Novi Sad J. Math. Vol. 49, No. 2, 2019, 123-138
https://doi.org/10.30755/NSJOM.08837

SUZUKI TYPE FIXED POINT RESULTS AND APPLICATIONS IN PARTIALLY ORDERED S_b - METRIC SPACES

Nebojša Dedović[1], Gajula Naveen Venkata Kishore[2][3], Daripally Ram Prasad[4], Jelena Vujaković[5]

Abstract. In this paper we give some applications to integral equations as well as homotopy theory via Suzuki type fixed point theorems in partially ordered complete S_b - metric space by using generalized contractive conditions. We also furnish an example which supports our main result.

AMS Mathematics Subject Classification (2010): 54H25; 47H10; 54E50

Key words and phrases: S_b-metric space; S_b-Cauchy sequence; S_b - completeness; fixed point; Suzuki type contraction

1. Introduction

Banach contraction principle in metric spaces is one of the most important results in fixed point theory and nonlinear analysis in general. Since 1922, when Stefan Banach [2] formulated the concept of contraction and posted his famous theorem, scientists around the world publish new results about generalization of metric space or with contractive mappings (see [1], [2], [3], [4], [5], [7], [6], [8], [9], [10], [22], [11], [12], [13], [17], [15], [14], [18], [16], [19], [20], [21]). Banach contraction principle is considered to be the initial result of the study of the fixed point theory in metric spaces.

Recently Sedghi et al. [15] defined S_b-metric spaces using the concept of S-metric spaces [14].

The aim of this paper is to prove some Suzuki type unique fixed point theorems for generalized contractive conditions in partially ordered S_b-metric spaces, also provide an application of integral equations as well as an application of Homotopy Theory. Throughout this paper $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ denote the set of all real numbers, non-negative real numbers and positive integers, respectively.

First we recall some definitions, lemmas and examples.

[1]Faculty of Agriculture, University of Novi Sad, Trg Dositeja Obradovića 8, Novi Sad, Serbia, e-mail: dedovicn@gmail.com

[2]Department of Engineering Mathematics, Sagi Rama Krishnam Raju Engineering College, Chinaa Amiram, Bhimavaram - 534 204, Andhra Pradesh, India, e-mail: kishore.apr2@gmail.com

[3]Corresponding Author

[4]Research Scholar Department of Mathematics, K L University, Vaddeswaram, Guntur - 522 502, Andhra Pradesh, India, e-mail: ramprasadmphil09@gmail.com

[5]Faculty of Sciences and Mathematics, University of Priština, Lole Ribara 29, Kosovska Mitrovica, 38220, Serbia, e-mail: enav@ptt.rs

Definition 1.1. *([14]) Let X be a non-empty set. An $S-$metric on X is a function $S : X^3 \to [0, +\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$,*

$(S1)$ $S(x, y, z) = 0$ *if and only if* $x = y = z$,

$(S2)$ $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ *for all* $x, y, z, a \in X$.

Then the pair (X, S) is called an S-metric space.

Definition 1.2. *([15]) Let X be a non-empty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \to [0, \infty)$ is a function satisfying the following properties :*

(S_b1) $S_b(x, y, z) = 0$ *if and only if* $x = y = z$,

(S_b2) $S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ *for all* $x, y, z, a \in X$.

Then the function S_b is called an S_b-metric on X and the pair (X, S_b) is called an S_b-metric space.

Remark 1.3. *([15]) It should be noted that the class of S_b-metric spaces is effectively larger than the class of S-metric spaces. Indeed each S-metric space is an S_b-metric space with $b = 1$.*

The following example shows that an S_b-metric on X need not be an S-metric on X.

Example 1.4. *([15]) Let (X, S) be an S-metric space, and $S_*(x, y, z) = (S(x, y, z))^p$, where $p > 1$ is a real number. Note that S_* is an S_b-metric with $b = 2^{2(p-1)}$. Also, (X, S_*) is not necessarily an S-metric space.*

Definition 1.5. *([15]) Let (X, S_b) be an S_b-metric space. Then, for $x \in X$, $r > 0$ we defined the open ball $B_{S_b}(x, r)$ and the closed ball $B_{S_b}[x, r]$ with center x and radius r as follows, respectively:*

$B_{S_b}(x, r) = \{y \in X : S_b(y, y, x) < r\}$ *and* $B_{S_b}[x, r] = \{y \in X : S_b(y, y, x) \leq r\}$.

Lemma 1.6. *([15]) In an S_b-metric space, we have* $S_b(x, x, y) \leq bS_b(y, y, x)$ *and* $S_b(y, y, x) \leq bS_b(x, x, y)$.

Lemma 1.7. *([15])In an S_b-metric space, we have*

$$S_b(x, x, z) \leq 2bS_b(x, x, y) + b^2 S_b(y, y, z)$$

Definition 1.8. *([15]) Let (X, S_b) be an S_b-metric space. A sequence $\{x_n\}$ in X is said to be:*

(1) *S_b-Cauchy if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$.*

(2) S_b-*convergent to a point* $x \in X$ *if, for each* $\epsilon > 0$, *there exists a positive integer* n_0 *such that* $S_b(x_n, x_n, x) < \epsilon$ *or* $S_b(x, x, x_n) < \epsilon$ *for all* $n \geq n_0$. *We denote by* $\lim_{n\to\infty} x_n = x$.

Definition 1.9. *([15]) An* S_b-*metric space* (X, S_b) *is called complete if every* S_b-*Cauchy sequence is* S_b-*convergent in* X.

Lemma 1.10. *([15]) Let* (X, S_b) *be an* S_b-*metric space with* $b \geq 1$ *and suppose that* $\{x_n\}$ *is* S_b-*convergent to* x, *then we have*

(i) $\frac{1}{2b} S_b(y, y, x) \leq \lim_{n\to\infty} \inf S_b(y, y, x_n) \leq \lim_{n\to\infty} \sup S_b(y, y, x_n)$
$\leq 2b S_b(y, y, x)$ *and*

(ii) $\frac{1}{b^2} S_b(x, x, y) \leq \lim_{n\to\infty} \inf S_b(x_n, x_n, y) \leq \lim_{n\to\infty} \sup S_b(x_n, x_n, y)$
$\leq b^2 S_b(x, x, y)$ *for all* $y \in X$

In particular, if $x = y$, *then we have* $\lim_{n\to\infty} S_b(x_n, x_n, y) = 0$.

