f)(t)=\frac{1}{\Gamma(\mu)}\int_a^t(t-\xi)^{\mu-1}f(\xi)d\xi,\qquad (I^{\mu}_{b-}f)(t)=\frac{1}{\Gamma(\mu)}\int_t^b(\xi-t)^{\mu-1}f(\xi)d\xi,$$

(2.2)
$$(D^{\mu}_{a+}f)(t)=\frac{d^m}{dt^m}(I^{m-\mu}_{a+}f)(t),\quad t>a,\quad (D^{\mu}_{b-}f)(t)=-\frac{d^m}{dt^m}(I^{m-\mu}_{b-}f)(t),\quad t<b.$$

Also, the left-sided and right-sided Caputo fractional derivative are defined as [31]
(2.3)
$$({}^CD^{\mu}_{a+}f)(t)=(I^{m-\mu}_{a+}\frac{d^m}{dt^m}f)(t)=\frac{1}{\Gamma(m-\mu)}\int_a^t(t-\xi)^{m-1-\mu}\frac{d^m}{dt^m}f(\xi)d\xi,\quad t>a,$$

(2.4)
$$({}^CD^{\mu}_{b-}f)(t)=-(I^{m-\mu}_{b-}\frac{d^m}{dt^m}f)(t)=\frac{1}{\Gamma(m-\mu)}\int_t^b(\xi-t)^{m-1-\mu}\frac{d^m}{dt^m}f(\xi)d\xi,\quad t<b.$$

Proposition 2.2. *The Riemann-Liouville fractional derivative of a constant is not equal to zero for* $0<\mu<1$, *[25]*

$$(D^{\mu}_{a+}1)(t)=\frac{(t-a)^{-\mu}}{\Gamma(1-\mu)},\qquad (D^{\mu}_{b-}1)(t)=\frac{(b-t)^{-\mu}}{\Gamma(1-\mu)}. \tag{2.5}$$

Proposition 2.3. *Let* $m-1<\mu<m,\ m\in\mathbb{N},\ p\geq 1,\ q\geq 1$ *and* $\frac{1}{p}+\frac{1}{q}\leq\mu+1$ $(p\neq 1, q\neq 1$ *in the case* $\frac{1}{p}+\frac{1}{q}=\mu+1)$. *If* $f(t)\in I^{\mu}_{b-}(L_p(a,b))$ *and* $g(t)\in I^{\mu}_{a+}(L_q(a,b))$, *then [3, 4]*

$$\begin{aligned}\int_a^b f(t)D^{\mu}_{a+}g(t)dt=&\int_a^b g(t)^CD^{\mu}_{b-}f(t)dt\\ &+\sum_{k=0}^{m-1}(-1)^kf^{(k)}(x)D^{m-k-1}I^{m-\mu}_{a+}g(x)|_a^b,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\int_a^b f(t)D^{\mu}_{b-}g(t)dt=&\int_a^b g(t)^CD^{\mu}_{a+}f(t)dt\\ &+\sum_{k=0}^{m-1}(-1)^{m-k}f^{(k)}(x)D^{m-k-1}I^{m-\mu}_{b-}g(x)|_a^b,\end{aligned} \tag{2.7}$$

$$\int_a^b f(t)D^{\mu}_{a+}g(t)dt=\int_a^b g(t)D^{\mu}_{b-}f(t)dt. \tag{2.8}$$

Lemma 2.4. *If* $f(t)\in C(a,b)\cap L(a,b)$, *we have [25]*

$$(D^{\mu}_{a+}I^{\mu}_{a+}f)(t)=f(t),\qquad (D^{\mu}_{b-}I^{\mu}_{b-}f)(t)=f(t),$$

$$({}^CD^{\mu}_{a+}I^{\mu}_{a+}f)(t)=f(t),\qquad ({}^CD^{\mu}_{b-}I^{\mu}_{b-}f)(t)=f(t),$$

and if $f(t)$ and its fractional derivative of order μ belongs to $C(a,b)\cap L(a,b)$, then for $c_j \in \mathbb{R}$ and $m-1<\mu\leq m$, we have

$$(I^{\mu}_{a+}D^{\mu}_{a+}f)(t)=f(t)-\sum_{j=1}^{m}c_j(t-a)^{\mu-j}, \tag{2.9}$$

$$(I^{\mu}_{b-}D^{\mu}_{b-}f)(t)=f(t)-\sum_{j=1}^{m}(-1)^{m-j}c_j(b-t)^{\mu-j}, \tag{2.10}$$

$$(I^{\mu}_{a+}\,{}^{C}D^{\mu}_{a+}f)(t)=f(t)-\sum_{j=0}^{m-1}c_j(t-a)^{j}, \tag{2.11}$$

$$(I^{\mu}_{b-}\,{}^{C}D^{\mu}_{b-}f)(t)=f(t)-\sum_{j=0}^{m-1}(-1)^{j}c_j(b-t)^{j}. \tag{2.12}$$

3. Main results

In this section, we consider a fractional nonlinear system including the Riemann-Liouville derivative of order μ and present some new Lyapunov-type inequalities for it. In the case $\mu=1$, we reduce the Lyapunov-type inequality of a fractional nonlinear system to the Lyapunov-type inequalities of a nonlinear system of integer order.

