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ON THE ZED-INTEGRAL

1. Introduction

We have introduced in paper [3] the notion of ZED integral of functions with values in a complete commutative semigroup with a special metric. We have used some ideas of J. Mikusiński [1]. So in special cases, our results reduce to the results for J. Mikusiński's HEM integral, Bochner and Lebesgue integrals.

In this paper, we continue our investigations of the ZED integral, now for the two dimensional case. Namely, we prove the generalization of the Fubini theorem. We obtain, by some additional assumptions, the two dimensional ZED -integral.

2. The ZED-integral

Let X be a commutative semigroup with a neutral element 0 and with a metric d, such that

$$d(x+y, x'+y') \leqslant d(x, x')+d(y, y') \qquad (d_+)$$

holds for each $x, x', y, y' \in X$. We assume that (X, d) is a complete metric space. Let K be an arbitrary set and let U be a family of functions from K to X. We assume that a function $\int$ (called integral) with values in X (or R) is defined on U. We assume that $U, \int$ satisfy the following axioms:

(Z) $0 \in U$ *and* $\int 0=0$;

(D) *If* $f, g \in U$, *then* $d(f, g) \in U$ *and* $d(\int f, \int g) \leqslant \int d(f, g)$;

(E) *If* $f_n \in U$ $n=1, 2, \ldots$, $\sum_n \int d(f_n, 0)<\infty$,

and the equality $f(x)=\sum_n f_n(x)$ *holds at every point* $x \in X$

at which $\sum_n d(f_n(x), 0) < \infty$, *then* $f \in U$ *and* $\int f = \sum_n \int f_n \cdot (f \simeq f_1 + f_2 + \dots)$,

$d(f, g)(t) = d(f(t), g(t))$. The integral $\int$ which satisfies axioms Z (zero property), D (distance property) and E (expansion property) we call ZED-integral. We consider the theory in two interpretations simultaneosly for X-valued and for real valued functions. In the real case, d is the usual distance. We extend the integral $\int$ on $-f$ ($f \in U$) for group valued function (so also for real valued function) in the following way

$$\text{(1)} \qquad I_f\, f \in U, \quad then\ -f \in U \quad and \int (-f) = -\int f.$$

In the following the integrals will be always ZED-integrals for X-valued functions and IED-integrals for real valued functions.

3. The generalized Fubini theorem

Let U_1 and U_2 be families of ZED-integrable functions on arbitrary sets K_1 and K_2 respectively and with values in a commutative complete semigroup X with a metric d for which holds (d_4).

Let G be a family of *basic functions* defined on $K_1 \times K_2$ such that hold:

1) If $f, g \in G$, then $d(f, g) \in G$ and $d(f, 0) \in G$;

2) If $f(x, y) \in G$, then $f(x_0, y) \in U_2$ for element $x_0 \in K_1$;

3) If $f(x, y) \in G$, then $\int f(x, y)\, dy \in U_1$.

f belongs to family U ,iff there are functions $f_n \in G$ $(n = 1, 2, \dots)$ such that hold

a) $\int dx \int d(f_1, 0)\, dy + \int dx \int d(f_2, 0)\, dy + \dots < \infty$;

b) $f(x, y) = f_1(x, y) + f_2(x, y) + \dots$ holds at those points (x, y) at which

$$\sum_n d(f_n(x, y), 0) < \infty.$$

Then we use the following notation

$$f \simeq f_1 + f_2 + \dots .$$

We define a two dimensional integral $\iint$ of $f \in U$ as $\iint f = \int dx \int f_1\, dy + \int dx \int f_2 dy + \dots$.

G e n e r a l i z e d F u b i n i t h e o r e m. *If* $f \in U$, *then the function* $\int f\, dy$ *is determined almost everywhere on* K_1 *and belongs to* U_1. *Morever*

$$\iint f = \int dx \int f\, dy.$$

The proof is similar to the proof of theorem 10.3.1 from [1] with some modifications. Namely, instaed of the norm we take $d(.,0)$ and instead of the necessary theorems from [1] we use some corresponding theorems from paper [3].

