Novi Sad J. Math. Vol. 49, No. 2, 2019, 1-16
https://doi.org/10.30755/NSJOM.04007

SEMILOCAL CONVERGENCE ANALYSIS AND COMPARISON OF ALTERNATIVE COMPUTATIONAL EFFICIENCY OF THE SIXTH-ORDER METHOD IN BANACH SPACES[1]

Jai Prakash Jaiswal[2][3][4]

Abstract. The purpose of the this paper is to discuss the semilocal convergence analysis of the sixth-order method for solving nonlinear equations in Banach spaces by using recurrence relations approach. The existence and uniqueness results have been derived, followed by error bound. The alternative computational efficiency of the considered algorithm with identical as well as unlike order schemes is also analyzed. Lastly, theoretical results have been verified by discussing the numerical example.

AMS Mathematics Subject Classification (2010): 65H10; 65J15; 47J25

Key words and phrases: nonlinear equation; Banach space; recurrence relation; semilocal convergence; alternative computational efficiency

1. Introduction

There are several types of convergence results are used to approximate solutions of nonlinear equations. The first, which we call a local convergence theorem, begins with the assumption that a particular solution x^* exists, and then asserts that there is a neighborhood U of x^* such that for all initial vectors in U the iterates generated by the process are well defined and converge to x^*. The second type of convergence theorem, which is called semilocal, does not require knowledge of the existence of a solution, but states that, starting from initial vectors for which some stiff conditions are satisfied, convergence to some solutions x^* is guaranteed. Moreover, theorems of this type usually include computable estimates for the error $x_n - x^*$, a possibility not afforded by the local convergence theorems.

Newton's method, which has quadratic rate of convergence, is one of the well established methods for solving nonlinear equations of the form $\Lambda(x) = 0$. The semilocal convergence of Newton's method in Banach spaces was established by Kantorovich in [8]. The convergence of the sequence obtained by the

[1]This work is supported by Science and Engineering Research Board (SERB), New Delhi, India under the scheme Start-up-Grant (Young Scientists) (Ref. No. YSS/2015/001507).

[2]Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal-462003

[3]Faculty of Science, Barkatullah University, Bhopal, M.P.462006, India

[4]Regional Institute of Education, Bhopal, M.P. 462013, India,
e-mail: asstprofjpmanit@gmail.com

iterative expression is derived from the convergence of majorizing sequences. In [11], Rall has suggested a different technique for the semilocal convergence of these methods, which is based on recurrence relations. It is worth mentioning that higher order convergence requires computation of derivatives of higher order which are very expansive in general. But higher order methods have their importance as in some applications involving stiff systems of equations faster convergence is required. It is obvious that, in order to increase the order of convergence, the operational cost must increase simultaneously. The main challenge in numerical analysis is to find the equilibrium between the convergence speed and operational cost. In other words, a scheme is better if it simultaneously improves the order and the efficiency index. For obtaining better efficiency, many higher order methods like third-order [1, 10], fourth-order [7, 15], fifth-order [2, 3] and sixth-order [13, 14] etc. are discussed in the literature. In this study, we discuss the semilocal convergence analysis of the sixth-order method proposed by Cordero et al. in [4], followed by its computational efficiency analysis in the sense of Grau et al. [6]. The method in Banach spaces is given by

$$\begin{aligned} z_k &= x_k - \frac{2}{3}\Gamma_k \Lambda(x_k), \\ y_k &= x_k - \frac{1}{2}[3\Lambda'(z_k) - \Lambda'(x_k)]^{-1}[3\Lambda'(z_k) + \Lambda'(x_k)]\Gamma_k \Lambda(x_k), \\ x_{k+1} &= y_k - 2[3\Lambda'(z_k) - \Lambda'(x_k)]^{-1}\Lambda(y_k), \end{aligned} \tag{1.1}$$

where $\Gamma_k = \Lambda'(x_k)^{-1}$.

In this paper, we study the semilocal convergence of the existing sixth-order scheme using the recurrence relations technique. For this, first we establish the system of recurrence relations and then prove the convergence result, followed by its error estimate. We also perform a comparative study of computational efficiency in the case of nonlinear systems, in which we compare the presented method with several previously existing schemes. Some of those previously existing schemes used for comparison are of the same convergence order, and some are of different convergence order. At last, numerical example is considered in order to verify the theoretical discussions.

2. Preliminary Results

Suppose B_1 and B_2 are two Banach spaces and $x_0 \in \mathfrak{D}$. Let the nonlinear operator $\Lambda : \mathfrak{D} \subseteq B_1 \to B_2$ be third-order Fréchet differentiable, where $\mathfrak{D}$ is an open subset of B_1. Consider the hypotheses

$(R1)||\Gamma_0 \Lambda(x_0)|| \leq \lambda$,

$(R2)||\Gamma_0|| \leq \zeta$,

$(R3)||\Lambda''(x)|| \leq K_1$, $x \in \mathfrak{D}$,

$(R4)||\Lambda'''(x)|| \leq K_2$, $x \in \mathfrak{D}$,

$(R5)$ $\exists$ a positive real number K_3 such that

$$||\Lambda'''(x) - \Lambda'''(y)|| \leq K_3||x - y||, \forall\ x, y \in \mathfrak{D}. \tag{2.1}$$

Initially, in the coming lemma we start with the some approximations of the nonlinear operator Λ, which will be used in the successive results.