Now we prove our main results.

2. Main Results

Definition 2.1. *Let* $(X, S_b, \preceq)$ *be a partially ordered complete* S_b *- metric space which is also regular, and* $f : X \to X$ *be mapping. We say that* f *is a Suzuki type generalized* φ *- contraction if there exists* $\varphi : [0, \infty) \to [0, \infty)$ *such that*

(2.1.1) f *is non-decreasing and* φ *is lower semi continuous,*

(2.1.2) $\varphi(t) = 0$ *if and only if* $t = 0$,

(2.1.3) $\frac{1}{4b^3} \min\{S_b(x, x, fx), S_b(y, y, fy)\} \leq S_b(x, x, y)$ *implies that*
$4b^4 \ S_b(fx, fx, fy) \leq M_f^i(x, y) - \varphi\left(M_f^i(x, y)\right)$,
for all $x, y \in X$, x *comparable to* y, $i = 3$ *or* 4 *or* 5. *Also*

$$M_f^5(x, y) = \max\left\{ \begin{array}{c} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ S_b(x, x, fy), S_b(y, y, fx) \end{array} \right\}.$$

$$M_f^4(x, y) = \max\left\{ \begin{array}{c} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ \frac{1}{4b^4}\left[S_b(x, x, fy) + S_b(y, y, fx)\right] \end{array} \right\}.$$

$$M_f^3(x, y) = \max\left\{ \begin{array}{c} S_b(x, x, y), \frac{1}{4b^4}\left[S_b(x, x, fx) + S_b(y, y, fy)\right], \\ \frac{1}{4b^4}\left[S_b(x, x, fy) + S_b(y, y, fx)\right] \end{array} \right\}.$$

Definition 2.2. *Suppose* $(X, \preceq)$ *is a partially ordered set, and* f *is a mapping of* X *into itself. We say that* f *is non-decreasing if for every* $x, y \in X$,

$$x \preceq y \text{ implies that } fx \preceq fy. \tag{2.1}$$

Definition 2.3. *Let $(X, S_b, \preceq)$ be a partially ordered complete S_b - metric space. $(X, S_b, \preceq)$ is said to be regular if every two elements of X are comparable, i.e., if $x, y \in X \Rightarrow$ either $x \preceq y$ or $y \preceq x$.*

Theorem 2.4. *[21] Let (X, d) be a complete metric space and let T be a mapping on X. Define a non increasing function θ from $[0, 1)$ into $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & if \quad 0 \le r \le \left(\sqrt{5} - 1\right)/2 \\ (1 - r)r^{-2}, & if \quad \left(\sqrt{5} - 1\right)/2 \le r \le 2^{-1/2} \\ (1 + r)^{-1}, & if \quad 2^{-1/2} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le d(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim\limits_{n \to \infty} T^n x = z$ for all $x \in X$.

Theorem 2.5. *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, which is also regular and let $f : X \to X$ be a Suzuki type generalized φ - contraction with $i = 5$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.*

Proof. Since f is a mapping from X into X, there exists a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, n = 0, 1, 2, 3, \ldots$$

Case (i): If $x_n = x_{n+1}$, then x_n is a fixed point of f.
Case (ii): Suppose $x_n \neq x_{n+1}$ for all n.
Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, it follows that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq f^3x_0 \preceq \cdots \preceq f^n x_0 \preceq f^{n+1}x_0 \preceq \cdots$$

Using $\frac{1}{4b^3} \min\{S_b(x_0, x_0, fx_0), S_b(x_1, x_1, fx_1)\} \le S_b(x_0, x_0, x_1)$, from (2.1.3) we have that

$$\begin{aligned} 4b^4 \quad & S_b\left(fx_0, fx_0, f^2x_0\right) \\ & = 4b^4 \ S_b\left(fx_0, fx_0, fx_1\right) \\ & \le M_f^5\left(x_0, x_1\right) - \varphi\left(M_f^5\left(x_0, x_1\right)\right), \\ & \le \max\left\{ \begin{array}{l} S_b\left(x_0, x_0, fx_0\right), S_b\left(fx_0, fx_0, f^2x_0\right), \\ S_b\left(x_0, x_0, f^2x_0\right) \end{array} \right\} \\ & \quad -\varphi\left(\max\left\{ \begin{array}{l} S_b\left(x_0, x_0, fx_0\right), S_b\left(fx_0, fx_0, f^2x_0\right), \\ S_b\left(x_0, x_0, f^2x_0\right) \end{array} \right\}\right) \\ & \le \max\left\{ \begin{array}{l} S_b\left(x_0, x_0, fx_0\right), S_b\left(fx_0, fx_0, f^2x_0\right), \\ S_b\left(x_0, x_0, f^2x_0\right) \end{array} \right\}. \end{aligned}$$

Based on above, we have that

$$S_b\left(fx_0,fx_0,f^2x_0\right)\le\max\left\{\begin{array}{l}\frac{1}{4b^4}\ S_b\left(x_0,x_0,fx_0\right),\\ \\ \frac{1}{4b^4}\ S_b\left(fx_0,fx_0,f^2x_0\right),\\ \\ \frac{1}{4b^4}\ S_b\left(x_0,x_0,f^2x_0\right)\end{array}\right\}. \tag{2.2}$$

But here, by Lemma 1.7,

$$\begin{aligned}\frac{1}{4b^4}\quad S_b\quad &\left(x_0,x_0,f^2x_0\right)\\ \le\quad &\frac{1}{4b^4}\left[2bS_b\left(x_0,x_0,fx_0\right)+b^2S_b\left(fx_0,fx_0,f^2x_0\right)\right]\\ \le\quad &\max\left\{\frac{1}{b^3}S_b\left(x_0,x_0,fx_0\right),\frac{1}{2b^2}S_b\left(fx_0,fx_0,f^2x_0\right)\right\}.\end{aligned}$$

