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LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS IN A FIELD II

In the first part [3], we studied the differential equation

$$(1)\qquad \sum_{\mu=0}^{M} \mathbf{A}_\mu \mathbf{x}^{(\mu)}(\lambda)=0, \qquad \lambda_1 \leqslant \lambda \leqslant \lambda_2$$

where the coefficients $\mathbf{A}_\mu$ were of the form:

$$(2')\qquad \mathbf{A}_\mu=\sum_{k=0}^{\infty} \mathbf{a}_{\mu,k} e^{-\tau_k^\mu \mathbf{s}}, \quad \mu=0, \ldots, M, \quad \mathbf{a}_{\mu,0}\neq 0,$$

$$(3')\qquad \mathbf{a}_{\mu,k}=\sum_{\nu=\nu_{\mu,k}}^{\infty} \alpha_{\mu,k,\nu} \mathbf{l}^{\nu/\sigma_{\mu,k}}, \qquad \alpha_{\mu,k},\ \nu_{\mu,k}\neq 0.$$

For every $\mu=0, \ldots, M$ the sequence $\{\tau_k^\mu\}$ is a strict monotone increasing and diverging, $\tau_0^\mu>-\infty$; $\nu_{\mu,k}>-\infty$; $\sigma_{\mu,\nu}\in N$. Here $\mathbf{s}$ is the differential operator, $\mathbf{l}$ the integral operator, and $e^{-\lambda \mathbf{s}}$, $\lambda>0$, the translation operator in the field M of Mikusinski operators [2].

We showed that the existence and the number of the linearly independent solutions to equation (1) can be read from a broken line which is easy to construct. In case $\tau_k^\mu=\alpha\cdot p_k^\mu$, $\alpha\in R^+$, p_k^μ an integer; $(\nu_{\mu,k})$ has a lower bound and $(\sigma_{\mu,k})$ has an upper bound, we gave a method for construction of the solutions and we analysed the character of the solutions (are they elements from $\mathcal{C}$, $\mathcal{L}$, $\mathcal{D}'$, or only from $\mathcal{M}$).

In this second part we shall leave alone the problems of the existence and the number of solutions to equation [1] and deal only with the construction of the solutions and their approximation (when such solutions exist), the error's estimation and the application of our results to the partial differential-difference equations.

In the following we shall suppose (see [3], p. 204) that our $\mathbf{A}_\mu$ (from equation (1)) has the form

$$(2)\qquad \mathbf{A}_\mu=\sum_{\nu=\nu_\mu}^{\infty} \mathbf{D}_{\mu,\nu} \mathbf{L}^\nu, \quad \mathbf{D}_{\mu,\nu_\mu}\neq 0, \qquad \nu_\mu \geqslant 0$$

$$(3)\qquad \mathbf{D}_{\mu,\nu}=\sum_{k=k_{\mu,\nu}}^{\infty} \beta_{\mu,\nu,k}\cdot \mathbf{h}^k, \qquad \beta_{\mu,\nu,k_{\mu,\nu}}\neq 0, \qquad k_{\mu,\nu}\geqslant 0$$

where $\mathbf{L}=\mathbf{l}^{1/\delta}$ and $\mathbf{h}=e^{-\alpha \mathbf{s}}$.

1. The Set $\mathscr{E}$

The set $\mathscr{E}$ is the set of a series:

$$\mathbf{E} \equiv \sum_{i=0}^{\infty} c_i \mathbf{H}_{i\alpha} \tag{4}$$

where c_i are complex numbers, α a fixed positive real number and $\mathbf{H}_{i\alpha}$ is the Heawiside's function:

$$\mathbf{H}_{i\alpha}(t) = \begin{cases} 0, & 0 \leqslant t < i\alpha \\ 1, & t \geqslant i\alpha. \end{cases}$$

Every series of the form (4) converges in $\mathscr{M}$ because the series (4) deduces to a finite sum on a bounded interval $[0, T]$.