Lemma 2.1. *[7] Suppose the nonlinear operator $\Lambda : \mathfrak{D} \subseteq B_1 \to B_2$ be continuously third-order Fréchet differentiable, then*

$$\begin{aligned}
&\Lambda(y_k) \\
= \;& \int_0^1 \Lambda''(r_k + r(y_k - r_k))(1-r)dr(y_k - r_k)^2 \\
&+ \int_0^1 \left[\Lambda''(x_k + r(r_k - x_k))(1-r) - \frac{1}{2}\Lambda''\left(x_k + \frac{2}{3}r(r_k - x_k)\right)\right] dr \\
&(r_k - x_k)^2 \\
&+ \int_0^1 \left[\Lambda''(x_k + r(r_k - x_k)) - \Lambda''\left(x_k + \frac{2}{3}r(r_k - x_k)\right)\right] dr(r_k - x_k) \\
&[\Lambda'(x_k) - 3\Lambda'(z_k)]^{-1} \int_0^1 \Lambda''\left(x_k + \frac{2}{3}r(r_k - x_k)\right) dr(r_k - x_k)^2, \qquad (2.2)
\end{aligned}$$

where $r_k = x_k - \Gamma_k \Lambda(x_k)$.

Lemma 2.2. *Suppose the hypotheses of Lemma 2.1 hold, then*

$$\begin{aligned}
&\Lambda(x_{k+1}) \\
= \;& \left\{\int_0^1 \left[\Lambda''\left(x_k + \frac{2}{3}r(r_k - x_k)\right) - \Lambda''(x_k + r(r_k - x_k))\right] dr(r_k - x_k)\right. \\
&\left. - \int_0^1 [\Lambda'(y_k + r(x_{k+1} - y_k)) - \Lambda'(r_k)]dr\right\} \left[I + \frac{3}{2}H(x_k)\right]^{-1} \Gamma_k \Lambda(y_k), \qquad (2.3)
\end{aligned}$$

where $H(x_k) = \Gamma_k[\Lambda'(z_k) - \Lambda'(x_k)]$.

Proof. Form Taylor expansion, one can deduce

$$\begin{aligned}
\Lambda(x_{k+1}) &= \Lambda(y_k) + \Lambda'(r_k)(x_{k+1} - y_k) \\
&\quad + \int_0^1 [\Lambda'(y_k + r(x_{k+1} - y_k)) - \Lambda'(r_k)]dr(x_{k+1} - y_k). \qquad (2.4)
\end{aligned}$$

By an application of the last sub-step of the considered scheme (1.1), we can write

$$\begin{aligned}
&\Lambda(y_k) + \Lambda'(r_k)(x_{k+1} - y_k) \\
&= \Lambda(y_k) - \Lambda'(r_k)\left[I + \frac{3}{2}H(x_k)\right]^{-1} \Gamma_k \Lambda(y_k) \\
&= \left[\{\Lambda'(x_k) - \Lambda'(r_k)\} + \frac{3}{2}\Lambda'(x_k)H(x_k)\right]\left[I + \frac{3}{2}H(x_k)\right]^{-1} \Gamma_k \Lambda(y_k) \\
&= \left\{\int_0^1 \left[\Lambda''\left(x_k + \frac{2}{3}r(r_k - x_k)\right) - \Lambda''(x_k + r(r_k - x_k))\right] dr(r_k - x_k)\right\} \\
&\left[I + \frac{3}{2}H(x_k)\right]^{-1} \Gamma_k \Lambda(y_k). \qquad (2.5)
\end{aligned}$$

Putting the expression (2.5) in the relation (2.4) followed by the final sub-step of the method (1.1), one can attain the expression (2.3). □

Now we consider the below mentioned functions. Let us denote

$$f_1(r) = \frac{r^2 - 2r + 2}{2(1-r)^2}, \tag{2.6}$$

$$f_2(r) = \frac{1}{1 - rg(r)}, \tag{2.7}$$

$$g(r,s,t) = \frac{r^3}{8(1-r)^2} + \frac{rs}{12(1-r)} + \frac{17}{216}t, \tag{2.8}$$

$$f_3(r,s,t) = \frac{g(r,s,t)}{(1-r)}\left[\frac{r^2}{2(1-r)} + \frac{s}{6} + \frac{r}{2(1-r)}g(r,s,t)\right]. \tag{2.9}$$

Denote $f^*(r) = f_1(r)r - 1$. Because $f^*(0) = -1$ and $f^*(1) = +\infty$, hence $f^*(r)$ contains at least one zero in $(0,1)$. Let ρ be the lowest positive zero $f^*(r) = 0$. Now, we are going to mention a few properties of f_1, f_2, f_3, which are given by relations (2.6), (2.7) and (2.9), respectively.

Lemma 2.3. *Let the functions f_1, f_2 and f_3 be given by equations (2.6), (2.7) and (2.9), respectively, then:*
(i) $f_1(r)$ and $f_2(r)$ are increasing and also both are greater than unity for $r \in (0,\rho)$,
(ii) $f_3(r,s,t)$ is increasing for $r \in (0,\rho)$, $s > 0$, $t > 0$.

Proof. The proof is straightforward. □

Now, we represent

$$\begin{aligned}
\lambda_0 &= \lambda, \\
\zeta_0 &= \zeta, \\
\alpha_0 &= K_1\zeta\lambda, \\
\beta_0 &= K_2\zeta\lambda^2, \\
\chi_0 &= K_3\zeta\lambda^3, \\
\varsigma &= 1/f_2(\alpha_0), \\
\varrho &= f_2^2(\alpha_0)f_3(\alpha_0,\beta_0,\chi_0), \\
\varrho &= \frac{f_1(\alpha_0)}{1 - f_2(\alpha_0)f_3(\alpha_0,\beta_0,\chi_0)}.
\end{aligned} \tag{2.10}$$