Definition 3.1. We consider the following fractional Emden-Fowler-type equations in the sense of Riemann-Liouville fractional derivative of order μ

$$\begin{cases} D^{\mu}x(t)=\alpha_1(t)x(t)+\beta_1(t)|u(t)|^{\gamma-2}u(t), & \gamma>1,\\ D^{\mu}u(t)=-\beta_2(t)|x(t)|^{\beta-2}x(t)-\alpha_1(t)u(t), & \beta>1,\end{cases} \tag{3.1}$$

with initial conditions

$$x(a)=x(b)=0, \qquad (I^{1-\mu}_{a+}u)(b)=(I^{1-\mu}_{b-}u)(a)=0, \qquad a,b\in\mathbb{R}\ (a<b),$$

and assume that the following hypotheses hold:

i) $\beta_1,\beta_2:[t_0,\infty)\subset\mathbb{R}\to\mathbb{R}$ are continuous functions such that $\beta_1(t)>0$ for $t\in[t_0,\infty)$.

ii) $\alpha_1:[t_0,\infty)\to\mathbb{R}$ is a continuous function.

iii) There exists a real solution $(x(t);u(t))$ of the fractional nonlinear system (3.1) such that $x(a)=x(b)=0$ $(a<b)$ and $x(t)\neq 0$ for $t\in(a,b)$.

We define a nontrivial solution $(x(t), u(t))$ of the system (3.1) on some infinite interval $[t_0, \infty)$ as a proper solution if

$$\sup\{|x(s)| + |u(s)| : t \le s < \infty\} > 0, \quad t \ge t_0.$$

A proper solution $(x(t), u(t))$ of the system (3.1) is called weakly oscillatory if at least one component has a sequence of zeros tending to infinity. This solution is said to be oscillatory if both components have sequences of zeros tending to infinity. If both components (at least one component) are different from zero for large t, then the solution $(x(t), u(t))$ is called nonoscillatory (weakly nonoscillatory). The system (3.1) is said to be oscillatory if all of its solutions are oscillatory.

Theorem 3.2. *The following inequality holds for the fractional nonlinear system* (3.1)

$$\begin{aligned} 2 \le & \frac{1}{\Gamma(\mu)} \int_a^b |t-\tau|^{\mu-1} |\alpha_1(t)| dt \\ & + \frac{M^{\frac{\beta}{\alpha}-1}}{\Gamma(\mu)} \Big(\int_a^b |t-\tau|^{\gamma(\mu-1)} \beta_1(t) dt \Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t) dt \Big)^{\frac{1}{\alpha}}, \end{aligned}$$

where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$, $M = \max |x(t)|_{a<t<b}$ *and* $\beta_2^+(t) = \max\{\beta_2(t), 0\}$.

Proof. Since $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$, there exists $\tau \in (a, b)$ such that $M = |x(\tau)| = \max |x(t)|_{a<t<b} > 0$. Separating the interval $[a, b]$ into two subintervals $[a, \tau]$ and $[\tau, b]$ and applying the left-sided and right-sided Riemann-Liouville fractional integral operator I^μ_{a+} and I^μ_{b-} on both sides of the first equation of the system (3.1) in the subintervals $[a, \tau]$ and $[\tau, b]$, respectively, we have

$$\Big(I^\mu_{a+} D^\mu_{a+} x(t)\Big)(\tau) = \Big(I^\mu_{a+} \big[\alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t)\big]\Big)(\tau),$$

$$\Big(I^\mu_{b-} D^\mu_{b-} x(t)\Big)(\tau) = \Big(I^\mu_{b-} \big[\alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t)\big]\Big)(\tau).$$

By employing the relations (2.9) and (2.10) for some real constants c_1 and d_1, we obtain

$$x(\tau) - c_1(\tau-a)^{\mu-1} = \frac{1}{\Gamma(\mu)} \int_a^\tau (\tau-t)^{\mu-1} \Big[\alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t)\Big] dt,$$

$$x(\tau) - d_1(b-\tau)^{\mu-1} = \frac{1}{\Gamma(\mu)} \int_\tau^b (t-\tau)^{\mu-1} \Big[\alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t)\Big] dt.$$

Because of $x(a) = x(b) = 0$ and $0 < \mu < 1$, the coefficients c_1 and d_1 have to be zero. At this point, by using the triangle inequality we get

$$|x(\tau)| \le \frac{1}{\Gamma(\mu)} \int_a^\tau |\tau-t|^{\mu-1} |\alpha_1(t)||x(t)| dt + \frac{1}{\Gamma(\mu)} \int_a^\tau |\tau-t|^{\mu-1} \beta_1(t)|u(t)|^{\gamma-1} dt,$$

$$|x(\tau)| \leq \frac{1}{\Gamma(\mu)} \int_\tau^b |t-\tau|^{\mu-1}|\alpha_1(t)||x(t)|dt + \frac{1}{\Gamma(\mu)} \int_\tau^b |t-\tau|^{\mu-1}\beta_1(t)|u(t)|^{\gamma-1}dt,$$

and then, by summing up two above inequalities, we have

$$2|x(\tau)| \leq \frac{1}{\Gamma(\mu)} \int_a^b |\tau-t|^{\mu-1}|\alpha_1(t)||x(t)|dt + \frac{1}{\Gamma(\mu)} \int_a^b |\tau-t|^{\mu-1}\beta_1(t)|u(t)|^{\gamma-1}dt. \tag{3.2}$$