From (2.2), we have that

$$S_b\left(fx_0,fx_0,f^2x_0\right)\le\max\left\{\begin{array}{c}\frac{1}{b^3}S_b\left(x_0,x_0,fx_0\right),\\ \frac{1}{2b^2}S_b\left(fx_0,fx_0,f^2x_0\right)\end{array}\right\}. \tag{2.3}$$

If $\frac{1}{2b^2}S_b\left(fx_0,fx_0,f^2x_0\right)$ is the maximum, we get a contradiction. Hence

$$S_b\left(fx_0,fx_0,f^2x_0\right)\le\frac{1}{b^3}S_b\left(x_0,x_0,fx_0\right). \tag{2.4}$$

Also, from $\frac{1}{4b^3}\min\{S_b(x_1,x_1,fx_1),S_b(x_2,x_2,fx_2)\}\le S_b(x_1,x_1,x_2)$ and (2.1.3), it follows

$$\begin{aligned}4b^4\quad &S_b\left(f^2x_0,f^2x_0,f^3x_0\right)\\ &=S_b\left(fx_1,fx_1,fx_2\right)\\ &\le M_f^5\left(x_1,x_2\right)-\varphi\left(M_f^4\left(x_1,x_2\right)\right),\\ &\le\max\left\{\begin{array}{l}S_b\left(fx_0,fx_0,f^2x_0\right),\\ S_b\left(f^2x_0,f^2x_0,f^3x_0\right),\\ S_b\left(fx_0,fx_0,f^3x_0\right)\end{array}\right\}\\ &\qquad-\varphi\left(\max\left\{\begin{array}{l}S_b\left(fx_0,fx_0,f^2x_0\right),\\ S_b\left(f^2x_0,f^2x_0,f^3x_0\right),\\ S_b\left(fx_0,fx_0,f^3x_0\right)\end{array}\right\}\right)\\ &\le\max\left\{\begin{array}{l}S_b\left(fx_0,fx_0,f^2x_0\right),\\ S_b\left(f^2x_0,f^2x_0,f^3x_0\right),\\ S_b\left(fx_0,fx_0,f^3x_0\right)\end{array}\right\}.\end{aligned}$$

Based on above, we have that

$$S_b\left(f^2x_0,f^2x_0,f^3x_0\right)\le\max\left\{\begin{array}{l}\frac{1}{4b^4}\ S_b\left(fx_0,fx_0,f^2x_0\right),\\ \\ \frac{1}{4b^4}\ S_b\left(f^2x_0,f^2x_0,f^3x_0\right),\\ \\ \frac{1}{4b^4}\ S_b\left(fx_0,fx_0,f^3x_0\right)\end{array}\right\}. \tag{2.5}$$

Here

$$\begin{aligned} \tfrac{1}{4b^4} \quad & S_b\left(fx_0, fx_0, f^3x_0\right) \\ & \leq \tfrac{1}{4b^4}\left[2bS_b\left(fx_0, fx_0, f^2x_0\right) + b^2 S_b\left(f^2x_0, f^2x_0, f^3x_0\right)\right] \\ & \leq \max\left\{\tfrac{1}{b^3} S_b\left(fx_0, fx_0, f^2x_0\right), \tfrac{1}{2b^2} S_b\left(f^2x_0, f^2x_0, f^3x_0\right)\right\}. \end{aligned}$$

From (2.5), we have that

$$S_b\left(f^2x_0, f^2x_0, f^3x_0\right) \leq \max\left\{\begin{array}{c} \tfrac{1}{b^3} S_b\left(fx_0, fx_0, f^2x_0\right), \\ \tfrac{1}{2b^2} S_b\left(f^2x_0, f^2x_0, f^3x_0\right) \end{array}\right\}. \tag{2.6}$$

If $\frac{1}{2b^2} S_b\left(f^2x_0, f^2x_0, f^3x_0\right)$ is maximum, we get a contradiction. After applying (2.4), we get

$$\begin{aligned} S_b\left(f^2x_0, f^2x_0, f^3x_0\right) & \leq \frac{1}{b^3} S_b\left(fx_0, fx_0, f^2x_0\right) \\ & \leq \frac{1}{\left(b^3\right)^2} S_b\left(x_0, x_0, fx_0\right). \end{aligned}$$

Continuing this process, we can conclude that

$$\begin{aligned} S_b\left(f^n x_0, f^n x_0, f^{n+1}x_0\right) & \leq \frac{1}{b^3} S_b\left(f^{n-1}x_0, f^{n-1}x_0, f^n x_0\right) \\ & \vdots \\ & \leq \frac{1}{\left(b^3\right)^{n-1}} S_b\left(fx_0, fx_0, f^2x_0\right) \\ & \leq \frac{1}{\left(b^3\right)^{n}} S_b\left(x_0, x_0, fx_0\right) \\ & \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.7}$$

As a consequence, we have

$$\lim_{n\to\infty} S_b\left(f^n x_0, f^n x_0, f^{n+1}x_0\right) = 0. \tag{2.8}$$

Now we must prove that $\{f^n x_0\}$ is an S_b-Cauchy sequence in $(X, S_b, \preceq)$. On the contrary, we suppose that $\{f^n x_0\}$ is not an S_b-Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

$$S_b\left(f^{m_k}x_0, f^{m_k}x_0, f^{n_k}x_0\right) \geq \epsilon \tag{2.9}$$

and

$$S_b\left(f^{m_k}x_0, f^{m_k}x_0, f^{n_k-1}x_0\right) < \epsilon. \tag{2.10}$$

Firstly, let us see that

$$\frac{1}{4b^3} \min\left\{\begin{array}{l} S_b\left(x_{m_k}, x_{m_k}, fx_{m_k}\right), \\ S_b\left(x_{n_k-1}, x_{n_k-1}, fx_{n_k-1}\right) \end{array}\right\} \leq S_b\left(x_{m_k}, x_{m_k}, x_{n_k-1}\right). \tag{2.11}$$