Additionally, we recursively define below the sequences for $k \geq 0$

$$\xi_{k+1} = \xi_k h(a_k), \tag{2.11}$$

$$\lambda_{k+1} = \lambda_k f_2(a_k) f_3(\alpha_k, \beta_k, \chi_k), \tag{2.12}$$

$$\alpha_{k+1} = \alpha_k f_2(a_k)^2 f_3(\alpha_k, \beta_k, \chi_k), \tag{2.13}$$

$$\beta_{k+1} = \beta_k f_2(a_k)^3 f_3^2(\alpha_k, \beta_k, \chi_k), \tag{2.14}$$

$$\chi_{k+1} = \chi_k f_2(a_k)^4 f_3^3(\alpha_k, \beta_k, \chi_k). \tag{2.15}$$

Lemma 2.4. *Let the functions f_1, f_2 and f_3 be given by equations* (2.6), (2.7) *and* (2.9)*, respectively. If* $0 < \alpha_0 < \rho$ *and*

$$f_2(\alpha_0)^2 f_3(\alpha_0, \beta_0, \chi_0) < 1, \tag{2.16}$$

then the following observations are true:
(i) $f_2(\alpha_k) > 1$ *and* $f_2(\alpha_k) f_3(\alpha_k, \beta_k, \chi_k) < 1$ *for* $k \geq 0$,
(ii) the sequences $\{\lambda_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ *and* $\{f_2(\alpha_k) f_3(\alpha_k, \beta_k, \chi_k)\}$ *are decreasing,*
(iii) $f_1(\alpha_k) a_k < 1$ *and* $f_2(\alpha_k)^2 f_3(\alpha_k, \beta_k, \chi_k) < 1$ *for* $k \geq 0$.

Proof. By the virtue of Lemma 2.3 and equations (2.12)-(2.16), it can be quickly proved that the results hold for $k = 0$, and then by using mathematical induction it can be easily shown that it also holds for all $k \geq 1$. □

Lemma 2.5. *Let the functions f_1, f_2 and f_3 be given by equations* (2.6), (2.7) *and* (2.9)*, respectively. Suppose* $\theta \in (0, 1)$*, then* $f_1(\theta r) < f_1(r)$, $f_2(\theta r) < f_2(r)$ *and* $f_3(\theta r, \theta^2 s, \theta^3 t) < \theta^5 f_3(r, s, t)$ *for* $r \in (0, \rho)$.

Proof. The lemma immediately follows by the virtue of equations (2.6)-(2.9) and fact that $\theta \in (0, 1)$ and $r \in (0, \rho)$. □

Lemma 2.6. *Suppose the hypotheses of Lemma* 2.4 *are true, then*

$$f_2(\alpha_k) f_3(\alpha_k, \beta_k, \chi_k) \leq \varsigma \varrho^{6^k}, k \geq 0, \tag{2.17}$$

and

$$\prod_{i=0}^{k} f_2(\alpha_i) f_3(\alpha_i, \beta_i, \chi_i) \leq \varsigma^{k+1} \varrho^{\left(\frac{6^{k+1}-1}{5}\right)}. \tag{2.18}$$

Proof. Since

$$\begin{aligned} \alpha_1 &= \alpha_0 f_2(\alpha_0)^2 f_3(\alpha_0, \beta_0, \chi_0) = \varrho \alpha_0, \\ \beta_1 &= \beta_0 f_2(\alpha_0)^3 f_3^2(\alpha_0, \beta_0, \chi_0) < \varrho^2 \beta_0 \text{ and} \\ \chi_1 &= \chi_0 f_2(\alpha_0)^4 f_3^3(\alpha_0, \beta_0, \chi_0) < \varrho^3 \chi_0 \end{aligned}$$

we can write

$$\begin{aligned} f_2(\alpha_1) f_3(\alpha_1, \beta_1, \chi_1) &< f_2(\varrho\alpha_0) f_3(\varrho\alpha_0, \varrho^2\beta_0, \varrho^3\chi_0) \\ &< \varrho^5 f_2(\alpha_0) f_3(\alpha_0, \beta_0, \chi_0) = \varrho^{6^1}. \end{aligned}$$

Assume $f_2(\alpha_i)f_3(\alpha_i,\beta_i,\chi_i) \leq \varsigma\varrho^{6^i}, i \geq 1$. By virtue of the Lemma 2.4, we can write

$$\begin{aligned}
&f_2(\alpha_{i+1})f_3(\alpha_{i+1},\beta_{i+1},\chi_{i+1}) \\
&\quad < f_2(\alpha_i)f_3(\alpha_i f_2^2(\alpha_i)f_3(\alpha_i,\beta_i,\chi_i), \beta_i f_2^3(\alpha_i)f_3^2(\alpha_i,\beta_i,\chi_i), \chi_i f_2^4(\alpha_i)f_3^3(\alpha_i,\beta_i,\chi_i)) \\
&\quad < f_2(\alpha_i)^5\{f_2(\alpha_i)f_3(\alpha_i,\beta_i,\chi_i)\}^6 < f_2(\alpha_i)^5\{\varsigma\varrho^{6^i}\}^6 = \varsigma\varrho^{6^{i+1}}.
\end{aligned}$$

Hence $f_2(\alpha_k)f_3(\alpha_k,\beta_k,\chi_k) \leq \varsigma\varrho^{6^k}$ is true for all $k \geq 0$. By an application of inequality (2.17), we attain

$$\prod_{i=0}^{k} f_2(\alpha_i)f_3(\alpha_i,\beta_i,\chi_i) \leq \prod_{i=0}^{k} \varsigma\varrho^{6^k} \leq \varsigma^{k+1}\varrho^{\sum_{i=0}^{k} 6^i} \leq \varsigma^{k+1}\varrho^{\left(\frac{6^{k+1}-1}{5}\right)},$$

which completes the proof. □

Lemma 2.7. *Suppose the hypotheses of Lemma* 2.4 *are true, then the sequence* $\{\lambda_k\}$ *satisfies the inequality*

$$\lambda_k \leq \lambda\varsigma^k\varrho^{\frac{6^k-1}{5}}, k \geq 0 \tag{2.19}$$

and hence the sequence $\{\lambda_k\}$ *converges to zero. Moreover, for any* $k \geq 0$, $m \geq 1$ *it satisfies*

$$\prod_{i=k}^{k+m} \lambda_i \leq \lambda\varsigma^k\varrho^{\frac{6^k-1}{5}} \left(\frac{1-\varsigma^{m+1}\varrho^{\frac{6^k(6^m+4)}{5}}}{1-\varsigma\varrho^{6^k}}\right). \tag{2.20}$$