Now, by using Hölder inequality for the second integral (3.2), we obtain

$$\begin{aligned}
&\int_a^b |\tau - t|^{\mu-1}\beta_1(t)|u(t)|^{\gamma-1}dt \\
&= \int_a^b |\tau - t|^{\mu-1}\beta_1^{\frac{1}{\gamma}}(t)\beta_1^{\frac{1}{\alpha}}(t)|u(t)|^{\gamma-1}dt \\
&\leq \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_1(t)|u(t)|^{\alpha(\gamma-1)}dt\Big)^{\frac{1}{\alpha}} \\
&= \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}},
\end{aligned} \tag{3.3}$$

where $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$. Substituting the relation (3.3) into (3.2), we get

$$\begin{aligned}
2|x(\tau)| \leq& \frac{1}{\Gamma(\mu)} \int_a^b |\tau - t|^{\mu-1}|\alpha_1(t)||x(t)|dt \\
&+ \frac{1}{\Gamma(\mu)}\Big(\int_a^b |\tau - t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}}.
\end{aligned} \tag{3.4}$$

Without loss of generality, if in the interval $[a,t]$, we multiply the first equation of the system (3.1) by $u(t)$ and the second one by $x(t)$, respectively, and then add the results, we find that

$$u(t)D_{a^+}^{\mu}x(t) + x(t)D_{a^+}^{\mu}u(t) = \beta_1(t)|u(t)|^{\gamma} - \beta_2(t)|x(t)|^{\beta}.$$

From the above equation, we integrate on the interval $[a,b]$ and use the relations

(2.8) and (2.2) for $0<\mu<1$ and $\tau, s \in (a,b)$ to obtain

$$\int_a^b \beta_1(t)|u(t)|^\gamma dt - \int_a^b \beta_2(t)|x(t)|^\beta dt = \int_a^b u(t) D^\mu_{a+}x(t)dt + \int_a^b x(t)D^\mu_{a+}u(t)dt$$
$$= \int_a^b u(t)D^\mu_{a+}x(t)dt + \int_a^b u(t)D^\mu_{b-}x(t)dt$$
$$= \int_a^b u(t)\Big[D^\mu_{a+}x(t) + D^\mu_{b-}x(t)\Big]dt$$
$$\leq \int_a^b u(t)\Big[\frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_a^t |t-s|^{-\mu}|x(s)|ds$$
$$+ \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_t^b |s-t|^{-\mu}|x(s)|ds\Big]dt$$
$$\leq \int_a^b u(t)\Big[\frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_a^t |t-s|^{-\mu}\max|x(s)|ds$$
$$+ \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_t^b |s-t|^{-\mu}\max|x(s)|ds\Big]dt$$
$$= \int_a^b u(t)|x(\tau)|\Big[\frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_a^t |t-s|^{-\mu}ds + \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_t^b |s-t|^{-\mu}ds\Big]dt$$
$$= (\pm 1)^{-\mu}M\int_a^b u(t)\Big[(D^\mu_{a+}1)(t) + (D^\mu_{b-}1)(t)\Big]dt,$$

where $M = |x(\tau)| = \max|x(s)|_{a<s<b}$. Using the relations (2.5) and (2.1) and taking into account $(I^{1-\mu}_{a^+}u)(b) = (I^{1-\mu}_{b^-}u)(a) = 0$, gives

$$(\pm 1)^{-\mu}M\int_a^b u(t)\Big[(D^\mu_{a+}1)(t) + (D^\mu_{b-}1)(t)\Big]dt$$
$$= (\pm 1)^{-\mu}M\Big[\frac{1}{\Gamma(1-\mu)}\int_a^b (t-a)^{-\mu}u(t)dt + \frac{1}{\Gamma(1-\mu)}\int_a^b (b-t)^{-\mu}u(t)dt\Big]$$
$$= (\pm 1)^{-\mu}M\Big[(I^{1-\mu}_{b^-}u)(a) + (I^{1-\mu}_{a^+}u)(b)\Big] = 0,$$

so

$$\int_a^b \beta_1(t)|u(t)|^\gamma dt \leq \int_a^b \beta_2(t)|x(t)|^\beta dt.$$

Therefore, from (3.4) we get

$$2|x(\tau)| \leq \frac{1}{\Gamma(\mu)}\int_a^b |\tau-t|^{\mu-1}|\alpha_1(t)||x(t)|dt$$
$$+ \frac{1}{\Gamma(\mu)}\Big(\int_a^b |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^b \beta_2(t)|x(t)|^\beta dt\Big)^{\frac{1}{\alpha}}. \tag{3.5}$$

Since $M = |x(\tau)| = \max|x(t)|_{a<t<b}$ and $\beta_2^+(t) = \max\{\beta_2(t), 0\}$, thus (3.5)

yields

$$
\begin{aligned}
2|x(\tau)| \leq & \frac{|x(\tau)|}{\Gamma(\mu)} \int_a^b |\tau - t|^{\mu-1} |\alpha_1(t)| dt \\
& + \frac{|x(\tau)|^{\frac{\beta}{\alpha}}}{\Gamma(\mu)} \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)} \beta_1(t) dt \Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t) dt \Big)^{\frac{1}{\alpha}},
\end{aligned}
$$

and finally

$$
\begin{aligned}
2 \leq & \frac{1}{\Gamma(\mu)} \int_a^b |\tau - t|^{\mu-1} |\alpha_1(t)| dt \\
& + \frac{M^{\frac{\beta}{\alpha}-1}}{\Gamma(\mu)} \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)} \beta_1(t) dt \Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t) dt \Big)^{\frac{1}{\alpha}}.
\end{aligned}
$$

□

Next, we state some illustrative consequences of Theorem 3.2.