On the contrary, suppose that

$$\frac{1}{4b^3}\min\left\{\begin{array}{c} S_b\left(x_{m_k},x_{m_k},fx_{m_k}\right), \\ S_b\left(x_{n_k-1},x_{n_k-1},fx_{n_k-1}\right)\end{array}\right\} > S_b\left(x_{m_k},x_{m_k},x_{n_k-1}\right). \tag{2.12}$$

Then

$$\begin{aligned}
\epsilon &\le S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\right) \\
&\le 2bS_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0\right)+b^2S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{n_k}x_0\right) \\
&< \frac{1}{2b^2}\min\left\{S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0\right),S_b\left(x_{n_k-1},x_{n_k-1},x_{n_k}\right)\right\} \\
&\quad +b^2S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{n_k}x_0\right).
\end{aligned}$$

Letting $k\to\infty$, it follows that $\epsilon\le 0$. It is a contradition. Thus, (2.11) holds.

Now, from (2.9) and (2.10), we have

$$\begin{aligned}
\epsilon &\le S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\right) \\
&\le 2bS_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0\right)+b^2S_b\left(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0\right).
\end{aligned}$$

Letting $k\to\infty$, we have

$$4b^2\epsilon \le \lim_{k\to\infty} 4b^4S_b\left(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0\right) \tag{2.13}$$

Now

$$\begin{aligned}
\lim_{k\to\infty} &\ 4\ b^4S_b\left(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0\right) \\
&= \lim_{k\to\infty} 4b^4S_b\left(x_{m_k+1},x_{m_k+1},x_{n_k}\right) \\
&= \lim_{k\to\infty} 4b^4\ S_b\left(fx_{m_k},fx_{m_k},fx_{n_k-1}\right) \\
&\le \lim_{k\to\infty} M_f^5\left(x_{m_k},x_{n_k-1}\right)-\lim_{k\to\infty}\varphi\left(M_f^5\left(x_{m_k},x_{n_k-1}\right)\right) \\
&\le \lim_{k\to\infty} M_f^5\left(x_{m_k},x_{n_k-1}\right) \\
&= \lim_{k\to\infty}\max\left\{\begin{array}{c}
S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0\right), \\
S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0\right), \\
S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{n_k}x_0\right), \\
S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\right), \\
S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k+1}x_0\right)
\end{array}\right\} \\
&< \lim_{k\to\infty}\max\left\{\begin{array}{c}
\epsilon,0,0,S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\right), \\
S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k+1}x_0\right)
\end{array}\right\} \\
&= \lim_{k\to\infty}\max\left\{\begin{array}{c}
\epsilon,S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\right), \\
S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{m_k+1}x_0\right)
\end{array}\right\}.
\end{aligned}$$

But

$$\begin{aligned}
\lim_{k\to\infty} &\ S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0\right) \\
&\le \lim_{k\to\infty}\left[\begin{array}{c}
2b\ S_b\left(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0\right) \\
+b^2S_b\left(f^{n_k-1}x_0,f^{n_k-1}x_0,f^{n_k}x_0\right)
\end{array}\right] \\
&< 2b\epsilon.
\end{aligned}$$

Also

$$\begin{aligned}\lim_{k\to\infty} \quad & S_b\left(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{m_k+1}x_0\right) \\ & \leq \lim_{k\to\infty}\left[\begin{array}{c} 2bS_b\left(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{m_k}x_0\right) \\ +b^2S_b\left(f^{m_k}x_0, f^{m_k}x_0, f^{m_k+1}x_0\right)\end{array}\right] \\ & < 2b^2\epsilon.\end{aligned}$$

Therefore from (2.13), we have that

$$4b^2\epsilon \quad \leq \quad \max\left\{\epsilon, 2b\epsilon, 2b^2\epsilon\right\} = 2b^2\epsilon.$$

It is a contradiction.

Hence $\{f^n x_0\}$ is an S_b-Cauchy sequence in the complete regular S_b - metric space $(X, S_b, \preceq)$. By completeness of (X, S_b), it follows that the sequence $\{f^n x_0\}$ converges to α in (X, S_b). Thus

$$\lim_{n\to\infty} f^n x_0 = \alpha = \lim_{n\to\infty} f^{n+1}x_0.$$

Next, we will need the following. For each $n \geq 1$, at least one of the following assertions holds:

$$\frac{1}{4b^3}S_b\left(x_{n+1}, x_{n+1}, x_n\right) \leq S_b(\alpha, \alpha, x_n)$$

or

$$\frac{1}{4b^3}S_b\left(x_n, x_n, x_{n-1}\right) \leq S_b(\alpha, \alpha, x_{n-1}).$$

On the contrary, suppose that

$$\frac{1}{4b^3}S_b\left(x_{n+1}, x_{n+1}, x_n\right) > S_b(\alpha, \alpha, x_n)$$

and

$$\frac{1}{4b^3}S_b\left(x_n, x_n, x_{n-1}\right) > S_b(\alpha, \alpha, x_{n-1}).$$

Now consider

$$\begin{aligned}S_b\left(x_{n-1}, x_{n-1}, x_n\right) &\leq 2bS_b\left(x_{n-1}, x_{n-1}, \alpha\right) + b^2S_b\left(\alpha, \alpha, x_n\right) \\ &< 2b^2S_b\left(\alpha, \alpha, x_{n-1}\right) + b^2\ \frac{1}{4b^3}S_b\left(x_{n+1}, x_{n+1}, x_n\right) \\ &< 2b^2\frac{1}{4b^3}S_b\left(x_n, x_n, x_{n-1}\right) + \frac{1}{4b}S_b\left(x_{n+1}, x_{n+1}, x_n\right) \\ &= \frac{1}{2b}\ b\ S_b\left(x_{n-1}, x_{n-1}, x_n\right) + \frac{1}{4b}\ b\ S_b\left(x_n, x_n, x_{n+1}\right) \\ &\leq \frac{1}{2}\ S_b\left(x_{n-1}, x_{n-1}, x_n\right) + \frac{1}{4b^3}S_b\left(x_{n-1}, x_{n-1}, x_n\right) \\ &= \frac{2b^3+1}{4b^3}S_b\left(x_{n-1}, x_{n-1}, x_n\right) \\ &\leq \frac{3}{4}S_b\left(x_{n-1}, x_{n-1}, x_n\right).\end{aligned}$$

It is a contradiction. Hence our claim is valid.