Proof. In view of the relation (2.12) and inequality (2.17), we obtain

$$\begin{aligned}
\lambda_k &= \lambda_{k-1}f_2(\alpha_{k-1})f_3(\alpha_{k-1},\beta_{k-1},\chi_{k-1}) \\
&= \lambda\prod_{i=0}^{k-1} f_2(\alpha_i)f_3(\alpha_i,\beta_i,\chi_i) \\
&\leq \varsigma^k\varrho^{\left(\frac{6^k-1}{5}\right)}.
\end{aligned}$$

Because $\varsigma, \varrho < 1$, it implies that $\lambda_k \to 0$ as $k \to \infty$. Indicate

$$\sigma = \sum_{i=k}^{k+m} \varsigma^i\varrho^{\frac{6^i}{5}}, where\ k \geq 0, m \geq 1.$$

The above expression can be also edited as

$$\begin{aligned}
\sigma &\leq \varsigma^i\varrho^{\frac{6^i}{5}} + \varrho^{6^i}\left(\sum_{i=k+1}^{k+m} \varsigma^i\varrho^{\frac{6^{i-1}}{5}}\right) \\
&= \varsigma^i\varrho^{\frac{6^i}{5}} + \varsigma\varrho^{6^i}\left(\sigma - \varsigma^{k+m}\varrho^{\frac{6^{k+m}}{5}}\right).
\end{aligned}$$

After manipulation, the above expression assumes the form

$$\sigma \le \varsigma^k \varrho^{\frac{6^k}{5}} \left(\frac{1-\varsigma^{m+1}\varrho^{\frac{6^k(6^m+4)}{5}}}{1-\varsigma\varrho^{6^k}} \right).$$

And hence

$$\begin{aligned} \sum_{i=k}^{k+m} \lambda_i &\le \lambda \left(\sum_{i=k}^{k+m} \varsigma^k \varrho^{\frac{6^k-1}{5}} \right) \\ &\le \lambda \varsigma^k \varrho^{\frac{6^k-1}{5}} \left(\frac{1-\varsigma^{m+1}\varrho^{\frac{6^k(6^m+4)}{5}}}{1-\varsigma\varrho^{6^k}} \right). \end{aligned}$$

And thus $\sum_{i=0}^{\infty} \lambda_i$ exists. □

3. Recurrence relations for the scheme

First, we denote $\mathfrak{T}(x,r) = \{y \in B_1 : ||y-x|| < r\}$ and $\overline{\mathfrak{T}(x,r)} = \{y \in B_1 : ||y-x|| \le r\}$. In this section, we are going to derive some recurrence relations for the considered scheme (1.1), keeping in mind that the hypothesis assumed in the earlier sections must hold.

When $k=0$, the existence of Γ_0 implies the existence z_0, r_0 and w_0. Also, we have

$$||z_0 - x_0|| \le \frac{2}{3}\lambda_0, \tag{3.1}$$

$$||r_0 - x_0|| \le \lambda_0. \tag{3.2}$$

and

$$||w_0 - x_0|| = \frac{1}{2}||\Gamma_0 \Lambda(x_0)|| \le \frac{1}{2}\lambda_0, \tag{3.3}$$

where $w_0 = x_0 - \frac{1}{2}\Gamma_0\Lambda(x_0)$. Hence z_0, r_0 and $w_0 \in \mathfrak{T}(x_0, k\lambda)$. If we represent $H(x_0) = \Gamma_0[\Lambda'(z_0) - \Lambda'(x_0)]$, then

$$||H(x_0)|| \le \frac{2}{3}\alpha_0. \tag{3.4}$$

Hence in view of the Banach Lemma and the fact that $\alpha_0 < 1$, $[I + \frac{3}{2}H(x_0)]^{-1}$ exists and satisfies

$$|| \left[I + \frac{3}{2}H(x_0) \right]^{-1} || \le \frac{1}{1-\alpha_0}. \tag{3.5}$$

Therefore

$$||y_0 - w_0|| \le \frac{1}{2(1-\alpha_0)}\lambda_0 \tag{3.6}$$

and hence

$$||y_0 - x_0|| \le \frac{2-\alpha_0}{2(1-\alpha_0)}\lambda_0. \tag{3.7}$$

It can be quickly calculated that

$$||\Lambda(y_0)|| \le \frac{\alpha_0}{2(1-\alpha_0)}\frac{\lambda_0}{\zeta_0}. \tag{3.8}$$

The last sub-step of the scheme (1.1) shows that

$$||x_1 - y_0|| \le \frac{\zeta}{1-\alpha_0}||\Lambda(y_0)||. \tag{3.9}$$

By virtue of the triangle inequality and equations (3.7)-(3.9), the above expression becomes

$$||x_1 - x_0|| \le f_1(\alpha_0)\lambda_0. \tag{3.10}$$

This implies that $x_1 \in \mathfrak{T}(x_0, k\lambda)$. Now

$$\begin{aligned} ||I - \Gamma_0\Lambda'(x_1)|| &\le ||\Gamma_0||||\Lambda'(x_0) - \Lambda'(x_1)|| \\ &\le \alpha_0 f_1(\alpha_0) < 1. \end{aligned}$$

Thus, the existence of $\Gamma_1 = [\Lambda'(x_1)]^{-1}$ is confirmed by the Banach Lemma and also

$$||\Gamma_1|| \le \frac{\zeta_0}{1-\alpha_0 f_1(\alpha_0)} = f_2(\alpha_0)\zeta_0 = \zeta_1. \tag{3.11}$$