Corollary 3.3. *Let the hypotheses of Definition 3.1 hold and α, γ and $\beta_2^+(t)$ are defined as before. In particular case $\beta = \alpha$, for the fractional nonlinear system* (3.1) *the following inequality holds*

$$
2 \leq \frac{1}{\Gamma(\mu)} \int_a^b |t-\tau|^{\mu-1} |\alpha_1(t)| dt + \frac{1}{\Gamma(\mu)} \Big(\int_a^b |t-\tau|^{\gamma(\mu-1)} \beta_1(t) dt \Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t) dt \Big)^{\frac{1}{\alpha}}.
$$

Corollary 3.4. *In the special cases $\beta = \alpha = 2$ and $\gamma = 2$, for the following fractional linear system*

$$
\begin{cases} D^\mu x(t) = \alpha_1(t) x(t) + \beta_1(t) u(t), \\ D^\mu u(t) = -\beta_2(t) x(t) - \alpha_1(t) u(t), \end{cases}
$$

with initial conditions

$$
x(a) = x(b) = 0, \qquad (I_{a^+}^{1-\mu} u)(b) = (I_{b^-}^{1-\mu} u)(a) = 0, \qquad a, b \in \mathbb{R} \ (a < b),
$$

the following inequality holds

$$
2 \leq \frac{1}{\Gamma(\mu)} \int_a^b |t-\tau|^{\mu-1} |\alpha_1(t)| dt + \frac{1}{\Gamma(\mu)} \Big(\int_a^b |t-\tau|^{2(\mu-1)} \beta_1(t) dt \Big)^{\frac{1}{2}} \Big(\int_a^b \beta_2^+(t) dt \Big)^{\frac{1}{2}}.
$$

Theorem 3.5. *In the special case of the system* (3.1)*, for the following fractional nonlinear system*

$$
\begin{cases} D^\mu x(t) = \beta_1(t) |u(t)|^{\gamma-2} u(t), \\ D^\mu u(t) = -\beta_2(t) |x(t)|^{\beta-2} x(t), \end{cases} \tag{3.6}
$$

with initial conditions

$$
x(a) = x(b) = 0, \qquad (I_{a^+}^{1-\mu} u)(b) = (I_{b^-}^{1-\mu} u)(a) = 0, \qquad a, b \in \mathbb{R} \ (a < b),
$$

the following inequalities hold for $0 < \mu < 1$ *and* $\tau \in (a, b)$

$$1 \leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\int_a^{\tau} |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^{\tau} \beta_2^{+}(t)dt\Big), \tag{3.7}$$

$$1 \leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\int_{\tau}^{b} |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_{\tau}^{b} \beta_2^{+}(t)dt\Big), \tag{3.8}$$

$$2^{\alpha} \leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\int_a^{b} |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^{b} \beta_2^{+}(t)dt\Big), \tag{3.9}$$

where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$, $M = \max|x(t)|_{a<t<b}$ *and* $\beta_2^{+}(t) = \max\{\beta_2(t), 0\}$.

Proof. Since $x(a) = x(b) = 0$ and x is not identically zero on $[a, b]$, there exists $\tau \in (a, b)$ such that $M = |x(\tau)| = \max|x(t)|_{a<t<b} > 0$. Using the relation (2.9) and applying the left-sided Riemann-Liouville fractional integral operator I^{μ}_{a+} on both sides of the first equation of the system (3.6) in the interval $[a, \tau]$ and taking into account $x(a) = 0$, we get

$$x(\tau) = \frac{1}{\Gamma(\mu)}\int_a^{\tau} (\tau - t)^{\mu-1}\beta_1(t)|u(t)|^{\gamma-2}u(t)dt,$$

and so

$$|x(\tau)| \leq \frac{1}{\Gamma(\mu)}\int_a^{\tau} |\tau - t|^{\mu-1}\beta_1(t)|u(t)|^{\gamma-1}dt. \tag{3.10}$$

Now, by using Hölder inequality on the right hand side of (3.10), we obtain

$$\begin{aligned}
&\int_a^{\tau} |\tau - t|^{\mu-1}\beta_1(t)|u(t)|^{\gamma-1}dt \\
&= \int_a^{\tau} |\tau - t|^{\mu-1}\beta_1^{\frac{1}{\gamma}}(t)\beta_1^{\frac{1}{\alpha}}(t)|u(t)|^{\gamma-1}dt \\
&\leq \Big(\int_a^{\tau} |\tau - t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^{\tau} \beta_1(t)|u(t)|^{\alpha(\gamma-1)}dt\Big)^{\frac{1}{\alpha}} \\
&= \Big(\int_a^{\tau} |\tau - t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^{\tau} \beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}},
\end{aligned}$$

where $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$. Therefore, the relation (3.10) implies that

$$|x(\tau)| \leq \frac{1}{\Gamma(\mu)}\Big(\int_a^{\tau} |\tau - t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^{\tau} \beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}}. \tag{3.11}$$

