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DISJOINT DISTRIBUTIONALLY CHAOTIC ABSTRACT PDEs

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Abstract. In this paper, we analyze disjoint distributionally chaotic abstract non-degenerate partial differential equations in Fréchet spaces, with integer or Caputo time-fractional derivatives. We present several illustrative examples and applications of our established results.

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1. Introduction and preliminaries

Linear topological dynamics of continuous operators in Banach and Fréchet spaces is an extremely popular field of functional analysis. Basic information about this subject can be obtained by consulting the monographs [1] by F. Bayart, E. Matheron and [16] by K.-G. Grosse-Erdmann, A. Peris.

The notion of distributional chaos was introduced by B. Schweizer and J. Smítal in [29] (1994) for interval maps and after that seriously reconsidered by a great number of authors including P. Oprocha [28] (2009). For linear continuous operators in Banach spaces, distributional chaos was firstly considered by J. Duan et al [14] (1999). N. C. Bernardes Jr. et al [5] (2013) were the first who systematically analyzed distributional chaos for linear continuous operators in Fréchet spaces (cf. also the reserach study of J. A. Conejero et al [10] (2016) for a correspoding study of linear not necessarily continuous operators). Some specific properties of distributionally chaotic operators in Banach spaces have been recently investigated by N. C. Bernardes Jr. et al [6] (2018).

Disjoint hypercyclic linear operators were introduced independently by L. Bernal–González [4] (2007) and J. Bès, A. Peris [8] (2007). Similar concepts, like disjoint mixing property and disjoint supercyclicity, have been analyzed by a great number of authors after that (for further information about disjoint hypercyclic operators and their generalizations, we refer the reader to [7], [18], [27] and references cited therein.

The main aim of this paper is to continue our recent research study [18] of disjoint distributional chaos in Fréchet spaces by investigating the abstract partial differential equations in Fréchet spaces with integer or Caputo time-fractional derivatives (concerning distributional chaos in metric and Fréchet

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spaces, one may refer e.g. to [3], [6], [28] and references cited therein). We focus our attention to the analysis of disjoint distributionally chaotic integrated C-semigroups, as a rather general concept for the investigations of abstract partial differential equations of the first order. We also consider disjoint distributionally chaotic properties of abstract time-fractional differential equations with Caputo derivatives; strictly speaking, we analyze disjoint distributional chaos for ζ-times C-regularized resolvent families ($\zeta \in (0,2) \setminus \{1\}$). For the sake of brevity, we consider only non-degenerate abstract partial differential equations here (for topological dynamics of abstract degenerate partial differential equations, the reader may consult our joint paper with V. Fedorov [15], the forthcoming monograph [25] and references cited therein).

The organization and main ideas of this paper can be described as follows. After giving some necessary explanations about the notation and general framework we are working in, we collect the basic material about integrated C-semigroups and ζ-times C-regularized resolvent families in two separate subsections, Subsection 1.1 and Subsection 1.2. The second section of the paper is devoted to the study of disjoint distributional chaos for integrated C-semigroups, while the third section of the paper is devoted to the study of disjoint distributional chaos for ζ-times C-regularized resolvent families ($\zeta \in (0,2) \setminus \{1\}$). Although not used explicitly, as for single operators [18], we also provide definitions of disjoint distributinally near to zero vectors, disjoint distributionally unbounded vectors and disjoint distributionally irregular vectors for these solution operator families. Without any doubt, the main result of paper is Theorem 2.3, which provides an efficient tool for proving several other structural results of ours. In addition to the above, a great deal of illustrative examples and applications is presented.

We use the standard notation in the sequel. By X and Y we denote two non-trivial Fréchet spaces over the same field of scalars $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and assume that the topologies of X and Y are induced by the fundamental systems $(p_n)_{n\in\mathbb{N}}$ and $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms, respectively (separability of X and Y will be assumed a priori in future). The translation invariant metric $d : X \times X \to [0,\infty)$, defined by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \ x,\ y \in X, \tag{1.1}$$

satisfies the following properties: $d(x+u, y+v) \le d(x,y)+d(u,v)$, $x,\ y,\ u,\ v \in X$; $d(cx,cy) \le (|c|+1)d(x,y)$, $c \in \mathbb{K},\ x,\ y \in X$, and $d(\alpha x, \beta x) \ge \frac{|\alpha-\beta|}{1+|\alpha-\beta|} d(0,x)$, $x \in X$, $\alpha,\ \beta \in \mathbb{K}$. Define the translation invariant metric $d_Y : Y \times Y \to [0,\infty)$ by replacing $p_n(\cdot)$ with $p_n^Y(\cdot)$ in (1.1). If $(X, \|\cdot\|)$ or $(Y, \|\cdot\|_Y)$ is a Banach space, then it will be assumed that the distance of two elements $x,\ y \in X$ ($x,\ y \in Y$) is given by $d(x,y) := \|x-y\|$ ($d_Y(x,y) := \|x-y\|_Y$). Keeping in mind this assumtation, our structural results clarified in Fréchet spaces remain true in the case that X or Y is a Banach space.

We assume that $N \in \mathbb{N}$ and $N \ge 2$. Then the fundamental system of increasing seminorms $(\mathbf{p}_n^{Y^N})_{n\in\mathbb{N}}$, where $\mathbf{p}_n^{Y^N}(x_1, \cdots, x_N) := \sum_{j=1}^{N} p_n^Y(x_j)$, $n \in \mathbb{N}$

($x_j \in Y$ for $1 \leq j \leq N$), induces the topology on the Fréchet space Y^N. The translation invariant metric

$$\mathrm{d}_{Y^N}(\vec{x}, \vec{y}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mathbf{p}_n(\vec{x} - \vec{y})}{1 + \mathbf{p}_n(\vec{x} - \vec{y})}, \quad \vec{x},\ \vec{y} \in Y^N,$$

is strongly equivalent with the metric

$$d_{Y^N}(\vec{x}, \vec{y}) := \max_{1 \leq j \leq N} d_Y(x_j, y_j), \quad \vec{x} = (x_1, \cdots, x_N) \in Y^N,\ \vec{y} = (y_1, \cdots, y_N) \in Y^N.$$

In the case that Y is a Banach space, then Y^N is likewise a Banach space and, in this case, it will be assumed that the distance in Y^N is given by $d_{Y^N}(\vec{x}, \vec{y}) = \max_{1 \leq j \leq N} \|x_j - y_j\|_Y$, $\vec{x} \in Y^N$, $\vec{y} \in Y^N$.