Using the equations proved in Lemmas 2.1 and 2.2, we can derive

$$||\Lambda(y_0)|| \le g(\alpha_0, \beta_0, \chi_0)\frac{\lambda_0}{\zeta_0}, \tag{3.12}$$

and

$$||\Lambda(x_{k+1})|| \le f_3(\alpha_0, \beta_0, \chi_0)\frac{\lambda_0}{\zeta_0}. \tag{3.13}$$

And thus

$$||r_1 - x_1|| = ||\Gamma_1\Lambda(x_1)|| \le f_2(\alpha_0)f_3(\alpha_0, \beta_0, \chi_0)\lambda_0 = \lambda_1. \tag{3.14}$$

Also, because $f_1(\alpha_0) > 1$ and using the triangle inequality, we get

$$||r_1 - x_0|| \le k\lambda. \tag{3.15}$$

which indicates $r_1, z_1, w_1 \in \mathfrak{T}(x_0, k\lambda)$. Moreover, we have

$$K_1||\Gamma_1||\ ||\Gamma_1\Lambda(x_1)|| \le f_2^2(\alpha_0)f_3(\alpha_0, \beta_0, \chi_0)\alpha_0 = \alpha_1, \tag{3.16}$$

$$K_2||\Gamma_1|| \; ||\Gamma_1\Lambda(x_1)||^2 \leq f_2^3(\alpha_0)f_3^2(\alpha_0,\beta_0,\chi_0)\beta_0 = \beta_1, \tag{3.17}$$

$$K_3||\Gamma_1|| \; ||\Gamma_1\Lambda(x_1)||^3 \leq f_2^4(\alpha_0)f_3^3(\alpha_0,\beta_0,\chi_0)\chi_0 = \chi_1, \tag{3.18}$$

By using the mathematical induction, we can derive the system of recurrence relations, which are listed in the lemma below:

Lemma 3.1. *Suppose the hypotheses of Lemma* 2.4 *are true and also the conditions* $(R1)-(R5)$ *hold, then the following relations hold* $\forall k \geq 0$*:*

$$\begin{aligned}
&(i) \text{There exists } \Gamma_k = [\Lambda'(x_k)]^{-1} \text{and } ||\Gamma_k|| \leq \zeta_k,\\
&(ii)||\Gamma_k\Lambda(x_k)|| \leq \lambda_k,\\
&(iii)K_1||\Gamma_k|| \; ||\Gamma_k\Lambda(x_k)|| \leq \alpha_k,\\
&(iv)K_2||\Gamma_k|| \; ||\Gamma_k\Lambda(x_k)||^2 \leq \beta_k,\\
&(v)K_3||\Gamma_k|| \; ||\Gamma_k\Lambda(x_k)||^3 \leq \chi_k,\\
&(vi)||x_{k+1}-x_k|| \leq f_1(\alpha_k)\lambda_k.\\
&(vii)||x_{k+1}-x_0|| \leq k\lambda.
\end{aligned} \tag{3.19}$$

Proof. The proof of $(i)-(vi)$ is straightforward. We prove only the part (vii). By the relation (vi) and Lemma 2.7, we have

$$\begin{aligned}
||x_{k+1}-x_0|| &\leq \sum_{i=0}^{k} ||x_{i+1}-x_i||\\
&\leq f_1(\alpha_0)\lambda\left(\frac{1-\varsigma^{k+1}\varrho^{\frac{6^k+4}{5}}}{1-f_2(\alpha_0)f_3(\alpha_0,\beta_0,\chi_0)}\right) \leq k\lambda.
\end{aligned}$$

□

4. Semilocal convergence

In this part, we will prove the main result, whose statement is as follows:

Theorem 4.1. *Suppose the nonlinear operator* $\Lambda : \mathfrak{D} \subseteq B_1 \to B_2$ *is continuously third-order Fréchet differentiable on* $\mathfrak{D}$*. Assume that* $x_0 \in \Omega$ *and all the conditions* $(R1)-(R5)$ *hold. Assume that inequality* (2.16) *is fulfilled;* f_1*,* f_2 *and* f_3 *are defined by* (2.6), (2.7) *and* (2.9), *respectively, and* $\overline{\mathfrak{T}(x_0, k\lambda)} \subseteq \mathfrak{D}$*. Then initiating from* x_0*, the sequence* $\{x_k\}$ *developed from the scheme* (1.1) *converges to a zero* x^* *of* $\Lambda(x)$ *with* $x_k, x^* \in \overline{B(x_0, k\lambda)}$ *and* x^* *is an exclusive solution of* $\Lambda(x) = 0$ *in* $\mathfrak{T}(x_0, \frac{2}{K_1\zeta} - k\lambda) \cap \mathfrak{D}$*. Furthermore, its error bound is given by*

$$||x_k - x^*|| \leq f_1(\alpha_0)\lambda\varsigma^k\varrho^{\frac{6^k-1}{5}}\left(\frac{1}{1-\varsigma\varrho^{6^k}}\right). \tag{4.1}$$

Proof. From the earlier section results, it is obvious that the sequence $\{x_k\}$ is well defined in $\overline{\mathfrak{T}(x_0, k\lambda)}$. Now since

$$
\begin{aligned}
||x_{k+m} - x_k|| &\leq \sum_{i=k}^{k+m-1} ||x_{i+1} - x_i|| \\
&\leq f_1(\alpha_0)\lambda\varsigma^k \varrho^{\frac{6^k-1}{5}} \left(\frac{1 - \varsigma^m \varrho^{\frac{6^k(6^{m-1}+4)}{5}}}{1 - \varsigma\varrho^{6^k}} \right). \qquad (4.2)
\end{aligned}
$$