Corollary. *If* $f_i \in U$ $(i=1,\dots,k)$ *then* $f_1+\dots+f_k \in U$ *and*

$$\iint(f_1+\dots+f_k)=\iint f_1+\dots+\iint f_k.$$

4. A property of series in the commutative complete semigroup

For further investigations on the double integral, we shall use a property of series in X.

We say that a series $a_1+a_2+\dots$ is composed of infinitely many series $a_{i1}+$ $+a_{i2}+\dots$ $(i=1,2,\dots)$, if there is a rearrangement of positive integers

$$p_{11}, p_{12}, \dots\ ,$$
$$p_{21}, p_{22}, \dots\ ,$$
$$\dots$$

such that $a_{ij}=a_{p_{ij}}$.

Theorem on series. *Let* $a_1+a_2+\dots$ *be a series such that* $\sum_n d(a_n,0)<\infty$ *hold and it be composed of an infinite number of convergent series*

$$s_1=a_{11}+a_{12}+\dots, \qquad s_2=a_{21}+a_{22}+\dots, \qquad \dots,$$

such that the series $s_1+s_2+\dots$ *converges. Then the series* $a_1+a_2+\dots$ *converges itself, and*

$$a_1+a_2+\dots=s_1+s_2+\dots\ .$$

Proof. First, we shall prove the theorem in a special case. Namely, when the series $a_1+a_2+\dots$ is composed of two series s_1 and s_2. The required formula follows from the inequality

$$d(a_1+\dots+a_n,\ a_{11}+\dots+a_{1n}+a_{21}+\dots+a_{2n})\leq$$
$$\leq d(a_{r_{n+1}},0)+d(a_{r_{n+2}},0)+\dots \qquad (by\ (d_+))$$

where r_{n+1} is the least postitive integer which does not appear in $1,2,\dots,n$ among $p_{11},\dots,p_{1n}$, $p_{21},\dots,p_{2n}$. The theorem is true, by induction, also in the case when a series is composed of any finite number of partial series.

Now we take the general case. For a given $\varepsilon>0$, there exists a positive integer k such that

$$(1) \qquad d(s_{1n}+\dots+s_{nn},\ a_1+\dots a_n)\leq d(a_{k+1},0)+d(a_{k+2},0)+\dots<\varepsilon$$

for $n>r$, where $s_{in}=a_{pi1}+\dots+a_{pin}$ $(i=1,2,\dots)$ and r is so chosen that $p_{ij}>k$ for $i,j>r$ and that among the p_{ij} with $i,j\leq r$ there appear all the numbers $1,\dots,k$.

Because the theorem is true in finite cases and by (1), we obtain the assertion of the theorem in the following way

$$d(s_1+\ldots+s_n,\, a_1+\ldots+a_n)\leqslant d(s_{1n}+\ldots+s_{nn},\, a_1+\ldots+a_n)+$$
$$+d(s_1, s_{1n})+\ldots+d(s_n, s_{nn})<2\varepsilon \quad \text{for} \quad n>r.$$

5. The double ZED-integral

We must introduce some additional conditions on the family of functions G and on the metric d to obtain the ZED-properties of the integral $\iint$.

Theorem on double integral. *If the metric on the complete commutative semigroup X satisfies (d_+) and*

$$|d(x, x')-d(y, y')|\leqslant d(x+y, x'+y') \qquad (d_-)$$

and the family G satisfies also the following conditions:

(z) $0\in G$; *for real valued function $f\in G$, $-f\in G$;*

(ed) *For each system $f_1, \ldots, f_k$ of functions from G, there exists in G another system of functions $g_1, \ldots, g_p$, having disjoint carriers (i. e., $g_i g_j=0$ for $i\neq j$), such that $f_1+\ldots+f_k=g_1+\ldots+g_p$ (property (em) in [1], p. 96);*

then the integral $\iint$ is a ZED-integral on U.