The addition is an interior operation for $\mathscr{E}$ but for the product and the quotient we have:

$$\mathbf{E}_1 \cdot \mathbf{E}_2 = \sum_{i=0}^{\infty} c_i^{(1)} \mathbf{H}_{i\alpha} \cdot \sum_{i=0}^{\infty} c_i^{(2)} \mathbf{H}_{i\alpha} = \mathbf{1} \sum_{i=0}^{\infty} d_i \mathbf{H}_{i\alpha},$$

where

$$d_i = \sum_{j=0}^{i} c_{i-j}^{(1)} c_j^{(2)}.$$

Let us suppose $c_0^{(2)} = \ldots = c_{k-1}^{(2)} = 0$ and $c_k^{(2)} \neq 0$, $k \geqslant 1$ then:

$$\frac{\mathbf{E}_1}{\mathbf{E}_2} = \mathbf{h}^{-k\mathbf{s}} \sum_{i=0}^{\infty} p_i \mathbf{H}_{i\alpha}, \quad h = e^{-\alpha s}.$$

In case $c_0^{(2)} \neq 0$:

$$\frac{\mathbf{E}_1}{\mathbf{E}_2} = \mathbf{s} \sum_{i=0}^{\infty} p_i \mathbf{H}_{i\alpha}, \qquad c_i^{(1)} = \sum_{j=0}^{i} c_{i-j}^{(2)} p_j.$$

Proposition 1. For a series $\sum_{i=0}^{\infty} \mathbf{E}_i \mathbf{L}^i$, $\mathbf{L} = \mathbf{1}^{1/\sigma}$, $\sigma > 0$, $\mathbf{E}_i \in \mathscr{E}$ *which converges in* $\mathcal{C}$ $\sum_{i=0}^{\infty} \mathbf{E}_i \mathbf{L}^i = 0 \Leftrightarrow \mathbf{E}_i = 0$, $i \geqslant 0$.

Proof. — Let us suppose that our series equals zero then

$$\mathbf{E}_0 = -\sum_{i=1}^{\infty} \mathbf{E}_i \mathbf{L}^i.$$

$\mathbf{E}_0$ as an element from $\mathscr{E}$ is a constant function or a step function. On the other side of the equality we have a continuous function of t. It follows that $\mathbf{E}_0$ has to be a constant. Now, when $t \to 0$, $-\sum_{i=1}^{\infty} \mathbf{E}_i \mathbf{L}^i$ tends to zero, consequently $\mathbf{E}_0 = 0$. We repeat the same discussion for $\mathbf{E}_1 = -\sum_{i=2}^{\infty} \mathbf{E}_i \mathbf{L}^{i-1}$, $\mathbf{E}_2, \ldots$.

2. The Construction of Solutions to Equation (1)

In the first part, we saw that the solutions to the characteristic equation:

(5) $$\sum_{\mu=0}^{M}\sum_{\nu=\nu_\mu}^{\infty} \mathbf{D}_{\mu,\nu}\mathbf{L}^{\nu}\mathbf{W}^{\mu}=0 \quad \text{where} \quad \mathbf{L}=\mathbf{l}^{1/\sigma} \quad \text{and} \quad \mathbf{E}_i=\mathbf{l}\mathbf{D}_i$$

were of the form:

$$\mathbf{W}=\sum_{i=i_0}^{\infty}\mathbf{D}_i\mathbf{L}^{i/\gamma}=\mathbf{L}^{i_0/\gamma-\sigma}\cdot\sum_{i=0}^{\infty}\mathbf{E}_{i_0+i}\mathbf{L}^{i/\gamma}=\mathbf{L}^{i_0/\gamma-\sigma}\mathbf{W}_0, \qquad \mathbf{W}_0\in\mathscr{E}.$$

The characteristic equation (5) is now:

(5′) $$\sum_{\mu=0}^{M}\sum_{\nu=\nu_\mu}^{\infty}\mathbf{E}_{\mu,\nu}\mathbf{L}^{\nu-\sigma+\mu(i_0/\gamma-\sigma)}\mathbf{W}_0^{\mu}=0.$$

With the notation $U=\mathbf{L}^{1/\gamma}$we have:

(6) $$P(\mathbf{U},\mathbf{W}_0)=\sum_{\mu=0}^{M}\sum_{\nu=\nu_\mu}^{\infty}\mathbf{s}\cdot\mathbf{E}_{\mu,\nu}\mathbf{U}^{\nu\gamma-\sigma\gamma+\mu i_0}\mathbf{s}^{\mu-1}\mathbf{W}_0^{\mu}=0.$$

Taking care of the product of two elements in $\mathscr{E}$, we can conclude that $P(\mathbf{U}\ \mathbf{W}_0)$ is a power series in $\mathbf{U}$ with coefficients in $\mathscr{E}$. We know [3] that it is always possible to find i_0 in such a way that:

(7) $$0=\nu_i\gamma+ii_0-\sigma\gamma=\nu_j\gamma+ji_0-\sigma\gamma=\ldots=$$
$$=\nu_t\gamma+ti_0-\sigma\gamma\leqslant\nu_\mu\gamma+\mu i_0-\sigma\gamma; \qquad \mu=0\ \ldots, M.$$

Using proposition 1 we have:

(8) $$P(0,\mathbf{E}_{i_0})\equiv\mathbf{s}\mathbf{E}_{i,\nu_i}\mathbf{s}^{i-1}\mathbf{E}_{i_0}^{i}+\mathbf{s}\mathbf{E}_{j,\nu_j}\mathbf{s}^{j-1}\mathbf{E}_{i_0}^{j}\ldots+\mathbf{s}\mathbf{E}_{t,\nu_t}\mathbf{s}^{t-1}\mathbf{E}_{i_0}^{t}=0.$$

Every addend in this sum is from $\mathscr{E}$. This equation is equivalent to:

(8′) $$\mathbf{D}_{i,\nu_i}\mathbf{D}_{i_0}^{i}+\mathbf{D}_{j,\nu_j}\mathbf{D}_{i_0}^{j}+\ldots+\mathbf{D}_{t,\nu_t}\mathbf{D}_{i_0}^{t}=0. \qquad (\mathbf{E}_{i_0}=\mathbf{l}\mathbf{D}_{i_0})$$

Let $i\leqslant j\leqslant\ldots\leqslant t$. Equation (8′) has $t-i$ solutions in $\mathbf{D}_{i0}$ because the field $\mathscr{H}(\alpha)$ which consists of the series which have the form $\sum_{i=i_0}^{\infty}\alpha_i\mathbf{h}^i$ is algebraically closed.

We can always suppose that $\mathbf{D}_{i_0}$ is a simple zero of polynomial (8′) and that polynomial (6) is irreducible. (If $\mathbf{D}_{i_0}$ is not a simple zero, the procedure is slightly different.)

$P(\mathbf{U},\mathbf{W}_0)$ can be expressed in powers of U and $\mathbf{W}_0-\mathbf{E}_{i_0}$:

$$P(\mathbf{U},\mathbf{W}_0)=\sum_{\mu=0}^{M}\sum_{j=0}^{\infty}\frac{1}{\mu!\,j!}P_{u^j,\ w_0^{\mu}}(0,\mathbf{E}_{i_0})(\mathbf{W}_0-\mathbf{E}_{i_0})^{\mu}\mathbf{U}^{j},$$

where we used, formally, the same notations as for the numerical polynomials. We know that $P(0, \mathbf{E}_i) \equiv P_{\mathbf{u}^0, \mathbf{w}_0^0}(0, \mathbf{E}_{i_0}) = 0$. The characteristic equation (6) allows us to write:

$$(9) \qquad P_{\mathbf{u}^0, \mathbf{w}_0^1}(0, \mathbf{E}_{i_0})(\mathbf{W}_0 - \mathbf{E}_{i_0}) = -\sum_{\mu=0}^{M}{}' \sum_{j=0}^{\infty}{}' \frac{1}{\mu!\, j!} P_{\mathbf{u}^j, \mathbf{w}^\mu}(0, \mathbf{E}_{i_0}) \mathbf{U}^j (\mathbf{W}_0 - \mathbf{E}_{i_0})^\mu$$