Without loss of generality, if in the interval $[a, t]$, we multiply the first equation of the system (3.6) by $u(t)$ and the second one by $x(t)$, respectively, and then add the results, we get

$$u(t)D^{\mu}_{a+}x(t) + x(t)D^{\mu}_{a+}u(t) = \beta_1(t)|u(t)|^{\gamma} - \beta_2(t)|x(t)|^{\beta}.$$

From the above equation, we integrate on the interval $[a,\tau]$ and use the relations (2.8) and (2.2) for $y \in (a,\tau)$ to obtain

$$
\begin{aligned}
&\int_a^\tau \beta_1(t)|u(t)|^\gamma dt - \int_a^\tau \beta_2(t)|x(t)|^\beta dt = \int_a^\tau u(t) D^\mu_{a+} x(t)dt + \int_a^\tau x(t) D^\mu_{a+} u(t)dt \\
&= \int_a^\tau u(t) D^\mu_{a+} x(t)dt + \int_a^\tau u(t) D^\mu_{\tau-} x(t)dt \\
&= \int_a^\tau u(t) \Big[D^\mu_{a+} x(t) + D^\mu_{\tau-} x(t) \Big] dt \\
&\leq \int_a^\tau |u(t)| \Big[\frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_a^t |t-y|^{-\mu} |x(y)| dy \\
&+ \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_t^\tau |y-t|^{-\mu} |x(y)| dy \Big] dt \\
&\leq \int_a^\tau |u(t)| \Big[\frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_a^t |t-y|^{-\mu} \max |x(y)| dy \\
&+ \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_t^\tau |y-t|^{-\mu} \max |x(y)| dy \Big] dt \\
&= \int_a^\tau |u(t)||x(\tau)| \Big[\frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_a^t |t-y|^{-\mu} dy + \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_t^\tau |y-t|^{-\mu} dy \Big] dt \\
&\leq \int_a^b |u(t)||x(\tau)| \Big[\frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_a^t |t-y|^{-\mu} dy + \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_t^b |y-t|^{-\mu} dy \Big] dt \\
&= (\pm 1)^{1-\mu} M \int_a^b u(t) \Big[(D^\mu_{a+} 1)(t) + (D^\mu_{b-} 1)(t) \Big] dt,
\end{aligned}
$$

where $M = |x(\tau)| = \max |x(y)|_{a<y<\tau<b}$. Using the relations (2.5) and (2.1), and taking into account $(I^{1-\mu}_{a+} u)(b) = (I^{1-\mu}_{b-} u)(a) = 0$, we get

$$
\begin{aligned}
&(\pm 1)^{1-\mu} M \int_a^b u(t) \Big[(D^\mu_{a+} 1)(t) + (D^\mu_{b-} 1)(t) \Big] dt \\
&= (\pm 1)^{1-\mu} M \Big[\frac{1}{\Gamma(1-\mu)} \int_a^b (t-a)^{-\mu} u(t) dt + \frac{1}{\Gamma(1-\mu)} \int_a^b (b-t)^{-\mu} u(t) dt \\
&= (\pm 1)^{1-\mu} M \Big[(I^{1-\mu}_{b-} u)(a) + (I^{1-\mu}_{a+} u)(b) \Big] = 0,
\end{aligned}
$$

so

$$\int_a^\tau \beta_1(t)|u(t)|^\gamma dt \leq \int_a^\tau \beta_2(t)|x(t)|^\beta dt.$$

Therefore, from the relation (3.11) we get

$$|x(\tau)| \leq \frac{1}{\Gamma(\mu)} \Big(\int_a^\tau |\tau - t|^{\gamma(\mu-1)} \beta_1(t) dt \Big)^{\frac{1}{\gamma}} \Big(\int_a^\tau \beta_2(t)|x(t)|^\beta dt \Big)^{\frac{1}{\alpha}}. \tag{3.12}$$

Since $M = |x(\tau)| = \max |x(t)|_{a<t<b}$ and $\beta_2^+(t) = \max\{\beta_2(t), 0\}$, thus the

relation (3.12) yields

$$|x(\tau)| \leq \frac{|x(\tau)|^{\frac{\beta}{\alpha}}}{\Gamma(\mu)}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}},$$

and we have

$$1 \leq \frac{M^{\frac{\beta}{\alpha}-1}}{\Gamma(\mu)}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}}. \tag{3.13}$$

Taking the α-th power of both sides of the inequality (3.13), we get

$$1 \leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^\tau \beta_2^+(t)dt\Big).$$