Now we have to prove that α is fixed point of f. Since $x_n,\ \alpha \in X$ and X is regular, it follows that either $x_n \preceq \alpha$ or $\alpha \preceq x_n$. Suppose $f\alpha \neq \alpha$. From (2.1.3) and Lemma 1.10, we have that

$$\begin{aligned} 4b^4\left(\frac{1}{2b}S_b(f\alpha, f\alpha, \alpha)\right) &\leq \liminf_{n\to\infty}\ 4b^4\left(S_b\left(f\alpha, f\alpha, f^{n+1}x_0\right)\right) \\ &\leq \liminf_{n\to\infty}\ M_f^5\left(\alpha, x_n\right) - \liminf_{n\to\infty}\ \varphi\left(M_f^5\left(\alpha, x_n\right)\right). \end{aligned} \tag{2.14}$$

Then, from Lemmas 1.6 and 1.10 we get

$$\begin{aligned} \liminf_{n\to\infty}\quad & M_f^5\left(\alpha, x_n\right) \\ & \leq \limsup_{n\to\infty}\ \max\left\{\ 0, S_b\left(\alpha, \alpha, f\alpha\right), 0, 0, S_b\left(x_n, x_n, f\alpha\right)\ \right\} \\ & \leq \max\left\{\ bS_b\left(f\alpha, f\alpha, \alpha\right), b^3S_b\left(f\alpha, f\alpha, \alpha\right)\ \right\} \\ & = b^3S_b\left(f\alpha, f\alpha, \alpha\right). \end{aligned}$$

Hence, from (2.14) and above calculations, we have

$$\begin{aligned} 2b^3S_b(f\alpha, f\alpha, \alpha) &\leq b^3\ S_b\left(f\alpha, f\alpha, \alpha\right) - \liminf_{n\to\infty}\ \varphi\left(M_f^5\left(\alpha, x_n\right)\right) \\ &\leq b^3\ S_b\left(f\alpha, f\alpha, \alpha\right). \end{aligned}$$

It is a contradiction. So α is a fixed point of f.

Finally, let us prove the uniqueness of the fixed point.
Suppose α^* is another fixed point of f such that $\alpha \neq \alpha^*$.
It is clear that $\frac{1}{4b^3}\min\left\{S_b\left(\alpha, \alpha, f\alpha\right), S_b\left(\alpha^*, \alpha^*, f\alpha^*\right)\right\} \leq S_b\left(\alpha, \alpha, \alpha^*\right)$.
Since $\alpha, \alpha^* \in X$ and X is regular we have that α and α^* are comparable.

From (2.1.3), we have

$$\begin{aligned} 4b^4S_b\left(\alpha, \alpha, \alpha^*\right) &\leq M_f^5\left(\alpha, \alpha^*\right) - \varphi\left(M_f^5\left(\alpha, \alpha^*\right)\right) \\ &= \max\{S_b\left(\alpha, \alpha, \alpha^*\right), S_b\left(\alpha^*, \alpha^*, \alpha\right)\} \\ &\quad -\varphi\left(\max\{S_b\left(\alpha, \alpha, \alpha^*\right), S_b\left(\alpha^*, \alpha^*, \alpha\right)\}\right) \\ &\leq bS_b\left(\alpha, \alpha, \alpha^*\right). \end{aligned}$$

It is a contradiction. Hence α is the unique fixed point of f in (X, S_b) and the proof is completed.

Example 2.6. *Let $X = [0,1]$ and $S_b : X^3 \to \mathbb{R}^+$ by $S_b(x, y, z) = (|y+z-2x| + |y-z|)^2$ and $\preceq$ by $a \preceq b \iff a \leq b$, then $(X, S_b, \preceq)$ is a complete ordered S_b - metric space with $b = 4$. Define $f : X \to X$ by $f(x) = \frac{x}{32\sqrt{2}}$. Also define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \frac{t}{2}$.*

Clearly for all $x, y \in X$, $\frac{1}{4b^3}\min\{S_b(x,x,fx), S_b(y,y,fy)\} \le S_b(x,x,y)$. *And*

$$\begin{aligned} 4b^4 \quad & S_b(fx,fx,fy) \\ & = 4b^4(|fx+fy-2fx|+|fx-fy|)^2 \\ & = 4b^4\left(2\left|\frac{x}{32\sqrt{2}} - \frac{y}{32\sqrt{2}}\right|\right)^2 \\ & = \tfrac{1}{2}S_b(x,x,y) \\ & \le \tfrac{1}{2}M_f^5(x,y) \\ & = M_f^5(x,y) - \varphi\left(M_f^5(x,y)\right), \end{aligned}$$

where

$$M_f^5(x,y) = \max\left\{ \begin{array}{c} S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), \\ S_b(x,x,fy), S_b(y,y,fx) \end{array} \right\}.$$

Hence from Theorem 2.5, 0 is the unique fixed point of f.

Theorem 2.7. *Let* $(X, S_b, \preceq)$ *be an ordered complete* S_b *metric space and let* $f : X \to X$ *be a Suzuki type generalized* φ *- contraction with* $i = 3$ *or* 4. *If there exists* $x_0 \in X$ *with* $x_0 \preceq fx_0$, *then* f *has a unique fixed point in* X.

Proof. If we replace $M_f^3(x,y)$ or $M_f^4(x,y)$ in place of $M_f^5(x,y)$, the rest of the proof follows from Theorem 2.5.