In this double sum with $'$ we noted that the addend which corresponds to the couple of indices $j=0$, $\mu=1$ is omitted. It is easy to see that $P_{\mathbf{u}^j, \mathbf{w}^\mu}(0, \mathbf{E}_{i_0}) = \mathbf{s}^\mu P_{j,\mu}$, where $P_{j,\mu}$ are elements from $\mathscr{C}$;
with this notation equation (9) is:

$$(9') \qquad \mathbf{s} P_{0,1}(\mathbf{W}_0 - \mathbf{E}_{i_0}) = -\sum_{\mu=0}^{M}{}' \sum_{j=0}^{\infty}{}' \frac{\mathbf{s}}{\mu!\, j!} P_{j,\mu} \mathbf{U}^j \mathbf{s}^{\mu-1} (\mathbf{W}_0 - \mathbf{E}_{i_0})^\mu$$

$P_{0,1}$ differs from zero because $\mathbf{E}_{i_0}$ is a simple zero of equation (8). As an element of $\mathscr{C}$, $P_{0,1}$ has the form $\sum_{i=0}^{\infty} \alpha_i \mathbf{h}^i$; $\mathbf{h} = e^{-\mathbf{s}}$. We can suppose that $\alpha_0 \neq 0$. (If we had $\alpha_0 = \alpha_1 = \ldots = \alpha_{k-1} = 0$ and $\alpha_k \neq 0$ we should have to multiply the relation (9′) with $\mathbf{h}^k$.)

Now the relation (9′) can be written in the form

$$(9'') \qquad \mathbf{W}_0 - \mathbf{E}_{i_0} = \sum_{\mu=0}^{M}{}' \sum_{j=0}^{\infty}{}' \mathbf{s} B_{j,\mu} \mathbf{U}^j \mathbf{s}^{\mu-1} (\mathbf{W}_0 - \mathbf{E}_{i_0})^\mu,$$

where $B_{j,\mu} = -1 \dfrac{P_{j,\mu}}{\mu!\, j!\, P_{0,1}}$ is an element from $\mathscr{C}$ too, or

$$(10) \qquad \sum_{i=1}^{\infty} \mathbf{E}_{i_0+i} \mathbf{U}^i = \sum_{\mu=0}^{M}{}' \sum_{j=0}^{\infty}{}' \mathbf{s} B_{j,\mu} \mathbf{U}^j \left(\sum_{i=1}^{\infty} \mathbf{E}_{i_0+i} \mathbf{U}^i \right)^\mu \mathbf{s}^{\mu-1} = \sum_{\mu=0}^{M} \sum_{j=0}^{\infty} \mathbf{s} B_{j,\mu} \mathbf{U}^j \sum_{i=1}^{\infty} \mathbf{E}_{i_0+i}^\mu \mathbf{U}^i,$$

where $\mathbf{E}_{i_0+i}^\mu$ are elements form $\mathscr{C}$. This relation (10) gives successively the coefficients $\mathbf{E}_{i_0+i}$, $(i = 1, 2, \ldots .)$.

3. Approximation of the Solutions to Equation (1)

Let us denote by $\mathscr{L}$ the ring of local integrable functions over $[0, \infty)$ with sum and convolution as two operations which give the structure of a ring in $\mathscr{L}$.

In $\mathscr{L}$ we define two binary relations:

$$f \leqslant g \Leftrightarrow f(t) \leqslant g(t), \qquad t \in [0, \infty)$$

$$f \leqslant_T g \Leftrightarrow f(t) \leqslant g(t), \qquad t \in [0, T].$$

For $f=\{f(t)\} \in \mathcal{L}$ by definition is: $|f|=\{|f(t)|\}$. If f and g are functions of class $\mathcal{L}$ then:

1. $|f+g| \leqslant |f|+|g|$
2. $|fg| \leqslant |f|\,|g|$
3. if for a fixed $f \in \mathcal{L}$ there exists $M_f=\sup\limits_{0 \leqslant t \leqslant T} |f(t)|$, then $|f| \leqslant_T M_f \mathbf{l}$.

In $\mathcal{M}$ we have no order relation so we have to define what is the approximation of an operator [4].