Proof. By the generalized Fubini theorem, property (z) and (Z)-property on U_1 and U_2, there follows the (Z)-property of the integral $\iint$.

Now, we shall prove that $\iint$ satisfies (D). First, we shall prove that $d(f, g)\in U$ for $f\in U$ and $g\in U$. We assume that

$$f\simeq f_1+f_2+\ldots \qquad (f_i\in G)$$
$$g\simeq g_1+g_2+\ldots \qquad (g_k\in G).$$

We denote by Z the set of all points from $K_1\times K_2$ where the equalities $f=f_1+$ $+f_2+\ldots$ and $g=g_1+g_2+\ldots$ hold (points at which the series $\sum_n d(f_n(x,y),0)<\infty$ and $\sum_n d(g_n(x,y),0)<\infty$ converge). We have

$$d(f, g)=\lim_{n\to\infty} d(s_n, u_n) \qquad \text{for} \qquad x\in Z,$$

where $s_n=f_1+\ldots+f_n$ and $u_n=g_1+\ldots+g_n$. We can write.

$$d(f, g)=d(s_1, u_1)+(d(s_2, u_2)-d(s_1, u_1))+\ldots \tag{5.1}$$

on Z. By the properites of G, we can write

$$d(s_1, u_1) = h_1,$$

(5.2) $$d(s_{n+1}, u_{n+1}) - d(s_n, u_n) = h_{p_n+1} + \ldots + h_{p_{n+1}}$$

for $n = 1, 2, \ldots$, where the functions h_i from G have disjoint carriers and $p_1 = 1$.

By (d_+) and (d_-), we obtain the following inequalities

$$|d(s_{n+1}, u_{n+1}) - d(s_n, u_n)| \leqslant d(s_{n+1} + u_n, u_{n+1} + s_n) =$$
$$= d(s_n + u_n + f_{n+1}, u_n + s_n + g_{n+1}) \leqslant d(f_{n+1}, g_{n+1}).$$

Hence and by (5.2), we obtain

$$d(h_{p_n+1}, 0) + \ldots + d(h_{p_{n+1}}, 0) \leqslant d(f_{n+1}, g_{n+1}) \leqslant$$
$$\leqslant d(f_{n+1}, 0) + d(g_{n+1}, 0) \qquad (n = 1, 2, \ldots).$$

Adding all these inequalities and also the inequality

$$d(h_1, 0) \leqslant d(f_1, 0) + d(g_1, 0)$$

we get

(5.3) $$d(h_1, 0) + d(h_2, 0) + \ldots \leqslant d(f_1, 0) + d(f_1, 0) + \ldots +$$
$$+ d(g_1, 0) + d(g_2, 0) + \ldots \quad \text{on } Z.$$

If we first integrate $\iint$ all these inequalities using the corollary and then sum them up we get

(5.4) $$\iint d(h_1, 0) + \iint d(h_2, 0) + \ldots \leqslant \iint d(f_1, 0) + \iint d(f_2, 0) + \ldots$$
$$\ldots + \iint d(g_1, 0) + \iint d(g_2, 0) + \ldots .$$

The series on the right side are convergent by assumptions

$$f \simeq f_1 + f_2 + \ldots \quad \text{and} \quad g \simeq g_1 + g_2 + \ldots$$

By (5.1) and (5.2), we obtain

$$d(f, g) = h_1 + h_2 + \ldots \quad \text{on} \quad Z. \tag{5.5}$$

The series on the right side, by (5.3), converges in the sense $\sum_n d(h_n, 0) < \infty$ on Z, but may be not only on Z. So we write

$$d(f, g) = h_1 + d(f_1, g_1) - d(f_1, g_1) + h_2 + \ldots .$$

Now, the series on the right side converges in the sense $d(h_1, 0) + d(f_1, g_1) + d(f_1, g_1) + d(h_2, 0) + \ldots < \infty$ on and only on Z. Then we can write

$$d(f, g) \simeq h_1 + d(f_1, g_1) - d(f_1, g_1) + h_2 + \ldots,$$

and so $d(f, g) \in U$.