Theorem 2.8. *Let* $(X, S_b, \preceq)$ *be an ordered complete* S_b *metric space and let* $f : X \to X$ *satisfy*
$\frac{1}{4b^3}\min\{S_b(x,x,fx), S_b(y,y,fy)\} \le S_b(x,x,y)$
$\Rightarrow S_b(fx,fx,fy) \le \lambda M_f^i(x,y)$,
where $\lambda \in \left[0, \frac{1}{4b^4}\right)$ *and* $i = 3$ *or* 4 *or* 5. *If there exists* $x_0 \in X$ *with* $x_0 \preceq fx_0$, *then* f *has a unique fixed point in* X.

3. Application to Integral Equations

In this section, we study the existence of a unique solution to an initial value problem, as an application of Theorem 2.5.

Theorem 3.1. *Consider the initial value problem*

$$x'(t) = T(t, x(t)), \quad t \in I = [0,1], \quad x(0) = x_0 \tag{3.1}$$

where $T : I \times \left[\frac{x_0}{4}, \infty\right) \to \left[\frac{x_0}{4}, \infty\right)$ *with*

$$\int_0^t T(x(s), y(s))ds = \min\left\{ \int_0^t T(s, x(s))ds, \int_0^t T(s, y(s))ds \right\}$$

and $x_0 \in \mathbb{R}$. *Then there exists a unique solution in* $C\left(I, \left[\frac{x_0}{4}, \infty\right)\right)$ *for the initial value problem (3.1).*

Proof. The integral equation corresponding to the initial value problem (3.1) is

$$x(t) = x_0 + \int_0^t T(s, x(s))ds.$$

Let $X = C\left(I, \left[\frac{x_0}{4}, \infty\right)\right)$ and $S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2$ *for* $x, y \in X$. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{3t}{4}$. Define $f : X \to X$ by

$$fx(t) \quad = \quad \frac{x_0}{4b^2} + \int_0^t T(x(s), y(s))ds. \tag{3.2}$$

Clearly for all $x, y \in X$, we have
$\frac{1}{4b^3} \min\{S_b(x, x, fx), S_b(y, y, fy)\} \leq S_b(x, x, y)$.
Now

$$\begin{aligned}
4b^4 \quad & S_b(fx(t), fx(t), fy(t)) \\
& = 4b^4 \{| \ fx(t) + fy(t) - 2fx(t) \ | + | \ fx(t) - fy(t) \ |\}^2 \\
& = 16b^4 \ | \ fx(t) - fy(t) \ |^2 \\
& = \tfrac{16b^4}{16b^4} \ | \ x_0 - y_0 \ |^2 \\
& \leq | \ x(t) - y(t) \ |^2 \\
& = \tfrac{1}{4} \ S_b(x, x, y) \\
& \leq M_f^5(x, y) - \varphi\left(M_f^5(x, y)\right),
\end{aligned}$$

where

$$M_f^5\,(x, y) = \max \left\{ \begin{array}{c} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ S_b(x, x, fy), S_b(y, y, fx) \end{array} \right\}.$$

Applying Theorem 2.5, we conclude that f has a unique fixed point in X.

4. Application to Homotopy

Theorem 4.1. *Let* (X, S_b) *be a complete* S_b *- metric space,* U *an open subset of* X *and* $\overline{U}$ *a closed subset of* X *such that* $U \subseteq \overline{U}$. *Suppose* $H : \overline{U} \times [0, 1] \to X$ *is an operator such that the following conditions are satisfied:*
(4.1.1) $x \neq H(x, \lambda)$ *for each* $x \in \partial U$ *and* $\lambda \in [0, 1]$,

(here ∂U *denotes the boundary of* U *in* X*),*

(4.1.2) $\frac{1}{4b^3} \min\left\{S_b\left(x, x, H(x, \lambda)\right), S_b\left(y, y, H(y, \lambda)\right)\right\} \leq S_b\left(x, x, y\right)$ *implies that*

$$4b^4 S_b(H(x, \lambda), H(x, \lambda), H(y, \lambda)) \leq S_b(x, x, y) - \varphi(S_b(x, x, y))$$

for all $x, y \in \overline{U}$ and $\lambda \in [0,1]$, where $f : [0,\infty) \to [0,\infty)$ is continuous, non-decreasing and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi continuous with $\varphi(t) > 0$ for $t > 0$,
(4.1.3) *there exists an $M \geq 0$ such that*

$$S_b(H(x,\lambda), H(x,\lambda), H(x,\mu)) \leq M|\lambda - \mu|,$$

for every $x \in \overline{U}$ and $\lambda, \mu \in [0,1]$.

Then $H(.,0)$ has a fixed point if and only if $H(.,1)$ has a fixed point.

Proof. Consider the set

$$A = \{\lambda \in [0,1] : x = H(x,\lambda) \text{ for some } x \in U\}.$$

Suppose that $H(.,0)$ has a fixed point in U. Then we have that $0 \in A$. So A is non-empty set. We will show that A is both open and closed in $[0,1]$ and so by the connectedness we have that $A = [0,1]$. As a result, $H(.,1)$ has a fixed point in U.

First we show that A is closed in $[0,1]$. To see this let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$. We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1,2,3,\ldots$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$.

Consider

$$\begin{aligned} S_b \quad & (x_n, x_n, x_{n+1}) \\ & = S_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_{n+1})) \\ & \leq 2bS_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)) \\ & \quad + b^2 S_b(H(x_{n+1},\lambda_n), H(x_{n+1},\lambda_n), H(x_{n+1},\lambda_{n+1})) \\ & \leq 2bS_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)) + b^2 M|\lambda_n - \lambda_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) \leq \lim_{n\to\infty} 2bS_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)) + 0.$$

Since

$$\frac{1}{4b^3} \min\left\{ \begin{array}{c} S_b\left(x_n, x_n, H(x_n,\lambda)\right), \\ S_b\left(x_{n+1}, x_{n+1}, H(x_{n+1},\lambda)\right) \end{array} \right\} \leq S_b\left(x_n, x_n, x_{n+1}\right),$$

from (4.1.2), we have that

$$\begin{aligned} \lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) \quad & \leq \lim_{n\to\infty} 4b^4 S_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n)) \\ & \leq \lim_{n\to\infty} \left[S_b(x_n, x_n, x_{n+1}) - \varphi(S_b(x_n, x_n, x_{n+1}))\right]. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0. \tag{4.1}$$