DEFINITION. The operator $\mathbf{a}$ approximates (approximates locally) the operator $\mathbf{b}$ with a factor $\mathbf{q} \in \mathcal{M}$ and a measure ε (measure $\varepsilon(T)$) if $\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b}) \in \mathcal{L}$ and $|\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b})| \leqslant \varepsilon \mathbf{l}$ ($|\mathbf{q}^{-1}(\mathbf{a}-\mathbf{b})| \leqslant_T \varepsilon(T)\mathbf{l}$).

If $\mathbf{q}=\mathbf{I}$, our definition gives the classical approximation in $\mathcal{L}$. ($\mathbf{I}$ is identical operator).

Proposition 2. For any element $\mathbf{E}=\sum\limits_{i=0}^{\infty} c_i \mathbf{H}_{i\alpha}$ *which belongs to* $\mathcal{E}$ *we have:*

$$|\mathbf{E}| \leqslant_T \left(\sum_{i=0}^{i'} |c_i|\right)\mathbf{l}, \qquad i'=\min\{i, i\alpha \geqslant T\}.$$

The proof follows from the fact that the series $\sum\limits_{i=0}^{\infty} c_i \mathbf{H}_{i\alpha}(t)$ deduces to a finite sum when $t \in [0, T]$, $T<\infty$.

Now we can construct the approximate solutions of equation (1) in the sense of our definition.

Let us suppose that for a solution $\mathbf{W}=\mathbf{L}^{i_0/\gamma-\sigma}\sum\limits_{i=0}^{\infty}\mathbf{E}_{i_0+i}\mathbf{L}^{i/\gamma}$ of the characteristic equation (5) we calculated the first $r+1$ coefficients $\mathbf{E}_{i_0}, \mathbf{E}_{i_0+1}, \ldots, E_{i_0+r}$ by the procedure we gave here, then the exact solution to equation (1) is $x(\lambda)=\exp(\lambda\mathbf{W})$. And an approximate solution to equation (1) is $\tilde{x}(\lambda)=\exp(\lambda\tilde{\mathbf{W}})$, where $\tilde{\mathbf{W}}=\mathbf{L}^{i_0/\gamma-\sigma}\sum\limits_{i=0}^{r}\mathbf{E}_{i_0+i}\mathbf{L}^{i/\gamma}$, $r<\infty$. The difference between the exact solution and the approximate one is:

$$x(\lambda)-\tilde{x}(\lambda)=\tilde{x}(\lambda)\left[\exp(\lambda\sum_{i=r+1}^{\infty}\mathbf{E}_{i_0+i}\mathbf{L}^{i/\gamma+i_0/\gamma-\sigma})-\mathbf{I}\right] \tag{11}$$

It is easy now to choose a factor and a measure of approximation.

Let us suppose that r is enough large, that means that $\frac{r+i_0}{\gamma}-\sigma>0$. In that case

$$\exp\left(\lambda\sum_{i=r+1}^{\infty}\mathbf{E}_{i_0+i}\mathbf{L}^{i/\gamma-i_0/\gamma-\sigma}\right)-\mathbf{I} \tag{12}$$

belongs to $\mathcal{L}$ (and something more, in $\mathcal{C}$) and can be used to define the measure of approximation of the exact solution.

Our aim is to deduce the problem of finding the measure of approximation from the expression (12) to finding the measure of approximation of the expression:

$$\exp\left(|\lambda| \sum_{i=r+1}^{\infty} c_i \mathbf{L}^{i/\gamma+i_0/\gamma}\right)-\mathbf{I}, \tag{13}$$

where $|\mathbf{E}_{i_0+i}| \leqslant_T c_i\mathbf{l}$. In this case we have the same situation as in our papers [4], [6] and we can use all the results we proved there.