This inequality implies that $\{x_k\}$ is a Cauchy sequence. Hence there exists an x^* such that $lim_{k\to\infty} x_k = x^*$. Let $k = 0$ and $m \to \infty$ in the relation (4.2), we get

$$||x^* - x_0|| \leq k\lambda. \qquad (4.3)$$

This confirms that $x^* \in \overline{\mathfrak{T}(x_0, k\lambda)}$. Now, it will be shown that x^* is a zero of $\Lambda(x) = 0$. Because

$$||\Gamma_0|| \; ||\Lambda(x_k)|| \leq ||\Gamma_k|| \; ||\Lambda(x_k)||. \qquad (4.4)$$

By letting $k \to \infty$ in the inequality (4.4) and imposing the continuity of Λ in $\mathfrak{D}$, we obtain $\Lambda(x^*) = 0$. At last, we check the uniqueness of x^* as a solution of the equation $\Lambda(x) = 0$ in $\mathfrak{T}(x_0, \frac{2}{K_1\zeta} - k\lambda) \cap \mathfrak{D}$. First, it is clear that

$$\frac{2}{K_1\zeta} - k\lambda = \left(\frac{2}{\alpha_0} - k \right) \lambda > \frac{1}{\alpha_0}\lambda > k\lambda$$

by the fact that $k < 1/\alpha_0$ and thus $x^* \in \overline{\mathfrak{T}(x_0, k\lambda)} \subseteq \mathfrak{T}(x_0, \frac{2}{K_1\zeta} - k\lambda) \cap \mathfrak{D}$. Suppose that z^* is another solution of $\Lambda(x) = 0$ in $\mathfrak{T}(x_0, \frac{2}{K_1\zeta} - k\lambda) \cap \mathfrak{D}$. By Taylor's expansion, we obtain

$$0 = \Lambda(z^*) - \Lambda(x^*) = \int_0^1 F'((1-r)x^* + rz^*)dr(z^* - x^*). \qquad (4.5)$$

Now, since

$$
\begin{aligned}
&||\Gamma_0|| \; || \int_0^1 [\Lambda'((1-r)x^* + rz^*) - \Lambda'(x_0)]dr|| \\
\leq\; & \frac{K_1\zeta}{2} \left[k\lambda + \frac{2}{K_1\zeta} - k\lambda \right] = 1,
\end{aligned}
$$

which implies that $\int_0^1 \Lambda'((1-r)x^* + rz^*)dr$ is invertible and thus $z^* = x^*$. Furthermore, taking $m \to \infty$ in the relation (4.2), we get the inequality (4.1) and it can be also written as

$$||x_k - x^*|| \leq \frac{f_1(\alpha_0)\lambda}{\varrho^{\frac{1}{5}}(1 - f_2(\alpha_0)f_3(\alpha_0, \beta_0, \chi_0))} (\varrho^{\frac{1}{5}})^{6^k}, \qquad (4.6)$$

which confirms that the scheme (1.1) has at least six R-order of convergence. □

5. Alternative Computational Efficiency

For analyzing efficiency of any considered iterative scheme for nonlinear systems, commonly used measures are the so called classical efficiency index initiated by Ostrowski [9] and the computational efficiency index suggested by Traub [12]. Recently Grau et al. [6] revisited computational efficiency concept. In this portion, we are going to compare Grau's alternative computational efficiency of the presented method with schemes of the same order, as well as schemes of different orders, whose semilocal convergence was previously discussed in the literature. After comparison, we observed that the presented method has better computational efficiency compared to schemes of the same order and of lower orders, which does not happen always in the case of nonlinear systems and that is the motivation behind considering this scheme in the present study.

The computational cost for a system of nonlinear systems of m variables is given by [6]

$$C(\mu_0, \mu_1, m) = \mu_0 a_0 m + \mu_1 a_1 m^2 + P(m), \tag{5.1}$$

where a_0 and a_1 represent the number of scalar functions of Λ and Λ' respectively, $P(m)$ is the number of product per iteration and μ_0 and μ_1 are the ratios between products and evaluations required to express the value of $C(\mu_0, \mu_1, m)$ in term of products. The expression for $P(m)$ is given by

$$\begin{aligned} &P(m) \\ &= \frac{m(2p_1m^2 + (3p_1(k+1) + 6p_2)m + 6p_0 + p_1(3k-5) + 6p_2(k-1))}{6}. \end{aligned} \tag{5.2}$$

where p_0, p_1 and p_2 denote the number of scalar products, the number of complete resolutions of the linear system and the number of resolution of two triangular systems per iteration, respectively, while k is the number of equivalent products for one division. The efficiency index (one can see [5]) of an iterative method is defined by $E = d^{1/C}$, where d is the convergence order and C is the computational cost per iteration. In order to compare the alternative computational efficiency of the presented method (1.1) denoted by $(M61)$, consider sixth-order method by Wang et al. [13] denoted by $(M62)$, fifth-order method by Chen et al. [2] denoted by $(M53)$ and fourth-order method by Hernández and Salanova [7] denoted by $(M44)$. Method $M62$ is given by

$$\begin{aligned} u_k &= x_k - \Gamma_k \Lambda(x_k), \\ y_k &= x_k + \frac{2}{3}(u_k - x_k), \\ z_k &= x_k - [6\Lambda'(y_k) - 2\Lambda'(x_k)]^{-1}[3\Lambda'(y_k) + \Lambda'(x_k)]\Gamma_k \Lambda(x_k), \\ x_{k+1} &= z_k - \left[\frac{3}{2}\Lambda'(y_k)^{-1} - \frac{1}{2}\Gamma_k\right] \Lambda(z_k). \end{aligned} \tag{5.3}$$