The inequality from condition (D) follows by the generalized Fubini theorem, the inequality property for real valued functions (*i. e.*, $f' \leqslant g'$ implies $\iint f' \leqslant \iint g'$) and corresponding properties on U_1 and U_2:

$$d\left(\iint f, \iint g\right)=d\left(\int dx \int f\,dy, \int dx \int g\,dy\right) \leqslant$$
$$\leqslant \int dx\, d\left(\int f\,dy, \int g\,dy\right) \leqslant \int dx \int d(f,g)\,dy = \iint d(f,g).$$

We remark that by (5.4) and (5.5) it follows that $f \simeq f_1+f_2+\ldots (f_i \in G)$ implies

(5.6.) $$\iint d(f,0) \leqslant \iint d(f_1,0)+\iint d(f_2,0)+\ldots\ .$$

Now we shall prove the property (E) of the integral $\iint$. We start with the representation

(5.7) $$f \simeq f_1+f_2+\ldots \qquad \text{for} \qquad f_i \in U$$

Let $\varepsilon_i > 0$ $i=1,2,\ldots$ such that the series $\varepsilon_1+\varepsilon_2+\ldots$ is convergent.

We shall prove that for each fixed i we can choose a represantation

(5.8) $$f_i \simeq f_{i1}+f_{i2}+\ldots \qquad \text{for} \qquad f_{ij} \in G,$$
such that

(5.9) $$\iint d(f_{i1},0)+\iint d(f_{i2},0)+\ldots < \varepsilon_i+\iint d(f_i,0).$$

Namely, for $f_i \simeq g_{i1}+g_{i2}+\ldots$ $(g_{ij} \in G)$ there exists r such that

$$\iint d(g_{ir+1},0)+\iint d(g_{ir+2},0)+\ldots < \frac{\varepsilon_i}{2}.$$

Now, there are functions $f_{i1},\ldots,f_{ir} \in G$ with disjoint carriers, such that

$$g_{i1}+\ldots+g_{ir}=f_{i1}+\ldots+f_{ir}.$$

Taking $f_{ik}=g_{ik}$ for $k>r$ we obtain the required expansions

$$f_i \simeq f_{i1}+f_{i2}+\ldots \qquad (i=1,2,\ldots).$$

This fact follows from the inequality

$$\iint d(f_{i1},0)+\ldots+\iint d(f_{i1},0) \leq \iint d(f_i,0)+\iint d(f_i, f_{i1}+\ldots+f_{ir})$$

by (5.9) and using the following fact

$$\iint d(f_i, f_{i1}+\ldots f_{ir}) \leqslant \iint d(g_{ir+1},0)+\iint d(g_{ir+2},0)+\ldots$$

(by (5.6)).

The required expansion of f we obtain by the composition of all the series (5.8) using the theorem on series from section 4., so we have

$$f \simeq g_1 + g_2 + \dots \qquad (g_i \in G).$$

The verification of this fact is similar to the end of the proof of theorem 10.4.1 in [1] with necessary modifications.

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O ZED-INTEGRALU

rezime

U radu [3] uveden je pojam ZED-integrala funkcija sa vrednostima u kompeltnoj komutativnoj polugrupi sa metrikom d koja zadovoljava uslov

$$(d_+) \qquad d(x+y, x'+y') \leq d(x, y) + d(x', y')$$

U ovom radu se nastavljaju započeta istraživanja, sada za dvodimenzionalni slučaj. Dokazuje se uopštena Fubinijeva teorema. Uz dodatni uslov za metriku

$$(d_-) \qquad |d(x, y) - d(x', y')| \leq d(x+y, x'+y')$$

i još neke pretpostavke dobija se dvostruki ZED-integral. U specijalnim slučajevima dobijaju se odgovarajući rezultati za J. Mikusinskijev HEM-integral, Bochnerov i Lebesgueov integral — [1]