Proposition 3. Let $\mathbf{W}=\mathbf{L}^{i_0/\gamma-\sigma} \sum_{i=0}^{\infty} \mathbf{E}_{i_0+i}\mathbf{L}^{i/\gamma}=\mathbf{L}^{i_0/\gamma-\sigma}\mathbf{W}_0$ *be a solution of the characteristic equation* (5′) *and* $B_{j,\mu}\in\mathcal{C}$ *the coeficijents in equation* (9″), *then* $|\mathbf{E}_{i_0+i}|\leqslant_T c_i\mathbf{l}$, $i=1, 2, \ldots$ *where* c_i *are the coeficijents of the numerical series* $w=\sum_{i=1}^{\infty} c_i u^i$ *which satisfies the equation:*

$$P(u, w)\equiv \sum_{\mu=0}^{M} \sum_{j=0}^{\infty} b_{j,\mu}u^j w^\mu=0 \tag{14}$$

$$b_{0,0}=0, \qquad b_{0,1}\neq 0, \qquad \textit{and} \qquad |B_{j,\mu}|\leqslant_T b_{j,\mu}\mathbf{l}.$$

Proof. — We saw that $\mathbf{W}_0$ satisfied equation (9″) and for E_{i_0+i}, $i=1, 2, \ldots$ we had equation (10) from which

$$E_{i_0+i}=\mathbf{l}R_i(\mathbf{s}E_{i_0},\ \mathbf{s}E_{i_0+1}, \ldots, \mathbf{s}E_{i_0+i-1};\ \mathbf{s}B_{j,\mu}).$$

Formally, we have the same situation as for a numerical implicit function [1], and we know that R_i is a polynomial in $\mathbf{s}E_{i_0}, \mathbf{s}E_{i_0+1}, \ldots, \mathbf{s}E_{i_0+i-1}$ and $\mathbf{s}B_{j,\mu}$. where $j+\mu<i$ (we know that all these elements are from the field $\mathcal{H}(\alpha)$ [3]). In such a way we obtain successively E_{i_0+i} as a polynomial Q_i in $\mathbf{s}B_{j,\mu}$ $(j+\mu<i)$ with positive coefficients:

$$E_{i_0+i}=\mathbf{l}Q_i(\mathbf{s}B_{j,\mu}).$$

We shall use now the same procedure to find $w=\sum_{i=1}^{\infty} c_i u^i$ which satisfies equation (14) and for the coefficients c_i we obtain the same relations:

$$c_i=R_i(c_0, c_1, \ldots, c_{i-1}, b_{j,\mu}), \qquad j+\mu<i \quad \text{and} \quad c_i=Q_i(b_{j,\mu}).$$

Taking care of the form of Q_i and that $|B_{j,\mu}|\leqslant_T b_{j,\mu}\mathbf{l}$, we have

$$|E_{i_0+i}|\leqslant_T \mathbf{l}Q_i(b_{j,\mu})\leqslant \mathbf{l}c_i.$$

4. Application to some Types of Numerical Equations

Our results can be applied to the equations of the form:

$$(15)\qquad \sum_{\mu=0}^{M}\sum_{\kappa=0}^{P\mu}\int_{0}^{t} a'_{\mu,k}(t-\tau)\frac{\partial^{\mu+k}}{\partial\lambda^{\mu}\partial t^{k}}x(\lambda,\ t-\tau_{k}^{\mu})\,d\tau=f(\lambda,t)$$

$\lambda_1\leqslant\lambda\leqslant\lambda_2$, $t\in R$, where $a'_{\mu,k}(t)=\sum\limits_{\nu=\nu_{\mu,\kappa}}^{\infty}\alpha'_{\mu,\kappa,\nu}t^{\nu/\sigma_{\mu,\kappa}+\kappa-1}$ $\quad \nu_{\mu,k}/\sigma_{\mu,k}>-k\cdot\sigma_{\mu,k}$ are rational numbers, $\alpha'_{\mu,\kappa,\nu}$ complex numbers. We suppose that

$$\mathbf{s}^{k}a'_{\mu,k}=\mathbf{a}_{\mu,k}=\sum_{\nu=\nu_{\mu,\kappa}}^{\infty}\alpha_{\mu,\kappa,\nu}\,\mathbf{l}^{\nu/\sigma_{\mu,k}},\qquad \alpha_{\mu,\kappa,\nu}=\Gamma(\nu/\sigma_{\mu,\kappa}+k)\alpha'_{\mu,\kappa,\nu}$$

is an algebraic operator which belongs to the field A (see [3]) and $x(\lambda,t)\equiv 0$, $t<0$.