At this point, using the relation (2.10) and applying the right-sided Riemann-Liouville fractional integral operator I_{b-}^{μ} on both sides of the first equation of the system (3.6) in the interval $[\tau, b]$ and taking into account $x(b)=0$, we have

$$|x(\tau)| \leq \frac{1}{\Gamma(\mu)}\int_\tau^b |t-\tau|^{\mu-1}\beta_1(t)|u(t)|^{\gamma-1}dt,$$

from which in a similar procedure to the previous process, we conclude (3.8). Since for $t>0$, the function $J(t)=t^{1-\alpha}$ is convex, so according to the Jensen inequality

$$J\Big(\frac{y+z}{2}\Big) \leq \frac{J(y)+J(z)}{2},$$

with $y=\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt$ and $z=\int_\tau^b |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt$, we obtain

$$\begin{aligned}
&\int_a^b \beta_2^+(t)dt \\
&= \int_a^\tau \beta_2^+(t)dt + \int_\tau^b \beta_2^+(t)dt \\
&\geq \frac{(\Gamma(\mu))^\alpha}{M^{\beta-\alpha}}\Bigg[\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{1-\alpha} + \Big(\int_\tau^b |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{1-\alpha}\Bigg] \\
&\geq \frac{(\Gamma(\mu))^\alpha}{M^{\beta-\alpha}}2\Bigg[\frac{1}{2}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\beta_1(t)dt + \int_\tau^b |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)\Bigg]^{1-\alpha} \\
&= \frac{(2\Gamma(\mu))^\alpha}{M^{\beta-\alpha}}\Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{1-\alpha},
\end{aligned}$$

and accordingly

$$2^\alpha \leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^b \beta_2^+(t)dt\Big).$$

Thus, we easily arrive at the following consequence of Theorem 3.5. □

Corollary 3.6. *In the particular case $\beta = \alpha$, the following inequalities hold*

$$1 \leq \frac{1}{(\Gamma(\mu))^{\alpha}}\Big(\int_a^{\tau} |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^{\tau} \beta_2^{+}(t)dt\Big),$$

$$1 \leq \frac{1}{(\Gamma(\mu))^{\alpha}}\Big(\int_{\tau}^{b} |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_{\tau}^{b} \beta_2^{+}(t)dt\Big),$$

$$2^{\alpha} \leq \frac{1}{(\Gamma(\mu))^{\alpha}}\Big(\int_a^{b} |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^{b} \beta_2^{+}(t)dt\Big).$$

Corollary 3.7. *In the special case $\beta = \alpha = 2$ and $\gamma = 2$, we get the following inequalities*

$$1 \leq \frac{1}{(\Gamma(\mu))^{2}}\Big(\int_a^{\tau} |t-\tau|^{2(\mu-1)}\beta_1(t)dt\Big)\Big(\int_a^{\tau} \beta_2^{+}(t)dt\Big),$$

$$1 \leq \frac{1}{(\Gamma(\mu))^{2}}\Big(\int_{\tau}^{b} |t-\tau|^{2(\mu-1)}\beta_1(t)dt\Big)\Big(\int_{\tau}^{b} \beta_2^{+}(t)dt\Big),$$

$$4 \leq \frac{1}{(\Gamma(\mu))^{2}}\Big(\int_a^{b} |t-\tau|^{2(\mu-1)}\beta_1(t)dt\Big)\Big(\int_a^{b} \beta_2^{+}(t)dt\Big).$$

Corollary 3.8. *In the special case $\mu = 1$, we can immediately deduce that the fractional nonlinear system with the Riemann-Liouville derivative* (3.1) *has the Lyapunov inequality*

$$2 \leq \int_a^b |\alpha_1(t)|dt + M^{\frac{\beta}{\alpha}-1}\Big(\int_a^b \beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^b \beta_2^{+}(t)dt\Big)^{\frac{1}{\alpha}}.$$

Remark 3.9. Let us consider the fractional nonlinear system (3.1) with the fractional Caputo derivative ${}^{C}D^{\mu}, 0 < \mu < 1$, (defined in (2.3) and (2.4)) and boundary conditions

$$x(a) = x(b) = 0, \qquad u(a) = u(b) = 0, \qquad (I_{a^+}^{1-\mu}u)(b) = (I_{b^-}^{1-\mu}u)(a) = 0,$$

where $a, b \in \mathbb{R}$ $(a < b)$. Then, by using the relations (2.11), (2.12), (2.6) and (2.7) and repeating the previous procedures for each mentioned theorems and corollaries, we obtain exactly similar Lyapunov-type inequalities to the fractional nonlinear system with the Caputo derivative.

4. Boundedness of Solutions

Theorem 4.1. *For $|\beta_2(t)| \in \mathbb{L}^p[t_0,\infty)$ and $1 \leq p < \infty$, if*

$$\int^{\infty} \beta_1(t)dt < \infty, \qquad \int^{\infty} |\beta_2(t)|dt < \infty,$$

then every weakly oscillatory proper solution of (3.6) *is bounded on $I = [t_0, \infty)$.*

Proof. Let $(x(t), u(t))$ be any nontrivial weakly oscillatory proper solution of the fractional nonlinear system (3.6) on I such that $x(t)$ has a sequence of zeros tending to infinity. If we assume that the conclusion is wrong and $\limsup |x(t)| = \infty$, then

$$\forall\ M_1 > 0, \qquad \exists\ 0 < T = T(M_1): \quad \forall\ t > T, \quad |x(t)| > M_1.$$

Since x is an oscillatory solution, there exists an interval (t_1, t_2) with $t_1 > T$ such that $x(t_1) = x(t_2) = 0$ and $|x(t)| > 0$ on (t_1, t_2). Therefore

$$\exists\ \tau \in (t_1, t_2): \quad M_1 < \max |x(t)|_{t_1<t<t_2} = |x(\tau)| = M.$$