$M53$ is written as

$$
\begin{aligned}
u_k &= x_k - \Gamma_k \Lambda(x_k),\\
y_k &= x_k + \frac{1}{2}(u_k - x_k),\\
z_k &= x_k - [\Lambda'(y_k)]^{-1}\Lambda(x_k),\\
x_{k+1} &= z_k - [3\Lambda'(y_k) - 2\Lambda'(x_k)]^{-1}\Lambda'(y_k)\Gamma_k\Lambda(z_k)
\end{aligned} \tag{5.4}
$$

and lastly Method $M44$ is defined as

$$
\begin{aligned}
y_k &= x_k - \Gamma_k \Lambda(x_k),\\
z_k &= x_k + \frac{2}{3}(y_k - x_k),\\
x_{k+1} &= y_k - \frac{3}{4}\left[I + \frac{3}{2}H(x_k)\right]^{-1} H(x_k)(y_k - x_k).
\end{aligned} \tag{5.5}
$$

Denoting the efficiency indices of the methods Mdi ($d = 6, 5, 4$ and $i = 1, 2, 3, 4$) by Edi and computational cost by Cdi, then taking into account the above considerations, we obtain
$C61 = m(4m^2 + (12\mu_1 + 6k + 8)m + 12\mu_0 + 18k + 2)/6$ and $E61 = 6^{1/C61}$,
$C62 = m(6m^2 + (12\mu_1 + 9k + 11)m + 12\mu_0 + 21k + 3)/6$ and $E62 = 6^{1/C62}$,
$C53 = m(6m^2 + (12\mu_1 + 9k + 10)m + 12\mu_0 + 15k - 3)/6$ and $E53 = 6^{1/C53}$,
$C44 = m(4m^2 + (12\mu_1 + 6k + 7)m + 6\mu_0 + 12k + 2)/6$ and $E44 = 6^{1/C44}$.

By virtue of the above values, we can state the following theorem:

Theorem 5.1. *For all $m \geq 2$, $\mu_0 > 0$, $\mu_1 > 0$ and $k \geq 1$ we have*
(i) $E61 > E62$,
(ii) $E61 > E53$,
(iii) $E61 > E44$.
Otherwise efficiency comparison depends upon m, μ_0, μ_1 and k.

Now we plot the graph for the results of the Theorem 5.1 for the particular triplet $(\mu_0, \mu_1, k) = (1, 1, 1)$. In the graph dotted lines represent method $M61$ and continuous lines represent method $M61$; red are for $M62$; green for $M53$ and pink for $M44$, respectively.

6. Numerical Results

In this portion, we start with the nonlinear integral equation $\Lambda(x) = 0$, where

$$
\Lambda(x)(s) = x(s) - \frac{4}{3} + \frac{1}{2}\int_0^1 s\ cos(x(r))dr, \tag{6.1}
$$

where $s \in [0, 1]$, $x \in \mathfrak{D} = \mathfrak{T}(0, 2) \subset B_1$ and $B_1 = C[0, 1]$ with the max-norm, and it is given by

$$
||x|| = \max_{s \in [0,1]} |x(s)|.
$$

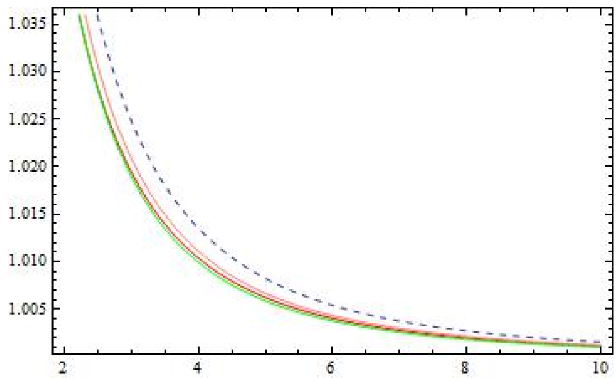

Figure 1: Plots for E-values of $M61$, $M62$, $M53$ and $M44$ for $m = 2, 3, ..., 10$.

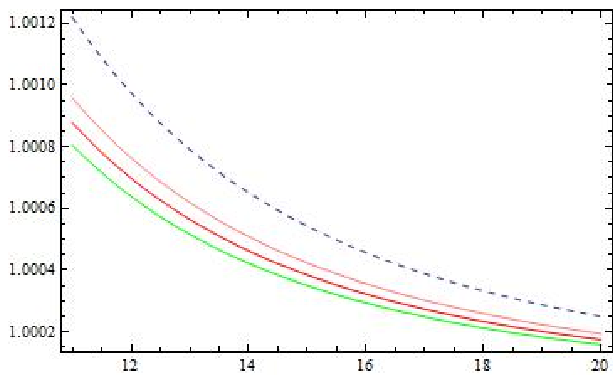

Figure 2: Plots for E-values of $M61$, $M62$, $M53$ and $M44$ for $m = 12, 14, ..., 20$.

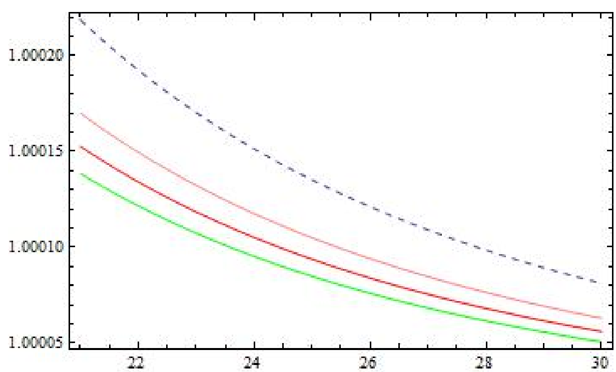

Figure 3: Plots for E-values of $M61$, $M62$, $M53$ and $M44$ for $m = 21, 22, ..., 30$.