To equation (15) corresponds, in the field $\mathcal{M}$, the equation

$$(16)\qquad \sum_{\mu=0}\sum_{\kappa=0}\mathbf{a}_{\mu,\kappa}\mathbf{e}^{\tau_{k}^{\mu}\mathbf{s}}\mathbf{x}^{(\mu)}(\lambda)=f(\lambda)+$$

$$+\sum_{\mu=0}^{M}\sum_{\kappa=0}^{P\mu}a'_{\mu,k}e^{-\tau_{k}^{\mu}\mathbf{s}}\sum_{i=0}^{n-1}\mathbf{s}^{n-i-1}\frac{\partial^{\mu+i}}{\partial\lambda^{\mu}\partial t^{i}}x(\lambda,t)\,|_{t=0}\quad(\lambda_1\leqslant\lambda\leqslant\lambda_2).$$

The homogeneous part of equation (16) is just of the form (1) that we discussed in the first part [3] and here. The linearly independent solutions to the homogeneous part of equation (1), as we know, can be used to obtain the solutions to the nonhomogeneous equation with supplementary conditions.

Let us remark that the solutions to equation (16) are not always the classical solutions to equation (15), even then the solutions to equation (16) are defined by functions; they are solutions to equation (15) in a generalized sense.

As an example we shall apply our results to the equation:

$$(17)\qquad \frac{\partial^2 x(\lambda,t)}{\partial\lambda\partial t}+\frac{\partial x(\lambda,t)}{\partial\lambda}=x(\lambda,t-1),\qquad \lambda\in R$$

with the conditions:

$$(18')\qquad x(\lambda,t)=0,\qquad t\leqslant 0,\qquad \lambda\in R;$$

$$(18'')\qquad x(0,t)=x_0\neq 0,\qquad t>0,\qquad x_0\in R.$$

To equation (17) with the condition (18′) corresponds the equation in the field $\mathcal{M}$ of the form

$$(19)\qquad (\mathbf{s}+1)\mathbf{x}'(\lambda)-e^{-\mathbf{s}}\mathbf{x}(\lambda)=0,\qquad \lambda\in R$$

and to the condition (18″) corresponds:

(19″) $$x(0)=x_0\mathbf{I}.$$

It is easy to see that in this case $\mathbf{A}_0=-e^{-\mathbf{s}}$, $\mathbf{A}_1=\mathbf{s}+1$, $\mathbf{A}_\mu=0$, $\mu\geqslant 2$; $\mathbf{D}_{0,0}=-e^{-\mathbf{s}}$, $\mathbf{D}_{0,\nu}=0$, $\nu\geqslant 1$, $\mathbf{D}_{1,-1}=1$, $\mathbf{D}_{1,0}=1$ $\mathbf{D}_{1,\nu}=0$ $\nu\geqslant 1$; $\sigma=1$.

By proposition 1 in the first part [3] the set of points $\{(\mu, \tau_0^\mu)$ consists of only two points (0,1) and (1,0) and we have $m=-1/2$. This proposition says that we have one solution of our equation (19). Let us suppose that the solution of the characteristic equation for equation (19) is of the form: $\mathbf{W}=\sum_{i=i_0}^{\infty}\mathbf{D}_i\mathbf{l}^i$.

As in our case $\nu_0=0$ and $\nu_1=-1$, system (7) is deduced to one equation $0=-\gamma+i_0$ and we can choose $\gamma=i_0=1$.