Suppose that $C \in L(X)$ is injective and A is a closed linear operator with domain and range contained in X. By $D(A)$, $R(A)$, $N(A)$ and $\sigma_p(A)$ we denote the domain, range, kernel space and the point spectrum of A, respectively. Set $p_n^C(x) := p_n(C^{-1}x)$, $n \in \mathbb{N}$, $x \in R(C)$. Then $p_n^C(\cdot)$ is a seminorm on $R(C)$ and the calibration $(p_n^C)_{n \in \mathbb{N}}$ induces a Fréchet locally convex topology on $R(C)$; we denote this space simply by $[R(C)]$. Let us recall that $[R(C)]$ is separable since X is as well as that $[R(C)]$ is a Banach space (complex Hilbert space) provided that X is. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \Big\{\lambda \in \mathbb{K} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(X)\Big\}.$$

Set, finally, $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$, $\mathbb{C}_- := \{z \in \mathbb{C} : \Re z < 0\}$, $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_- := (-\infty, 0)$, $\mathbb{K}_+ := \{\mathbb{C}_+,\ \mathbb{R}_+\}$, $\mathbb{K}_- := \{\mathbb{C}_-,\ \mathbb{R}_-\}$, $\Sigma_\alpha := \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| < \alpha\}$ ($\alpha \in (0, \pi]$), $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ and $\mathbb{N}_n := \{1, \cdots, n\}$ ($s \in \mathbb{R}$, $n \in \mathbb{N}$), $g_\zeta(t) := t^{\zeta - 1}/\Gamma(\zeta)$ ($t > 0$, $\zeta > 0$) and recall that the upper density of a set $D \subseteq [0, \infty)$ is defined by

$$\overline{dens}(D) := \limsup_{t \to +\infty} \frac{m(D \cap [0, t])}{t},$$

where m denotes the Lebesgue measure on $[0, \infty)$.

We need the following notion from [18]:

Definition 1.1. Suppose that, for every $j \in \mathbb{N}_N$ and $k \in \mathbb{N}$, $A_{j,k} : D(A_{j,k}) \subseteq X \to Y$ is a linear operator and $\tilde{X}$ is a closed linear subspace of X. Then we say that the sequence $((A_{j,k})_{k \in \mathbb{N}})_{1 \leq j \leq N}$ is disjoint $\tilde{X}$-distributionally chaotic, $(d, \tilde{X})$-distributionally chaotic for short, iff there exist an uncountable set $S \subseteq \bigcap_{j=1}^{N} \bigcap_{k=1}^{\infty} D(A_{j,k}) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have

$$\overline{dens}\Bigg(\bigcap_{j \in \mathbb{N}_N} \{k \in \mathbb{N} : d_Y\big(A_{j,k}x, A_{j,k}y\big) \geq \sigma\}\Bigg) = 1, \text{ and}$$

$$\overline{dens}\Bigg(\bigcap_{j \in \mathbb{N}_N} \{k \in \mathbb{N} : d_Y\big(A_{j,k}x, A_{j,k}y\big) < \epsilon\}\Bigg) = 1.$$

The sequence $((A_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ is said to be densely $(d,\tilde{X})$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. A finite sequence $(A_j)_{1\leq j\leq N}$ of closed linear operators on X is said to be (densely) $\tilde{X}$-distributionally chaotic iff the sequence $((A_{j,k} \equiv A_j^k)_{k\in\mathbb{N}})_{1\leq j\leq N}$ is. The set S is said to be $(d,\sigma_{\tilde{X}})$-scrambled set ((d,σ)-scrambled set in the case that $\tilde{X}=X$) of $((A_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ $((A_j)_{1\leq j\leq N})$; in the case that $\tilde{X}=X$, then we also say that the sequence $((A_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N}$ $((A_j)_{1\leq j\leq N})$ is disjoint distributionally chaotic, d-distributionally chaotic for short.

1.1. Integrated C-semigroups

The following definition is fundamental in the theory of abstract ill-posed differential equations of first order (cf. [22, 19] for more details on the subject):

Definition 1.2. Suppose that $\alpha\geq 0$ and A is a closed linear operator. If there exists a strongly continuous operator family $(S_\alpha(t))_{t\geq 0}\subseteq L(X)$ such that:

(i) $S_\alpha(t)A\subseteq AS_\alpha(t)$, $t\geq 0$,

(ii) $S_\alpha(t)C=CS_\alpha(t)$, $t\geq 0$,

(iii) for all $x\in X$ and $t\geq 0$: $\int_0^t S_\alpha(s)x\,ds\in D(A)$ and

$$A\int_0^t S_\alpha(s)x\,ds=S_\alpha(t)x-g_{\alpha+1}(t)Cx,$$

then it is said that A is a subgenerator of a (global) α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$.

If $\alpha=0$, then $(S_0(t))_{t\geq 0}$ is also said to be a C-regularized semigroup with subgenerator A (we refer the reader to [22] for definition of an entire C-regularized group and its integral generator (subgenerator)). The integral generator of $(S_\alpha(t))_{t\geq 0}$ is defined by

$$\hat{A}:=\Biggl\{(x,y)\in X\times X:S_\alpha(t)x-g_{\alpha+1}(t)Cx=\int_0^t S_\alpha(s)y\,ds,\ t\geq 0\Biggr\}.$$

Let us recall that the integral generator of $(S_\alpha(t))_{t\geq 0}$ is a closed linear operator which extends any subgenerator of $(S_\alpha(t))_{t\geq 0}$. Furthermore, for any subgenerator A of $(S_\alpha(t))_{t\geq 0}$, the following equality holds $\hat{A}=C^{-1}AC$.

Denote by $Z_1(A)$ the space consisting of those elements $x\in X$ for which there exists a unique X-valued continuous mapping satisfying $\int_0^t u(s,x)\,ds\in D(A)$ and $A\int_0^t u(s,x)\,ds=u(t,x)-x$, $t\geq 0$, i.e., the unique mild solution of the corresponding Cauchy problem (ACP_1) :

$$(ACP_1):u'(t)=Au(t),\ t\geq 0,\ u(0)=x.$$

If A is a subgenerator (the integral generator) of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$, then there is only one (trivial) mild solution of (ACP_1) with $x=0$, so that $Z_1(A)$ is a linear subspace of X. Moreover, for every number $\beta>\alpha$, the operator A is a subgenerator (the integral generator) of a global β-times integrated C-semigroup $(S_\beta(t)\equiv(g_{\beta-\alpha}*S_\alpha\cdot)(t))_{t\geq 0}$. As it is well known, the space $Z_1(A)$ consists exactly of those elements $x\in X$ for which the mapping $t\mapsto C^{-1}S_{\lceil\alpha\rceil}(t)x$, $t\geq 0$ is well defined and $\lceil\alpha\rceil$-times continuously differentiable on $[0,\infty)$; see e.g. [22]. As it is usually done in the theory of C-distribution semigroups, we set

$$\mathcal{G}(\varphi)x:=(-1)^{\lceil\alpha\rceil}\int\limits_0^\infty \varphi^{(\lceil\alpha\rceil)}(t)S_{\lceil\alpha\rceil}(t)x\,dt,\quad \varphi\in\mathcal{D}_{\mathbb{K}},\ x\in X$$

and

$$G\bigl(\delta_t\bigr)x:=\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}C^{-1}S_{\lceil\alpha\rceil}(t)x,\quad t\geq 0,\ x\in Z_1(A);$$

here $\mathcal{D}_{\mathbb{K}}$ denotes the space of $\mathbb{K}$-valued smooth test functions with compact support contained in K. Then the following holds: $G(\delta_t)(Z_1(A))\subseteq Z_1(A),\ t\geq 0$, $G(\delta_t)C\subseteq CG(\delta_t)$, $t\geq 0$ and

$$G\bigl(\delta_s\bigr)G\bigl(\delta_t\bigr)x=G\bigl(\delta_{t+s}\bigr)x,\ t,\ s\geq 0,\ x\in Z_1(A). \tag{1.2}$$