In view of the equation (6.1), we can easily calculate

$$\Lambda'(x)y(s) = y(s) - \frac{1}{2}\int_0^1 s\ sin(x(r))y(r)dr,\ y \in \mathfrak{D},$$

$$\Lambda''(x)yz(s) = -\frac{1}{2}\int_0^1 s\ cos(x(r))y(r)z(r)dr,\ y, z \in \mathfrak{D},$$

$$\Lambda'''(x)yzw(s) = -\frac{1}{2}\int_0^1 s\ cos(x(r))y(r)z(r)w(r)dr,\ y, z, w \in \mathfrak{D}.$$

Obviously

$$||\Lambda''(x)|| \leq \frac{1}{2} = K_1,\ x \in \mathfrak{D},$$

$$||\Lambda'''(x)|| \leq \frac{1}{2} = K_2,\ x \in \mathfrak{D},$$

and with with $K_3 = 1/2$

$$||\Lambda'''(x) - \Lambda'''(y)|| \leq \frac{1}{2}||x - y||,\ x, y \in \mathfrak{D}.$$

If we assume $x_0 = 4/3$, then

$$||\Lambda(x_0)|| \leq \frac{1}{2}cos\frac{4}{3}$$

and

$$||I - \Lambda'(x_0)|| \leq \frac{1}{2}sin\frac{4}{3}.$$

By virtue of the above inequalities, one can find

$$||\Gamma_0|| \leq \zeta = 1.94541, ||\Gamma_0\Lambda(x_0)|| \leq \lambda = 0.228817, \alpha_0 = 0.222571,$$
$$\beta_0 = 0.0509278, \chi_0 = 0.0116531.$$

Now

$$f^*(\alpha_0) = \alpha_0 f_1(\alpha_0) - 1 = -0.705 < 0.$$

and

$$f_2^2(\alpha_0)f_3(\alpha_0, \beta_0, \chi_0) = 0.000469 < 1.$$

The above relations verified that hypotheses of the Theorem 4.1 hold. The recurrence relations for the method (1.1) are mentioned in the Table 1. Moreover, the solution $x^* \in \overline{B(x_0, K\lambda)} = \overline{\mathfrak{T}(4/3, 0.30380...)} \subset \mathfrak{D}$ and it is unique in $\mathfrak{T}(4/3, 1.4486...) \cap \mathfrak{D}$.

Table 1: Results of recurrence relations

k	λ_k	ζ_k	α_k	β_k	χ_k	$f_2(\alpha_k)f_3(\alpha_k,\beta_k,\chi_k)$
0	0.229	1.945	0.223	0.051	0.012	0.001
1	7.553e-5	2.761	1.043e-4	7.877e-9	5.950e-13	1.735e-21
2	1.311e-25	2.761	1.810e-25	2.372e-50	3.110e-75	2.731e-125
3	3.580-150	2.761	4.942e-150	1.770e-299	6.332e-499	4.148e-748

7. Concluding Remarks

In the present article, we have discussed the semilocal convergence analysis of the sixth order iterative method for solving nonlinear equation in Banach spaces. The analysis was done using recurrence relation technique. The existence and uniqueness theorem was established along with its error bound. The comparison of the alternative computational efficiency was also done with algorithms of the similar convergence order, as well as algorithms of different convergence order. Finally, a numerical example has been presented to validate the theoretical discussions.

Acknowledgment

The authors would like to express sincere gratitude to the reviewers for their constructive criticism of this article which lead to a much better preparation of it in its current version.

References

[1] Amat, S., and Busquier, S. Third-order iterative methods under Kantorovich conditions. *J. Math. Anal. Appl. 336*, 1 (2007), 243–261.

[2] Chen, L., Gu, C., and Ma, Y. Semilocal convergence for a fifth-order Newton's method using recurrence relations in Banach spaces. *J. Appl. Math.* (2011), Art. ID 786306, 15.

[3] Cordero, A., Hernández-Verón, M. A., Romero, N., and Torregrosa, J. R. Semilocal convergence by using recurrence relations for a fifth-order method in Banach spaces. *J. Comput. Appl. Math. 273* (2015), 205–213.

[4] Cordero, A., H. J. L. M. E., and Torregrosa, J. R. A modified newton-jarratts composition. *Numer Algor 55* (2010), 87–99.

[5] Gautschi, W. *Numerical analysis.* Birkhäuser Boston, Inc., Boston, MA, 1997. An introduction.

[6] Grau-Sánchez, M., Grau, A., and Noguera, M. On the computational efficiency index and some iterative methods for solving systems of nonlinear equations. *J. Comput. Appl. Math. 236*, 6 (2011), 1259–1266.

[7] Hernández, M. A., and Salanova, M. A. Sufficient conditions for semilocal convergence of a fourth order multipoint iterative method for solving equations in Banach spaces. *Southwest J. Pure Appl. Math.*, 1 (1999), 29–40.

[8] Kantorovich, L. V., and Akilov, G. P. *Functional analysis*, second ed. Pergamon Press, Oxford-Elmsford, N.Y., 1982. Translated from the Russian by Howard L. Silcock.

[9] Ostrowski, A. M. *Solution of equations and systems of equations.* Second edition. Pure and Applied Mathematics, Vol. 9. Academic Press, New York-London, 1966.

[10] Parida, P. K., and Gupta, D. K. Recurrence relations for a Newton-like method in Banach spaces. *J. Comput. Appl. Math. 206*, 2 (2007), 873–887.

[11] Rall, L. B. *Computational solution of nonlinear operator equations.* Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1979. Corrected reprint of the 1969 original.

[12] Traub, J. F. *Iterative Methods for the Solution of Equations.* Chelsea Publishing Company, New York, 1982.

[13] Wang, X., Kou, J., and Gu, C. Semilocal convergence of a sixth-order Jarratt method in Banach spaces. *Numer. Algorithms 57*, 4 (2011), 441–456.

[14] Zheng, L., and Gu, C. Semilocal convergence of a sixth-order method in Banach spaces. *Numer. Algorithms 61*, 3 (2012), 413–427.

[15] Zheng, L., and Gu, C. Fourth-order convergence theorem by using majorizing functions for super-Halley method in Banach spaces. *Int. J. Comput. Math. 90*, 2 (2013), 423–434.

Received by the editors February 25, 2016
First published online January 23, 2019