Equation (8′) gives $-\mathbf{e}^{-\mathbf{s}}+\mathbf{D}_1=0$ $\mathbf{D}_1=\mathbf{e}^{-\mathbf{s}}$ and $\mathbf{W}=e^{-\mathbf{s}}\dfrac{1}{1+\mathbf{s}}=\mathbf{H}_1\dfrac{1}{1+\mathbf{l}}=$ $=\mathbf{H}_1\sum_{i=0}^{\infty}(-1)^i\mathbf{l}^i$

The solution to equation (19) with the condition (19″) is

$$x(\lambda)=x_0\exp\left(\lambda\mathbf{H}_1\frac{1}{1+\mathbf{l}}\right)$$

The solution to equation (17) with the conditions (18′) and (18″) is

$$x(\lambda,t)=\begin{cases}0, & t\leqslant 0, \quad \lambda\in R\\ x_0\exp\left(\lambda\mathbf{H}_1\dfrac{1}{1+\mathbf{l}}\right), & t>0, \quad \lambda\in R\end{cases}$$

we see that $x(\lambda,t)=x_0$, $0<t<1$, $\lambda\in R$.

Let us suppose now that we took only the approximate solution to the characteristic equation:

$$\tilde{\mathbf{W}}=\mathbf{H}_1\sum_{i=0}^{r}(-1)^i\mathbf{l}^i.$$

The measure of the approximation $\varepsilon(T)$ can be calculated from:

$$\exp\left(\lambda\mathbf{H}_1\sum_{i=r+1}^{\infty}(-1)^i\mathbf{l}^i\right)-I\leqslant\exp(|\lambda|\cdot\mathbf{H}_1\mathbf{l}^{r+1})-I\leqslant_T|\lambda|\,\mathbf{H}_1\mathbf{l}^{r+1}e^{\lambda(T-1)^r/r!}=\varepsilon(T),\quad T>1.$$

The error's estimation is

$$|x_0e^{\lambda\mathbf{W}}-x_0e^{\lambda\tilde{\mathbf{W}}}|\leqslant_T|x_0|\,|e^{\lambda\tilde{\mathbf{W}}}|\,|\lambda|\,\mathbf{H}_1e^{|\lambda|(T-1)^r/r!}\mathbf{l}^{r+1}\leqslant$$

$$\leqslant_T|\lambda|\cdot e^{|\lambda|}\cdot|x_0|\cdot\frac{(T-1)^{r+1}}{(r+1)!}e^{|\lambda|(T-1)^r/r!}.$$

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LINEARNE DIFERENCIJALNE JEDNAČINE SA KOEFICIJENTIMA U POLJU II

Rezime

U ovom radu data je konstrukcija rešenja, aproksimacija rešenja kao i ocena greške za diferencijalnu jednačinu:

$$\sum_{\mu=0}^{M} \mathbf{A}_\mu x^{(\mu)}(\lambda)=0, \qquad \lambda_1 \leqslant \lambda \leqslant \lambda_2$$

gde su koeficijenti $\mathbf{A}_\mu$ oblika:

$$\mathbf{A}_\mu=\sum_{k=0}^{\infty} \mathbf{a}_{\mu,k} e^{-\tau_k^\mu \mathbf{s}}, \quad \mu=0,\ldots,M, \qquad \mathbf{a}_{\mu,0}\neq 0$$

$$\mathbf{a}_{\mu,k} \sum_{\nu=\nu_{\mu,k}}^{\infty} \alpha_{\mu,k,\nu} \mathbf{l}^{\nu/\sigma_{\mu,k}}, \qquad \alpha_{\mu,k,\nu_{\mu,k}}\neq 0$$

Za svako μ niz $\{\tau_k^\mu\}$ je striktno monotono rastući i divergira, $\tau_0>-\infty$; $\nu_{\mu,k}>-\infty$; $\sigma_{\mu,\nu}\in \mathrm{N}$. Ovde su $\mathbf{l}$, $\mathbf{s}$, $\mathbf{e}^{-\lambda \mathbf{s}}$, $\lambda>0$, operatori integraljenja, diferenciranja i translacije u polju $\mathcal{M}$ operatora Mikusinskog.

Opšti postupak za rešavanje prikazan je naprimeru parcijalne diferencijalno-diferentne jednačine:

$$\frac{\partial^2 \chi(\lambda,t)}{\partial\lambda\,\partial t}+\frac{\partial \chi(\lambda,t)}{\partial \lambda}=\chi(\lambda,t-1), \qquad \lambda\in \mathrm{R}$$