Is is also known that the solution space $Z_1(A)$ is independent of the choice of $(S_\alpha(t))_{t\geq 0}$ in the following sense: If $C_1\in L(X)$ is another injective operator with $C_1A\subseteq AC_1$, $\gamma\geq 0$, $x\in X$ and A is a subgenerator (the integral generator) of a global γ-times integrated C_1-semigroup $(S^\gamma(t))_{t\geq 0}$, then the mapping $t\mapsto C^{-1}S_{\lceil\alpha\rceil}(t)x$, $t\geq 0$ is well defined and $\lceil\alpha\rceil$-times continuously differentiable on $[0,\infty)$ iff the mapping $t\mapsto C_1^{-1}S^{\lceil\gamma\rceil}(t)x$, $t\geq 0$ is well defined and $\lceil\gamma\rceil$-times continuously differentiable on $[0,\infty)$. In this case, we have $u(t;x):=G(\delta_t)x=\frac{d^{\lceil\gamma\rceil}}{dt^{\lceil\gamma\rceil}}C_1^{-1}S^{\lceil\gamma\rceil}(t)x$, $t\geq 0$ is a unique mild solution of the corresponding Cauchy problem (ACP_1).

The notions of exponential equicontinuity and analyticity of integrated C-semigroups are well known; the basic results about integrated C-cosine functions can be found in [22, 19], as well.

1.2. ζ-Times C-regularized resolvent families ($\zeta\in(0,2)\setminus\{1\}$)

The following definition has been introduced by M. Li, Q. Zheng and J. Zhang in [26] (see [25, 22, 19] for more details about abstract time-fractional differential equations):

Definition 1.3. Suppose that $\zeta>0$ and A is a closed linear operator on X. A strongly continuous operator family $(R_\zeta(t))_{t\geq 0}$ is said to be a ζ-times C-regularized resolvent family having A as a subgenerator iff the following holds:

(i) $R_\zeta(t)A\subseteq AR_\zeta(t),\ t\geq 0$, $R_\zeta(0)=C$ and $CA\subseteq AC$,

(ii) $R_\zeta(t)C=CR_\zeta(t),\ t\geq 0$ and

(iii) $R_\zeta(t)x = Cx + \int_0^t g_\zeta(t-s)AR_\zeta(s)x\,ds,\ t \geq 0,\ x \in D(A)$.

In the case $C = I$, then we also say that $(R_\zeta(t))_{t\geq 0}$ is a ζ-times regularized resolvent family with subgenerator A.

The integral generator of $(R_\zeta(t))_{t\geq 0}$ is defined by

$$\hat{A} := \left\{(x,y) \in X \times X : R_\zeta(t)x - Cx = \int_0^t g_\zeta(t-s)R_\zeta(s)y\,ds \text{ for all } t \geq 0\right\},$$

and it is a closed linear operator which extends any subgenerator of $(R_\zeta(t))_{t\geq 0}$.

Let $m := \lceil \zeta \rceil$. The Caputo fractional derivative $\mathbf{D}_t^\zeta u(t)$ is defined for those functions $u \in C^{m-1}([0,\infty) : X)$ for which $g_{m-\zeta} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty) : X)$, by

$$\mathbf{D}_t^\zeta u(t) := \frac{d^m}{dt^m}\left[g_{m-\zeta} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

The abstract evolution equation

$$\mathbf{D}_t^\zeta u(t) = Au(t),\ t > 0,\ u(0) = x,\ u^{(k)}(0) = 0,\ k = 1, \cdots, m-1, \tag{1.3}$$

is well posed in the sense of [2, Definition 2.2] iff the abstract Volterra equation

$$u(t;x) = x + \int_0^t g_\zeta(t-s)Au(s;x)\,ds,\ t \geq 0, \tag{1.4}$$

is well posed in the usual sense ([22]). Suppose that A is a subgenerator of an ζ-times C-regularized resolvent family $(R_\zeta(t))_{t\geq 0}$, and

$$R_\zeta(t)x = Cx + A\int_0^t g_\zeta(t-s)R_\zeta(s)x\,ds,\ t \geq 0,\ x \in X. \tag{1.5}$$

Denote by $Z_\zeta(A)$ the set consisting of those vectors $x \in X$ such that $R_\zeta(t)x \in R(C)$, $t \geq 0$ and the mapping $t \mapsto C^{-1}R_\zeta(t)x$, $t \geq 0$ is continuous. Then $R(C) \subseteq Z_\zeta(A)$, and $x \in Z_\zeta(A)$ iff there exists a unique strong solution of (1.4); if this is the case, the unique strong solution of (1.4) is given by $u(t;x) = C^{-1}R_\zeta(t)x$, $t \geq 0$. In the sequel, we assume the validity of (1.5) a priori.

Denote by $E_\beta(z)$ the Mittag-Leffler function $E_\beta(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta n+1)}$, $z \in \mathbb{C}$, where $\beta > 0$. Suppose, further, that $\zeta \in (0,2) \setminus \{1\}$ and $l \in \mathbb{N} \setminus \{1\}$. We will use the following asymptotic formulae for the Mittag-Leffler functions ([2]):

$$E_\zeta(z) = \frac{1}{\zeta}e^{z^{1/\zeta}} + \varepsilon_\zeta(z),\ |\arg(z)| < \zeta\pi/2, \tag{1.6}$$

and

$$E_\zeta(z) = \varepsilon_\zeta(z),\ |\arg(-z)| < \pi - \zeta\pi/2, \tag{1.7}$$

where

$$\varepsilon_\zeta(z) = \sum_{n=1}^{l-1} \frac{z^{-n}}{\Gamma(1-\zeta n)} + O(|z|^{-l}),\ |z| \to \infty. \tag{1.8}$$

2. Disjoint distributionally chaotic properties of abstract PDEs of the first order

In [18], we have introduced and analyzed twelve different types of disjoint distributional chaos for multivalued linear operators in Fréchet spaces. For the sake of simplicity, we will consider here only one type of disjoint distributional chaos for integrated C-semigroups, disjoint distributional chaos of type 1. This is the most intriguing type of disjoint distributional chaos considered in [18] because it is the strongest one and implies all others (we will not particularly emphasize further that this is disjoint distributional chaos of type 1):