According to the inequalities in Theorem 3.5 for every $t_1 \geq T$, we let

$$\int_{t_1}^{\infty} |\beta_2(t)| dt < 1, \tag{4.1}$$

$$\int_{t_1}^{\infty} \beta_1^{\nu}(t) dt < \Big(M^{\alpha-\beta}(\Gamma(\mu))^{\alpha}\delta^{\frac{1-\alpha}{\rho}}\Big)^{\frac{\nu}{\alpha-1}}, \tag{4.2}$$

where $\delta = \frac{|b-\tau|^{\rho\gamma(\mu-1)+1}+|\tau-a|^{\rho\gamma(\mu-1)+1}}{\rho\gamma(\mu-1)+1}$. By employing the Hölder inequality on the right hand side of (3.9) with indices ρ and ν, we get

$$\begin{aligned}
2^{\alpha} &\leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^b \beta_2^{+}(t)dt\Big)\\
&\leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\int_a^b |t-\tau|^{\rho\gamma(\mu-1)}dt\Big)^{\frac{\alpha-1}{\rho}}\Big(\int_a^b \beta_1^{\nu}(t)dt\Big)^{\frac{\alpha-1}{\nu}}\Big(\int_a^b \beta_2^{+}(t)dt\Big)\\
&= \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\frac{|b-\tau|^{\rho\gamma(\mu-1)+1}+|\tau-a|^{\rho\gamma(\mu-1)+1}}{\rho\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{\rho}}\\
&\quad\times\Big(\int_a^b \beta_1^{\nu}(t)dt\Big)^{\frac{\alpha-1}{\nu}}\Big(\int_a^b \beta_2^{+}(t)dt\Big).
\end{aligned}\tag{4.3}$$

Substituting (4.1) and (4.2) into the inequality (4.3), we obtain

$$\begin{aligned}
2^{\alpha} &\leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\frac{|b-\tau|^{\rho\gamma(\mu-1)+1}+|\tau-a|^{\rho\gamma(\mu-1)+1}}{\rho\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{\rho}}\\
&\quad\times\Big(\int_{t_1}^{t_2} \beta_1^{\nu}(t)dt\Big)^{\frac{\alpha-1}{\nu}}\Big(\int_{t_1}^{t_2} \beta_2^{+}(t)dt\Big)\\
&\leq \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^{\alpha}}\Big(\frac{|b-\tau|^{\rho\gamma(\mu-1)+1}+|\tau-a|^{\rho\gamma(\mu-1)+1}}{\rho\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{\rho}}\\
&\quad\times\Big(\int_{t_1}^{\infty} \beta_1^{\nu}(t)dt\Big)^{\frac{\alpha-1}{\nu}}\Big(\int_{t_1}^{\infty} |\beta_2(t)|dt\Big) \leq 1,
\end{aligned}$$

so $|x(t)|$ is bounded on I. Thus

$$\forall\ t \in I, \quad \exists\ K > 0: \ |x(t)| \leq K.$$

Now, we show that $|u(t)|$ is bounded. For this purpose, we apply the left-sided Riemann-Liouville fractional integral operator $I^{\mu}_{\tau+}$ on both sides of the second equation of the system (3.6) in the interval $[\tau, t]$ $(\tau \leq t \leq t_2)$,

$$u(t) - c_1(t-\tau)^{\mu-1} = -\frac{1}{\Gamma(\mu)}\int_{\tau}^{t}(t-s)^{\mu-1}\beta_2(s)|x(s)|^{\beta-2}x(s)ds.$$

So

$$|u(t)| \leq \frac{1}{\Gamma(\mu)}\int_{\tau}^{t}|t-s|^{\mu-1}|\beta_2(s)||x(s)|^{\beta-1}ds + |c_1||t-\tau|^{\mu-1},$$

which by using the Hölder inequality, we get

$$\begin{aligned}
|u(t)| &\leq \frac{1}{\Gamma(\mu)}\int_{\tau}^{t}|t-s|^{\mu-1}|\beta_2(s)||x(s)|^{\beta-1}ds + |c_1||t-\tau|^{\mu-1}\\
&\leq \frac{\left(\int_{\tau}^{t}|t-s|^{q(\mu-1)}ds\right)^{\frac{1}{q}}}{\Gamma(\mu)}\Big(\int_{\tau}^{t}|\beta_2(s)|^{p}|x(s)|^{p(\beta-1)}ds\Big)^{\frac{1}{p}} + |c_1||t-\tau|^{\mu-1}\\
&\leq \frac{1}{\Gamma(\mu)}\Big(\frac{|\tau-s|^{q(\mu-1)+1}}{q(\mu-1)+1}\Big)^{\frac{1}{q}}K^{\beta-1}\Big(\int_{\tau}^{t}|\beta_2(s)|^{p}ds\Big)^{\frac{1}{p}} + |c_1||t-\tau|^{\mu-1}\\
&\leq \frac{1}{\Gamma(\mu)}\Big(\frac{|\tau-s|^{q(\mu-1)+1}}{q(\mu-1)+1}\Big)^{\frac{1}{q}}K^{\beta-1}\Big(\int_{t_1}^{\infty}|\beta_2(s)|^{p}ds\Big)^{\frac{1}{p}} + |c_1||t-\tau|^{\mu-1}.
\end{aligned}$$