Now we prove that $\{x_n\}$ is an S_b-Cauchy sequence in (X, S_b). On the contrary, suppose that $\{x_n\}$ is not S_b-Cauchy. There exists an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$S_b(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon \tag{4.2}$$

and

$$S_b(x_{m_k}, x_{m_k}, x_{n_k-1}) < \epsilon. \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\begin{aligned} \epsilon &\leq S_b(x_{m_k}, x_{m_k}, x_{n_k}) \\ &\leq 2bS_b(x_{m_k}, x_{m_k}, x_{m_k+1}) + b^2 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}). \end{aligned}$$

Letting $k \to \infty$, we have that

$$\frac{\epsilon}{b^2} \leq \lim_{n\to\infty} S_b\left(x_{m_k+1}, x_{m_k+1}, x_{n_k}\right).$$

But

$$\begin{aligned} \lim_{n\to\infty} \; & S_b\left(x_{m_k+1}, x_{m_k+1}, x_{n_k}\right) \\ &\leq 4b^4 \lim_{n\to\infty} S_b\left(H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{n_k}, \lambda_{n_k})\right) \\ &\leq \lim_{n\to\infty} \left[S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) - \varphi(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}))\right]. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} S_b\left(x_{m_k+1}, x_{m_k+1}, x_{n_k}\right) = 0.$$

Therefore,

$$\epsilon = 0, \tag{4.4}$$

which is a contradiction. Hence $\{x_n\}$ is an S_b-Cauchy sequence in (X, S_b) and by the completeness of (X, S_b), there exists an $\alpha \in U$ with

$$\lim_{n\to\infty} x_n = \alpha = \lim_{n\to\infty} x_{n+1}. \tag{4.5}$$

Since

$$\frac{1}{4b^3} \min\left\{S_b\left(\alpha, \alpha, H(\alpha, \lambda)\right), S_b\left(x_n, x_n, H(x_n, \lambda)\right)\right\} \leq S_b\left(\alpha, \alpha, x_n\right),$$

we have

$$\begin{aligned} \frac{1}{2b} \; & S_b\left(H(\alpha, \lambda), H(\alpha, \lambda), \alpha\right) \\ &\leq \lim_{n\to\infty} \inf \; \frac{1}{2b} S_b\left(H(\alpha, \lambda), H(\alpha, \lambda), H(x_n, \lambda)\right) \\ &\leq \lim_{n\to\infty} \inf \; 4b^4 S_b\left(H(\alpha, \lambda), H(\alpha, \lambda), H(x_n, \lambda)\right) \\ &\leq \lim_{n\to\infty} \inf\left[S_b\left(\alpha, \alpha, x_n\right) - \varphi(S_b(\alpha, \alpha, x_n))\right] \\ &= 0. \end{aligned}$$

It follows that $\alpha = H(\alpha, \lambda)$. Thus $\lambda \in A$. Hence A is closed in [0, 1].

Now, let us prove that A is open in [0, 1]. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since U is open, then there exists $r > 0$ such that $B_{S_b}(x_0, r) \subseteq U$. Choose $\lambda \in (\lambda_0 - \tilde{\epsilon}, \lambda_0 + \tilde{\epsilon})$ such that $|\lambda - \lambda_0| \leq \frac{1}{M^n} < \tilde{\epsilon}$. Then for $x \in \overline{B_p(x_0, r)} = \{x \in X | S_b(x, x, x_0) \leq r + b^2 S_b(x_0, x_0, x_0)\}$. Also

$$\frac{1}{4b^3} \min \{S_b(x, x, H(x, \lambda)), S_b(x_0, x_0, H(x_0, \lambda))\} \leq S_b(x, x, x_0).$$

$$\begin{aligned} S_b \quad & (H(x,\lambda), H(x,\lambda), x_0) \\ & = S_b(H(x,\lambda), H(x,\lambda), H(x_0,\lambda_0)) \\ & \leq 2bS_b(H(x,\lambda), H(x,\lambda), H(x,\lambda_0)) + b^2 S_b(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)) \\ & \leq 2bM|\lambda - \lambda_0| + b^2 S_b(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)) \\ & \leq \tfrac{2b}{M^{n-1}} + b^2 S_b(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} S_b(H(x,\lambda), H(x,\lambda), x_0) & \leq b^2 S_b(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)) \\ & \leq 4b^4 S_b(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)) \\ & \leq S_b(x,x,x_0) - \varphi(S_b(x,x,x_0)) \\ & \leq S_b(x,x,x_0). \end{aligned}$$

$$\begin{aligned} S_b(H(x,\lambda), H(x,\lambda), x_0) & \leq S_b(x,x,x_0) \\ & \leq r + b^2 S_b(x_0,x_0,x_0). \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \tilde{\epsilon}, \lambda_0 + \tilde{\epsilon})$, $H(x, \lambda) \in \overline{B_p(x_0, r)}$ implies $H(., \lambda) : \overline{B_p(x_0, r)} \to \overline{B_p(x_0, r)}$. Since also (4.1.2) holds and φ is continuous with $\varphi(t) > 0$ for $t > 0$, then all conditions of Theorem 2.5 are satisfied.

Thus we deduce that $H(., \lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in U since (4.1.1) holds. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \tilde{\epsilon}, \lambda_0 + \tilde{\epsilon})$. Hence $(\lambda_0 - \tilde{\epsilon}, \lambda_0 + \tilde{\epsilon}) \subseteq A$ and therefore A is open in [0, 1]. For the reverse implication, we use the same strategy.