Definition 2.1. Let $\alpha_j \geq 0$, let $C_j \in L(X)$ be injective for all $j \in \mathbb{N}_N$ and let $(S_{\alpha_j}(t))_{t\geq 0}$ be a global α_j-times integrated C_j-semigroup with the integral generator A_j $(j \in \mathbb{N}_N)$. Suppose that $\tilde{X}$ is a closed linear subspace of X. Denote by $t \mapsto G_j(\delta_t)x$, $t \geq 0$ the unique mild solution of the corresponding Cauchy problem (ACP_1), with the operator A replaced by A_j therein $(j \in \mathbb{N}_N)$. Then we say that $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ are disjoint $\tilde{X}$-distributionally chaotic, $(d, \tilde{X})$-distributionally chaotic in short, iff there exist an uncountable set $S \subseteq \bigcap_{j=1}^{N} Z_1(A_j) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $j \in \mathbb{N}_N$ and $t \geq 0$ we have that

$$\overline{dens}\Biggl(\bigcap_{j\in\mathbb{N}_N} \bigl\{t \geq 0 : d_Y\bigl(G_j(\delta_t)x, G_j(\delta_t)y\bigr) \geq \sigma\bigr\}\Biggr) = 1,\ \text{and}$$
$$\overline{dens}\Biggl(\bigcap_{j\in\mathbb{N}_N} \bigl\{t \geq 0 : d_Y\bigl(G_j(\delta_t)x, G_j(\delta_t)y\bigr) < \epsilon\bigr\}\Biggr) = 1.$$

The sequence $(S_{\alpha_j}(t))_{t\geq 0}$ is said to be densely $(d, \tilde{X})$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. The set S is said to be $(d, \sigma_{\tilde{X}})$-scrambled set ((d, σ)-scrambled set in the case that $\tilde{X} = X$) of $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$; in the case that $\tilde{X} = X$, then we also say that the sequence $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ is (densely) disjoint distributionally chaotic, (densely) d-distributionally chaotic in short.

Now we introduce the notion of disjoint distributionally irregular vectors for integrated C-semigroups:

Definition 2.2. Let $\alpha_j \geq 0$, let $C_j \in L(X)$ be injective for all $j \in \mathbb{N}_N$ and let $(S_{\alpha_j}(t))_{t\geq 0}$ be a global α_j-times integrated C_j-semigroup with the integral generator A_j $(j \in \mathbb{N}_N)$. Suppose that $\tilde{X}$ is a closed linear subspace of X, $m \in \mathbb{N}$ and $x \in \bigcap_{j=1}^{N} Z_1(A_j) \cap \tilde{X}$. Denote by $t \mapsto G_j(\delta_t)x$, $t \geq 0$ the unique mild solution of the corresponding Cauchy problem (ACP_1), with the operator A replaced by A_j therein $(j \in \mathbb{N}_N)$. Then we say that:

(i) x is disjoint distributionally near to 0 for $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ iff there exists $A \subseteq [0,\infty)$ such that $\overline{Dens}(A) = 1$ and $\lim_{s\to\infty, s\in A} G_j(\delta_s)x = 0$ for all $j \in \mathbb{N}_N$;

(ii) x is disjoint distributionally m-unbounded for $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ iff there exists $B \subseteq [0,\infty)$ such that $\overline{Dens}(B) = 1$ and $\lim_{s\to\infty, s\in B} p_m(G_j(\delta_s)x) = 0$ for all $j \in \mathbb{N}_N$; x is disjoint distributionally unbounded for the tuple $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ iff there exists $q \in \mathbb{N}$ such that x is disjoint distributionally q-unbounded for $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$;

(iii) x is a disjoint $\tilde{X}$-distributionally irregular vector for $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ (disjoint distributionally irregular vector for $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ simply, in the case that $\tilde{X} = X$) iff x is both disjoint distributionally near to 0 and disjoint distributionally unbounded.

The following important result is a continuous analogue of [18, Theorem 4.3]. It also provides an extension of [10, Theorem 4.1] for disjoint distributional chaos:

Theorem 2.3. *Suppose that X_0 is a dense linear subspace of X, $(T_j(t))_{t\geq 0} \subseteq L(X,Y)$ is a strongly continuous operator family for each $j \in \mathbb{N}_N$, as well as:*

(a) $\lim_{t\to\infty} T_j(t)x = 0$, $x \in X_0$, $j \in \mathbb{N}_N$,

(b) there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq [0,\infty)$ such that $\overline{Dens}(B) = 1$, and $\lim_{t\to\infty, t\in B} p_m(T_j(t)x) = \infty$ for each $j \in \mathbb{N}_N$, resp. $\lim_{t\to\infty, t\in B} \|T_j(t)x\| = \infty$ for each $j \in \mathbb{N}_N$, if X is a Banach space.

Then there exist a dense linear subspace S of X and a number $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{Dens}\Biggl(\bigcap_{j\in\mathbb{N}_N} \Bigl\{s \geq 0 : d_Y\bigl(T_j(s)x, T_j(s)y\bigr) \geq \sigma\Bigr\}\Biggr) = 1$$

and

$$\overline{Dens}\Biggl(\bigcap_{j\in\mathbb{N}_N} \Bigl\{s \geq 0 : d_Y\bigl(T_j(s)x, T_j(s)y\bigr) < \epsilon\Bigr\}\Biggr) = 1.$$

Proof. The proof is very similar to those of [5, Theorem 15] and [10, Theorem 4.1], so that we will only outline the main points of the proof. Consider first the case in which X and Y are Frechét spaces. If so, the family $(T_j(t))_{t\geq 0} \subseteq$

$L(X, Y)$ is locally equicontinuous for all $j \in \mathbb{N}_N$. Hence, for every $l,\ n \in \mathbb{N}$, there exist $c_{l,n} > 0$ and $a_{l,n} \in \mathbb{N}$ such that $p_l^Y(T_j(t)x) \leq c_{l,n} p_{a_{l,n}}(x)$, $x \in X$, $t \in [0, n]$, $j \in \mathbb{N}_N$. Suppose, for the time being, that:

$$p_k^Y(T_j(t)x) \leq p_{k+\lceil t \rceil}(x), \quad x \in X,\ t \geq 0,\ k \in \mathbb{N},\ j \in \mathbb{N}_N. \tag{2.1}$$

We may assume that $m = 1$. Then there exist a sequence $(x_k)_{k \in \mathbb{N}}$ in X_0 such that $p_k(x_k) \leq 1$, $k \in \mathbb{N}$ and a strictly increasing sequence of positive real numbers $(t_k)_{k \in \mathbb{N}}$ tending to infinity such that:

$$\overline{Dens}\Big(\Big\{1 \leq s \leq t_k : p_1\big(T_j(s)x_k\big) > k2^k\Big\}\Big) \geq t_k\Big(1 - k^{-2}\Big)$$

and

$$\overline{Dens}\Big(\Big\{1 \leq s \leq t_k : p_k\big(T_j(s)x_l\big) < k^{-1}\Big\}\Big) \geq t_k\Big(1 - k^{-2}\Big), \quad l = 1, \cdots, k-1,$$

for any $j \in \mathbb{N}_N$. Furthermore, it is clear that there is a strictly increasing sequence $(r_s)_{s \in \mathbb{N}}$ of positive integers satisfying that:

$$r_{s+1} \geq 1 + r_s + \lceil t_{r_s+1} \rceil, \quad s \in \mathbb{N}.$$

Arguing as in [5, Theorem 15], we get that there exists a dense linear subspace S of X such that, for every $x \in S$, there exist two sets $A_x,\ B_x \subseteq [0, \infty)$ such that $\overline{Dens}(A) = \overline{Dens}(B) = 1$, $\lim_{t \to \infty, t \in A_x} T_j(t)x = 0$ and $\lim_{t \to \infty, t \in B_x} p_1(T_j(t)x) = \infty$. Now the final conlusion of the theorem follows as in the discrete case. Finally, a few words about the process of renorming. Introducing recursively the following fundamental system of increasing seminorms $p'_n(\cdot)$ $(n \in \mathbb{N})$ on X :