Since $|\beta_2(t)| \in \mathbb{L}^p[t_0,\infty)$, thus

$$\Big(\int_{t_1}^{\infty}|\beta_2(t)|^{p}dt\Big)^{\frac{1}{p}} \leq \infty,$$

which implies that $|u(t)|$ is bounded on I and

$$|u(t)| \leq \infty.$$

□

Theorem 4.2. *For $\beta_2^{+}(t) \in \mathbb{L}^p[t_0,\infty)$ and $1 \leq p < \infty$, if $(x(t), u(t))$ is any weakly oscillatory proper solution of the fractional nonlinear system* (3.6) *with $\beta_1(t) = 1$, then the distance between consecutive zeros of the first component of $(x(t), u(t))$ tends to infinity as $t \to +\infty$.*

Proof. We suppose that $(x(t), u(t))$ is a nontrivial weakly oscillatory proper solution of the fractional nonlinear system (3.6) with $\beta_1(t) = 1$ on I such that $x(t)$ has a sequence of zeros tending to $+\infty$ and the conclusion is wrong. Then the sequence $\{t_n\}$ of zeros of $x(t)$ has a subsequence $\{t_{n_m}\}$ such that

$$\exists N > 0\ \forall m\ |t_{n_{m+1}} - t_{n_m}| \leq N.$$

Let

$$\forall\ m\ \ \exists s_{n_m} \in (t_{n_{m+1}}, t_{n_m}):\quad \max|x(t)| = |x(s_{n_m})| = M,$$

then $\forall\, m \quad |s_{n_m} - t_{n_m}| < N$. Suppose

$$(4.4)\quad \Big(\int_{t_{n_m}}^{\infty} (\beta_2^+(s))^\mu ds\Big)^{\frac{1}{\mu}} \le M^{\alpha-\beta}(\Gamma(\mu))^\alpha \Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{1-\alpha}{q}} N^{\frac{1-\alpha}{p}-\frac{1}{q}}.$$

Applying the Hölder inequality for the right hand side of (3.7) with indices p and q, we have

$$\begin{aligned}
1 &\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\int_a^\tau |t-\tau|^{\gamma(\mu-1)}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^\tau \beta_2^+(t)dt\Big)\\
&\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\int_a^\tau |t-\tau|^{q\gamma(\mu-1)}dt\Big)^{\frac{\alpha-1}{q}}\Big(\int_a^\tau \beta_1^p(t)dt\Big)^{\frac{\alpha-1}{p}}\Big(\int_a^\tau \beta_2^+(t)dt\Big)\\
&\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}}\Big(\int_a^\tau \beta_1^p(t)dt\Big)^{\frac{\alpha-1}{p}}\Big(\int_a^\tau \beta_2^+(t)dt\Big),
\end{aligned}$$

from which by using the fact that $\beta_1(t) = 1$, we get

$$(4.5)\qquad 1 \le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}}\Big(s_{n_m}-t_{n_m}\Big)^{\frac{\alpha-1}{p}}\int_{t_{n_m}}^{s_{n_m}} \beta_2^+(t)dt.$$

Now, by employing the Hölder inequality for the right hand side of (4.5) with indices p and q, we have

$$\begin{aligned}
1 &\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}}\Big(s_{n_m}-t_{n_m}\Big)^{\frac{\alpha-1}{p}}\int_{t_{n_m}}^{s_{n_m}} \beta_2^+(t)dt\\
&\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}} \frac{\Big(\int_{t_{n_m}}^{s_{n_m}}(\beta_2^+(t))^p dt\Big)^{\frac{1}{p}}\Big(\int_{t_{n_m}}^{s_{n_m}} dt\Big)^{\frac{1}{q}}}{\Big(s_{n_m}-t_{n_m}\Big)^{\frac{1-\alpha}{p}}}\\
&= \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}}\Big(s_{n_m}-t_{n_m}\Big)^{\frac{\alpha-1}{p}+\frac{1}{q}}\Big(\int_{t_{n_m}}^{s_{n_m}}(\beta_2^+(t))^p dt\Big)^{\frac{1}{p}}\\
(4.6)\quad &\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}}\Big(s_{n_m}-t_{n_m}\Big)^{\frac{\alpha-1}{p}+\frac{1}{q}}\Big(\int_{t_{n_m}}^{\infty}(\beta_2^+(t))^p dt\Big)^{\frac{1}{p}}.
\end{aligned}$$

Substituting the inequality (4.4) into (4.6), we get

$$\begin{aligned}
1 &\le \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}}\Big(s_{n_m}-t_{n_m}\Big)^{\frac{\alpha-1}{p}+\frac{1}{q}}\Big(\int_{t_{n_m}}^{\infty}(\beta_2^+(t))^p dt\Big)^{\frac{1}{p}}\\
&< \frac{M^{\beta-\alpha}}{(\Gamma(\mu))^\alpha}\Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{\alpha-1}{q}} N^{\frac{\alpha-1}{p}+\frac{1}{q}} M^{\alpha-\beta}(\Gamma(\mu))^\alpha\\
&\times \Big(\frac{|\tau-a|^{q\gamma(\mu-1)+1}}{q\gamma(\mu-1)+1}\Big)^{\frac{1-\alpha}{q}} N^{\frac{1-\alpha}{p}-\frac{1}{q}} = 1.
\end{aligned}$$

This contradiction completes the proof. □

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