Corollary 4.2. *Let (X, p) be a complete partial metric space, U is an open subset of X and $H : \overline{U} \times [0, 1] \to X$ with the following properties:*

(1) *$x \neq H(x, t)$ for each $x \in \partial U$ and each $\lambda \in [0, 1]$ (here ∂U denotes the boundary of U in X),*

(2) *there exist $x, y \in \overline{U}$ and $\lambda \in [0, 1], L \in \left[0, \frac{1}{4b^4}\right)$, such that*

$$S_b(H(x,\lambda), H(x,\lambda), H(y,\mu)) \leq L S_b(x, x, y),$$

(3) *there exists $M \geq 0$, such that*

$$\frac{1}{4b^3} \min \{S_b(x, x, H(x, \lambda)), S_b(y, y, H(y, \lambda))\} \leq S_b(x, x, y) \text{ implies that}$$

$$S_b(H(x,\lambda), H(x,\lambda), H(x,\mu)) \leq M|\lambda - \mu)|$$

for all $x \in \overline{U}$ and $\lambda, \mu \in [0, 1]$.

If $H(., 0)$ has a fixed point in U, then $H(., 1)$ has a fixed point in U.

Proof. Proof follows by taking $f(x) = x, \varphi(x) = x - Lx$ with $L \in \left[0, \frac{1}{4b^4}\right)$ in Theorem 4.1.

5. Conclusions

In this paper we conclude some applications of fixed point theorems in partially ordered S_b - metric spaces.

Acknowledgement

The work of the first author is supported by Ministry of Education, Science and Technological Development of the Republic of Serbia under the project TR-37017.

References

[1] Abbas, M., Ali Khan, M., and Radenović, S. Common coupled fixed point theorems in cone metric spaces for w-compatible mappings. *Appl. Math. Comput. 217*, 1 (2010), 195–202.

[2] Banach, S. *Théorie des opérations linéaires.* Éditions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original.

[3] Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostraviensis 1* (1993), 5–11.

[4] Gholidahneh, A., Sedghi, S., Došenović, T., and Radenović, S. Ordered s-metric spaces and coupled common fixed point theorems of integral type contraction. *Mathematics Interdisciplinary Research 2* (2017), 71–84.

[5] Hima Bindu, V. M. L., and Kishore, G. N. V. Suzuki type unique common tripled fixed point theorem for weak ϕcontraction in multiplicative metric spaces. *Int. J. Pure Appl. Math. 108*, 3 (2016), 561–580.

[6] Hima Bindu, V. M. L., Kishore, G. N. V., and Rao, K. P. R. $\psi - \phi$ contraction on suzuki type unique common coupled fixed point theorem in partially ordered multiplicative metric spaces. *Int. J. Pure Appl. Math. 109*, 3 (2016), 583–600.

[7] Hima Bindu, V. M. L., Kishore, G. N. V., and Rao, K. P. R. A suzuki type unique common coupled fixed point theorem for jungck type maps in partial metric spaces. *Far East J. Math. Sci. (FJMS) 100*, 4 (2016), 521–536.

[8] Kishore, G. N. V., Rao, K. P. R., and Hima Bindu, V. M. L. Suzuki type unique common fixed point theorem in partial metric spaces by using (C): condition with rational expressions. *Afr. Mat. 28*, 5-6 (2017), 793–803.

[9] Kishore, G. N. V., Rao, K. P. R., Panthi, D., Srinivasa Rao, B., and Satyanaraya, S. Some applications via fixed point results in partially ordered s_b metric spaces. *Fixed Point Theory Appl. 2017* (2017), 10.

[10] Lakshmikantham, V., and Ćirić, L. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal. 70*, 12 (2009), 4341–4349.

[11] Ram Prasad, D., Kishore, G. N. V., and Priyanka, K. Applications of fixed point theory in integral equation and homotophy using partially ordered s_b metric spaces. *International J. Engin. Techn.(UAE) 7*, 3.3 (2018), 146 – 150.

[12] Rao, K. P. R., Kishore, G. N. V., and Hima Bindu, V. M. L. Suzuki type fixed point theorem for rational contraction in partial metric spaces. *Glob. J. Pure Appl. Math. 11*, 4 (2015), 2223–2231.

[13] Rohen, Y., Došenović, T., and Radenović, S. A note on the paper "a fixed point theorems in s_b- metric spaces". *Filomat 31*, 11 (2017), 3335–3346.

[14] Sedghi, S., Altun, I., Shobe, N., and Salahshour, M. Some properties of s-metric space and fixed point results. *KyungpookMath. J. 54*, 1 (2014), 113–122.

[15] Sedghi, S., Gholidahneh, A., Došenović, T., Esfahani, J., and Radenović, S. Common fixed point of four maps in s_b- metric spaces. *J. Linear and Topological Algebra 5*, 2 (2016), 93–104.

[16] Sedghi, S., Rezaee, M. M., Došenović, T., and Radenović, S. Common fixed point theorems for contractive mappings satisfying ϕ-maps in s-metric spaces. *Acta Sci. Math. 8*, 2 (2016), 298–311.

[17] Sedghi, S., Shobe, N., and Aliouche, A. A generalization of fixed point theorem in $s-$metric spaces. *Mat. Vesnik 64*, 3 (2012), 258–266.

[18] Sedghi, S., Shobe, N., and Došenović, T. Fixed point results in $s-$ metric spaces. *Nonlinear Funct. Anal. Appl. 20*, 1 (2015), 55–67.

[19] Shatanawi, W., Rao, K. P. R., Kishore, G. N. V., Abodayeh, K., and Ram Prasad, D. A new suzuki type common coupled fixed point result along with $(\psi - \phi)$ - contractive condition in sb metric spaces. *Nonlinear Funct. Anal. Appl. 23*, 2 (2018), 225–245.

[20] Srinuvasa Rao, B., Kishore, G. N. V., Hari Krishna, Y., and Sarwar, M. Generalized (α, β) rational contractions in ordered s_b metric spaces with applications. *Intern. J. Mechan. Engin. Techn. 9*, 3 (2018), 272–282.

[21] Suzuki, T. A generalized banach contraction principle which characterizes metric completeness. *Proc. Amer. Math. Soc. 136*, 5 (2008), 1861–1869.

[22] Zaed, M., and Brailey, S. A new approach to generalized metric spaces. *J. Nonlinear Convex Anal. 7*, 2 (2006), 289–297.

Received by the editors December 7, 2018
First published online January 12, 2019