$$\begin{aligned}
&p'_1(x) \equiv p_1(x), \quad x \in X,\\
&p'_2(x) \equiv p'_1(x) + c_{1,1}p_{a_{1,1}}(x) + p_2(x), \quad x \in X,\\
&\cdots\\
&p'_{n+1}(x) \equiv p'_n(x) + c_{1,n}p_{a_{1,n}}(x) + \cdots + c_{n,1}p_{a_{n,1}}(x) + p_{n+1}(x), \quad x \in X,\\
&\cdots,
\end{aligned}$$

we may assume without loss of generality that (2.1) holds; hence, the assertion is proved in the case that X and Y are Frechét spaces. If X or Y is a Banach space, say Y, then we can 'renorm' it, by endowing Y with the following increasing family of seminorms $p_n^Y(y) := n\|y\|_Y$ $(n \in \mathbb{N},\ y \in Y)$, which turns the space Y into a linearly and topologically homeomorphic Fréchet space. This completes the proof of the theorem. $\square$

It is clear that Theorem 2.3 can be directly applied to strongly continuous semigroups and thus provides a great number of concrete examples of disjoint distributionally chaotic single operators (see [22, Chapter 3] and [11] for more details). In the remaining part of paper, we will primarily examine possible applications of Theorem 2.3 to the abstract ill-posed abstract Cauchy problems.

For any injective operator $C \in L(X)$, any closed linear operator A commuting with C and any positive integer $n \in \mathbb{N}$, we endow the space $C(D(A^n))$ with the following family of seminorms $p_{m,n}(Cx) := p_m(x)+p_m(Ax)+\cdots+p_m(A^n x)$, $m \in \mathbb{N}$, $x \in D(A^n)$ $(n \in \mathbb{N})$. Of course, if X is a Banach space, then the space $C(D(A^n))$ carries the topology induced by the norm $\|Cx\|_n := \|x\| + \|Ax\| + \cdots + \|A^n x\|$, $x \in D(A^n)$. Denote this space by $[C(D(A^n))]$.

Now we will reconsider the assertion of [10, Theorem 5.4] for disjoint distributional chaos:

Theorem 2.4. *Suppose that $\alpha_j \geq 0$, $t_j > 0$ and A_j subgenerates a global α_j-times integrated C_j-semigroup $(S_{\alpha_j}(t))_{t\geq 0}$ on X ($j \in \mathbb{N}_N$). Let $n_j := \lceil \alpha_j \rceil$ for any $j \in \mathbb{N}_N$, let $C \in L(X)$ be injective, and let $[R(C)]$ be continuously embedded in the space $[C_j(D(A_j^{n_j}))]$ for all $j \in \mathbb{N}_N$. Furthermore, suppose that the following conditions hold:*

(i) There exists a dense subset X_0' of $[R(C)]$ such that $\lim_{t\to\infty} G_j(\delta_t)x = 0$, $x \in X_0'$, $j \in \mathbb{N}_N$.

(ii) There exist $x \in R(C)$ and $m \in \mathbb{N}$ such that $\lim_{t\to\infty} p_m(G_j(\delta_t)x) = \infty$, $j \in \mathbb{N}_N$ ($\lim_{t\to\infty} \|G_j(\delta_t)x\| = \infty$, $j \in \mathbb{N}_N$ in the case that X is a Banach space).

Then $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $G_1(\delta_{t_1})$, $G_2(\delta_{t_2}), \cdots, G_N(\delta_{t_N})$ are disjoint distributionally chaotic; if $R(C)$ is dense in X, then $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $G_1(\delta_{t_1})$, $G_2(\delta_{t_2}), \cdots, G_N(\delta_{t_N})$ are densely disjoint distributionally chaotic.

Proof. It is clear that $[R(C)]$ is separable. Let us recall that $C_j(D(A_j^{n_j})) \subseteq Z_1(A_j)$ for all $j \in \mathbb{N}_N$; furthermore, if $x = C_j y \in C_j(D(A_j^{n_j}))$, then for every $t \geq 0$ we have:

$$G_j(\delta_t)x = S_{\alpha_j}(t)A_j^{n_j}y + \sum_{i=0}^{n_j-1} \frac{t^{n_j-i-1}}{(n_j-i-1)!} C_j A^{n_j-1-i}y, \quad j \in \mathbb{N}_N.$$

Since $[R(C)]$ is continuously embedded in the space $[C_j(D(A_j^{n_j}))]$ for all $j \in \mathbb{N}_N$, we have that, for every $t \geq 0$, the mapping $G(\delta_t) : [R(C)] \to X$ is linear and continuous. Furthermore, the family $(G(\delta_t))_{t\geq 0} \subseteq L([R(C)], X)$ is strongly continuous. We define $T_{j,k} \equiv G(\delta_{kt_j}) : [R(C)] \to X$ ($j \in \mathbb{N}_N$, $k \in \mathbb{N}$). Then $((T_{j,k})_{k\in\mathbb{N}})_{1\leq j\leq N} \subseteq L([R(C)], X)$ and (1.2) yields that $T_{j,k}x = G_j(\delta_{t_j})^k x$, $x \in R(C)$. Now an application of [18, Theorem 4.4] yields that the operators $G_1(\delta_{t_1})$, $G_2(\delta_{t_2}), \cdots, G_N(\delta_{t_N})$ are disjoint distributionally chaotic, while an application of Theorem 2.3 yields that $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ are disjoint distributionally chaotic. Finally, if $R(C)$ is dense in X, then it almost trivially follows from the foregoing that $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $G_1(\delta_{t_1})$, $G_2(\delta_{t_2}), \cdots, G_N(\delta_{t_N})$ are densely disjoint distributionally chaotic. $\square$

Remark 2.5. (i) If $\lambda_j \in \rho_{C_j}(A_j)$ for all $j \in \mathbb{N}_N$, then the choice $C := \prod_{j=1}^N C_j((\lambda_j - A_j)^{-1}C_j)^n$ can always be made.

(ii) If $\lambda_j \in \sigma_p(A_j)$ and $A_j x = \lambda_j x$ for some $x \in X \setminus \{0\}$ and $j \in \mathbb{N}_N$, then $x \in Z_1(A_j)$ and $G_j(\delta_t)x = e^{\lambda_j t}x$, $t \geq 0$. In particular, $\lim_{t\to\infty} G_j(\delta_t)x = 0$ if $\lambda \in \mathbb{K}_-$, and there exists $m \in \mathbb{N}$ such that $\lim_{t\to\infty} p_m(G_j(\delta_t)x) = \infty$ ($\lim_{t\to\infty} \|G_j(\delta_t)x\| = \infty$ in the case that X is a Banach space), if $\lambda \in \mathbb{K}_+$.

(iii) Assume now that all requirements of Theorem 2.4 stated before the formulation of (i)-(iii) hold true, as well as that $X_0 := \{Cx : (\forall j \in \mathbb{N}_N)\,(\exists \lambda_{j,-} \in \mathbb{K}_-)\, A_j Cx = \lambda_{j,-}Cx\}$. Suppose that

 (a) $\tilde{X} := \overline{X_0}^{[R(C)]}$ is non-trivial subspace of $[R(C)]$, and
 (b) there exist a vector $Cx \in \tilde{X}$ and the scalars $\lambda_{j,+} \in \mathbb{K}_+$ such that $A_j Cx = \lambda_{j,+}Cx$ for all $j \in \mathbb{N}_N$.

 Repeating literally the arguments given in the proof of Theorem 2.4, with the spaces $[R(C)]$ and X_0' replaced with the spaces $\tilde{X}$ and X_0 therein, we get that $((S_{\alpha_j}(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $G_1(\delta_{t_1})$, $G_2(\delta_{t_2})$, $\cdots$, $G_N(\delta_{t_N})$ are disjoint $\tilde{X}$-distributionally chaotic. The question whether we can make a choice $\tilde{X} = R(C)$ has an affirmative answer in the case that the operators A_j have nice supplies of eigenfunctions (see e.g. the proof of Desch-Schappacher-Webb criterion for strongly continuous semigroups [13, Theorem 3.1], as well as Example 2.6 below).

The interested reader may try to formulate an analogue of [10, Theorem 5.9] for disjoint distributional chaos of entire C-regularized groups.

We proceed by providing two illustrative examples.

Example 2.6. ([12]) Assume that $\mathbb{K} = \mathbb{C}$, ω_1, ω_2, V_{ω_2,ω_1}, $Q_j(z)$, $Q_j(B)$, h_μ and X possess the same meaning as in [12, Section 5], as well as that the number $L \in \mathbb{N}$ is sufficiently large and takes the role of number N from this section. Let $t_j > 0$ and let the following two conditions hold:

(A) there exists a non-empty subset Ω' of $\operatorname{int}(V_{\omega_2,\omega_1})$ which has a cluster point in $\operatorname{int}(V_{\omega_2,\omega_1})$ and satisfies that, for every $z \in \Omega'$ and for every $j \in \mathbb{N}_N$, we have $Q_j(z) \in \mathbb{C}_-$;

(B) there exists $z \in \operatorname{int}(V_{\omega_2,\omega_1})$ such that, for every $j \in \mathbb{N}_N$, we have $Q_j(z) \in \mathbb{C}_+$.

Then $\pm Q_j(B)h_\mu = \pm Q_j(\mu)h_\mu$, $e^{-(-B^2)^L}h_\mu = e^{-(-\mu^2)^L}h_\mu$, $\mu \in \operatorname{int}(V_{\omega_2,\omega_1})$ and the operator $Q_j(B)$ is the integral generator of the $C \equiv (e^{-(-z^2)^L})(B)$-regularized semigroup $(W_{Q_j}(t) \equiv z \mapsto e^{tQ_j(z)}e^{-(-z^2)^L})(B))_{t\geq 0}$ on X ($j \in \mathbb{N}_N$). Furthermore, the set $R(C)$ is dense in X. The validity of (A)-(B) yields that Theorem 2.4 and Remark 2.5(iii) can be applied, showing that the C-regularized semigroups $((W_{Q_j}(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $e^{t_1 Q(B)}$, $e^{t_2 Q(B)}, \cdots, e^{t_N Q(B)}$ are densely disjoint distributionally chaotic.

Example 2.7. (cf. [9, Example 2.13]) Let us assume that $\zeta \geq 0$, $-A \notin L(X)$, $-A$ generates an exponentially equicontinous ζ-times integrated cosine function $(C_\zeta(t))_{t\geq 0}$, $N \in \mathbb{N}$, $N \geq 2$ and $P_j(z) = \sum_{i=0}^{n_j} a_{i,j}z^i$ is a non-zero complex polynomial with $a_{n_j,j} > 0$ ($j \in \mathbb{N}_N$). Assume, further, that there are an open connected subset Ω of $\mathbb{C}$ and an analytic mapping $f : \Omega \to X \setminus \{0\}$ such that $\sigma_p(-A) \supseteq \Omega$ and $f(\lambda) \in N(-A-\lambda) \setminus \{0\}$, $\lambda \in \Omega$ (e.g., let $a > 0$, let $\rho(x) := e^{-a|x|}$, $x \in \mathbb{R}$, $X := L^p_\rho(\mathbb{R})$, $D(B) := \{f \in X \mid f(\cdot)$ is loc. abs. continuous, $f' \in E\}$ and $Af := f'$, $f \in D(B)$; then A generates C_0-group on X and the above holds with $A = -B^2$, $\Omega = \{z^2 : |\Re z| < a\}$ and $f(z^2) = e^{z\cdot}$ for $|\Re z| < a$; cf. [13] for the notion).

Set $\mathcal{A} := \left(\begin{smallmatrix} 0 & I \\ -A & 0 \end{smallmatrix}\right)$, and suppose further that Ω' is a non-empty open connected subset of $\mathbb{C}$ such that $\lambda^2 \in \Omega$ for all $\lambda \in \Omega'$. Define $F : \Omega' \to (X \times X) \setminus \{(0,0)\}$ by $F(\lambda) := [f(\lambda^2)\ \lambda f(\lambda^2)]^T$, $\lambda \in \Omega'$. Then we know that $F(\cdot)$ is analytic, $\sigma_p(\mathcal{A}) \supseteq \Omega'$ and $F(\lambda) \in N(\mathcal{A}-\lambda) \setminus \{(0,0)\}$, $\lambda \in \Omega'$. Further on, the operator $\mathcal{A}$ generates an exponentially equicontinuous $(\zeta+1)$-times integrated semigroup $(S_{\zeta+1}(t))_{t \geq 0}$ in $X \times X$, which is given by

$$S_{\zeta+1}(t) := \begin{pmatrix} \int_0^t C_\zeta(s)\,ds & \int_0^t (t-s) C_\zeta(s)\,ds \\ C_\zeta(t) - g_{\zeta+1}(t)C & \int_0^t C_\zeta(s)\,ds \end{pmatrix}, \quad t \geq 0.$$

On the other hand, the operator $\mathcal{A}^2$ generates an exponentially equicontinuous, analytic $(\zeta/2)$-times integrated semigroup of angle $\pi/2$. Set $Q_1(z) := z$ and $Q_j(z) := -P_j(-z^2)$ ($z \in \mathbb{C}$, $2 \leq j \leq N$), as well as $A_j := Q_j(\mathcal{A})$. Then the operator A_j generates an exponentially equicontinuous, analytic η-times integrated semigroup $(S^j_\eta(t))_{t\geq 0}$ of angle $\pi/2$, for $2 \leq j \leq N$. Suppose that the conditions (A) and (B) hold with the set $\operatorname{int}(V_{\omega_2,\omega_1})$ replaced by the set Ω'. These conditions ensure that Theorem 2.4 and Remark 2.5(iii) are applicable, so that the integrated semigroups $(S_{\zeta+1}(\cdot), (S^j_\eta(\cdot))_{2\leq j\leq N})$ are densely disjoint distributionally chaotic, which also holds for corresponding tuples of single operators.

Finally, at the end of this section, we would like to propose an interesting problem for our readers:

Example 2.8. (cf. also [19, Example 3.1.35(i)], [20, Example 38] and [10, Example 5.12, Example 5.13]). Let us assume that $n \in \mathbb{N}$, $\rho(t) := \frac{1}{t^{2n}+1}$, $t \in \mathbb{R}$, $Af := f'$, $D(A) := \{f \in C_{0,\rho}(\mathbb{R}) : f' \in C_{0,\rho}(\mathbb{R})\}$, $X_n := (C_{0,\rho}(\mathbb{R}))^{n+1}$, $D(A_n) := D(A)^{n+1}$ and $A_n(f_1, \cdots, f_{n+1}) := (Af_1 + Af_2, Af_2 + Af_3, \cdots, Af_n + Af_{n+1}, Af_{n+1})$, $(f_1, \cdots, f_{n+1}) \in D(A_n)$. Then we already know that $\pm A_n$ generate global polynomially bounded n-times integrated semigroups $(S_{n,\pm}(t))_{t\geq 0}$, and neither A_n nor $-A_n$ generates a local $(n-1)$-times integrated semigroup. By [19, Proposition 2.1.17], the above implies that A_n^2 generates a polynomially bounded n-times integrated cosine function $(C_n(t) \equiv 1/2(S_{n,+}(t) + S_{n,-}(t)))_{t\geq 0}$. Due to [19, Corollary 2.4.9], we have that A_n^2 generates a polynomially bounded $(n/2)$-times integrated semigroup $(S_{n/2}(t))_{t\geq 0}$. We would like to ask whether $(S_{n/2}(t))_{t\geq 0}$ is densely disjoint distributionally chaotic and whether $(S_{n,\pm}(t))_{t\geq 0}$ and $(S_{n/2}(t))_{t\geq 0}$ are densely disjoint distributionally chaotic.

3. Disjoint distributionally chaotic properties of abstract fractional PDEs

Let us recall that $\zeta \in (0,2) \setminus \{1\}$. We start this section by providing analogues of Definition 2.1 and Definition 2.2 for fractional resolvent families:

Definition 3.1. Let $\alpha_j \geq 0$, let $C_j \in L(X)$ be injective for all $j \in \mathbb{N}_N$ and let $(R_j(t))_{t\geq 0}$ be a global ζ-times C_j-regularized resolvent family with the integral generator A_j $(j \in \mathbb{N}_N)$. Suppose that $\tilde{X}$ is a closed linear subspace of X. Let $Z_{j,\zeta}(A_j)$ the set consisting of those vectors $x \in X$ such that $R_j(t)x \in R(C_j)$, $t \geq 0$ and the mapping $t \mapsto C_j^{-1}R_j(t)x$, $t \geq 0$ is continuous. Denote by $t \mapsto C_j^{-1}R_j(t)x$, $t \geq 0$ the unique mild solution of the corresponding Cauchy problem (1.3), with the operator A replaced by A_j therein $(j \in \mathbb{N}_N)$. Then we say that $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ are disjoint $\tilde{X}$-distributionally chaotic, $(d, \tilde{X})$-distributionally chaotic in short, iff there exist an uncountable set $S \subseteq \bigcap_{j=1}^N Z_{j,\zeta}(A_j) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that for each $j \in \mathbb{N}_N$ and $t \geq 0$ we have that

$$\overline{dens}\Bigg(\bigcap_{j\in\mathbb{N}_N}\big\{t \geq 0 : d_Y\big(C_j^{-1}R_j(t)x, C_j^{-1}R_j(t)y\big) \geq \sigma\big\}\Bigg) = 1, \text{ and}$$

$$\overline{dens}\Bigg(\bigcap_{j\in\mathbb{N}_N}\big\{t \geq 0 : d_Y\big(C_j^{-1}R_j(t)x, C_j^{-1}R_j(t)y\big) < \epsilon\big\}\Bigg) = 1.$$

The sequence $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ is said to be densely $(d, \tilde{X})$-distributionally chaotic iff S can be chosen to be dense in $\tilde{X}$. The set S is said to be $(d, \sigma_{\tilde{X}})$-scrambled set ((d,σ)-scrambled set in the case that $\tilde{X} = X$) of the tuple $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$; in the case that $\tilde{X} = X$, then we also say that the sequence $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ is (densely) disjoint distributionally chaotic, (densely) d-distributionally chaotic in short.

Definition 3.2. Let $\alpha_j \geq 0$, let $C_j \in L(X)$ be injective for all $j \in \mathbb{N}_N$ and let $(R_j(t))_{t\geq 0}$ be a global ζ-times C_j-regularized resolvent family with the integral generator A_j $(j \in \mathbb{N}_N)$. Suppose that $\tilde{X}$ is a closed linear subspace of X. Let $Z_{j,\zeta}(A_j)$ be defined as above, and let $t \mapsto C_j^{-1}R_j(t)x$, $t \geq 0$ be the unique mild solution of the corresponding Cauchy problem (1.3), with the operator A replaced by A_j therein $(j \in \mathbb{N}_N)$. Let $m \in \mathbb{N}$ and $x \in \bigcap_{j=1}^N Z_{j,\zeta}(A_j) \cap \tilde{X}$. Then we say that:

(i) x is disjoint distributionally near to 0 for $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ iff there exists $A \subseteq [0,\infty)$ such that $\overline{Dens}(A) = 1$ and $\lim_{s\to\infty, s\in A} C_j^{-1}R_j(s)x = 0$ for all $j \in \mathbb{N}_N$;

(ii) x is disjoint distributionally m-unbounded for $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ iff there exists a set $B \subseteq [0,\infty)$ satisfying that $\overline{Dens}(B) = 1$ and

$$\lim_{s\to\infty, s\in B} p_m\big(C_j^{-1}R_j(s)x\big) = 0$$

for all $j \in \mathbb{N}_N$; x is disjoint distributionally unbounded for the tuple $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ iff there exists $q \in \mathbb{N}$ such that x is disjoint distributionally q-unbounded for $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$;

(iii) x is a disjoint $\tilde{X}$-distributionally irregular vector for $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ (disjoint distributionally irregular vector for $((R_j(t))_{t\geq 0})_{1\leq j\leq N}$ simply, in the case that $\tilde{X} = X$) iff x is both disjoint distributionally near to 0 and disjoint distributionally unbounded.

Concerning disjoint distributional chaos of abstract time-fractional differential equations, the theoretical aspects are basically the same as for the abstract differential equations of the first order and almost anything reasonable relies on possible applications of Theorem 2.3. Here we will formulate only one simple result regarding this theme:

Theorem 3.3. *Let $\alpha_j \geq 0$, $t_j > 0$, let $C_j \in L(X)$ be injective for all $j \in \mathbb{N}_N$, and let $(R_j(t))_{t\geq 0}$ be a global ζ-times C_j-regularized resolvent family with the integral generator A_j ($j \in \mathbb{N}_N$). Let for each i, $j \in \mathbb{N}_N$ such that $i \neq j$, we have $C_iA_j \subseteq A_jC_i$, $C_iR_j(t) = R_j(t)C_i$, $t \geq 0$ and $R_j(t)A_i \subseteq A_iR_j(t)$, $t \geq 0$. Set $C := \prod_{j=1}^{N} C_j$. Then $(\mathrm{R}_j(t) \equiv R_j(t)\prod_{1\leq i\leq N, i\neq j} C_i)_{t\geq 0}$ is a global ζ-times C-regularized resolvent family with the integral generator A_j ($j \in \mathbb{N}_N$). Suppose, further, that there exists a dense linear subspace X_0 of X such that the following holds:*

(a) $\lim_{t\to\infty} \mathrm{R}_j(t)x = 0$, $x \in X_0$, $j \in \mathbb{N}_N$,

(b) there exist $x \in X$ and $m \in \mathbb{N}$ such that $\lim_{t\to\infty} p_m(\mathrm{R}_j(t)x) = \infty$ for each $j \in \mathbb{N}_N$, resp. $\lim_{t\to\infty} \|\mathrm{R}_j(t)x\| = \infty$ for each $j \in \mathbb{N}_N$, if X is a Banach space.

Then $((\mathrm{R}_j(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $C^{-1}\mathrm{R}_1(t_1), C^{-1}\mathrm{R}_2(t), \cdots, C^{-1}\mathrm{R}_N(t_N)$ are disjoint distributionally chaotic; if, moreover, $R(C)$ is dense in X, then $((\mathrm{R}_j(t))_{t\geq 0})_{1\leq j\leq N}$ and the operators $C^{-1}\mathrm{R}_1(t_1)$, $C^{-1}\mathrm{R}_2(t_2), \cdots, C^{-1}\mathrm{R}_N(t_N)$ are densely disjoint distributionally chaotic.

Proof. Since for each i, $j \in \mathbb{N}_N$ such that $i \neq j$, we have $C_iA_j \subseteq A_jC_i$, $C_iR_j(t) = R_j(t)C_i$, $t \geq 0$ and $R_j(t)A_i \subseteq A_iR_j(t)$, $t \geq 0$, it follows immediately from definition that $(\mathrm{R}_j(t))_{t\geq 0}$ is a global ζ-times C-regularized resolvent family with the integral generator A_j ($j \in \mathbb{N}_N$), where C is defined as above. Now the final conclusion follows from Theorem 2.3, by considering the sequence $((C^{-1}\mathrm{R}_j(t))_{t\geq 0})_{1\leq j\leq N}$ of strongly continuous families consisting of linear continuous mappings between the spaces $[R(C)]$ and X. $\square$

Now we will provide two illustrative applications of Theorem 3.3, in which the regularizing operator C is the identity operator (for general C, we can modify Example 2.6; see also [21, Example 2.5(iv)]):

Example 3.4. ([13], [21]) Let $a,\ b,\ c > 0$, $\zeta \in (1,2)$, $c < \frac{b^2}{2a} < 1$ and

$$\Lambda := \left\{ \lambda \in \mathbb{C} : \left| \lambda - c + \frac{b^2}{4a} \right| \leq \frac{b^2}{4a},\ \Im(\lambda) \neq 0 \text{ if } \Re(\lambda) \leq c - \frac{b^2}{4a} \right\}.$$

Then the operator $-A$ with domain $D(-A) = \{f \in W^{2,2}([0,\infty)) : f(0) = 0\}$, generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2}$ in the space $X = L^2([0,\infty))$; the same holds in the case that the operator $-A$ acts on $X = L^1([0,\infty))$ with domain $D(-A) = \{f \in W^{2,1}([0,\infty)) : f(0) = 0\}$. In both cases, $-\Lambda \subseteq \sigma_p(A)$. Suppose that $\theta \in (\zeta\frac{\pi}{2} - \pi, \pi - \zeta\frac{\pi}{2})$ and $P_j(z) = \sum_{l=0}^{n} a_{l,j} z^l$ is a non-constant complex polynomial such that $a_{l,n} > 0$ and the following two conditions hold:

(A)' there exists a non-empty subset Ω' of $-\Lambda$ which has a cluster point in $-\Lambda$ and satisfies that, for every $z \in \Omega'$ and for every $j \in \mathbb{N}_N$, we have $-e^{i\theta}P_j(z) \notin \overline{\Sigma_{\zeta\pi/2}}$;

(B)' there exists $z \in -\Lambda$ such that, for every $j \in \mathbb{N}_N$, we have $-e^{i\theta}P_j(z) \in \Sigma_{\zeta\pi/2}$.

We know that the operator $-e^{i\theta}P_j(A)$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P_j}(t))_{t\geq 0}$ of angle $\frac{\pi-|\theta|}{\zeta} - \frac{\pi}{2}$; cf. [22] for the notion. Here, the requirements needed for applying Theorem 3.3 are satisfied, which can be verified with the help of asymptotic expansion formuale (1.6)-(1.8) and the conditions (A)'-(B)'. As a consequence, we have that $((R_{\zeta,\theta,P_j}(t))_{t\geq 0})_{1\leq j\leq N}$ are densely disjoint distributionally chaotic.

(ii) ([17], [21]) Let X be a symmetric space of non-compact type and rank one, let $p > 2$, let the parabolic domain P_p and the positive real number c_p possess the same meaning as in [17], and let $P^j(z) = \sum_{l=0}^{n} a_{l,j} z^l$, $z \in \mathbb{C}$ be a non-constant complex polynomial with $a_{l,n} > 0$ ($j \in \mathbb{N}_N$). Suppose that $\zeta \in (1,2)$, $\pi - n\arctan\frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} > 0$ and $\theta \in (n\arctan\frac{|p-2|}{2\sqrt{p-1}} + \zeta\frac{\pi}{2} - \pi, \pi - n\arctan\frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2})$. We know that $-e^{i\theta}P^j(\Delta^{\natural}_{X,p})$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P^j}(t))_{t\geq 0}$ of angle $\frac{1}{\zeta}(\pi - n\arctan\frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} - |\theta|)$, for any $j \in \mathbb{N}_N$. Using the fact that $\mathrm{int}(P_p) \subseteq \sigma_p(\Delta^{\natural}_{X,p})$, the validity of conditions (A)'-(B)' with the set $-\Lambda$ and polynomials $-e^{i\theta}P_j(z)$ replaced therein with the set $\mathrm{int}(P_p)$ and polynomials $-e^{i\theta}P^j(z)$ ensures that $(R_{\zeta,\theta,P^j}(t))_{t\geq 0}$ are densely disjoint distributionally chaotic.

We close the paper with the observation that distributionally chaotic properties of abstract multi-term fractional differential equations have been considered by the author in [23]. Applying Theorem 2.3, we can simply deduce several extensions of results established in this section for corresponding fractional resolvent operator families governing solutions of such equations.

In [15] and [24], we have followed slightly different approaches to the concepts of disjoint hypercyclicity, disjoint topologically mixing property and the usual distributional chaos for abstract (multi-term) fractional differential equations. Disjoint distributionally chaotic solutions of such equations can be analyzed by following this approach, as well. Related results will appear somewhere else.

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