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TWO RANDOM FIXED POINT THEOREMS

Abstract In this paper we shall prove two fixed point theorems for mapping $H:M\to S\,(M\subset S)$ where $(S, \mathcal{F}, t)$ is the random normed space $(\mathcal{U}, \mathcal{F}, T_m)$, $\mathcal{U}$ is the set of classes of random variables $\xi:\Omega\to X$ which are equal with probability one, $(\Omega, \mathcal{K}, P)$ is a complete probability measure space, $(X, \| \|)$ is a separable Banach space and $\mathcal{F}:X\to\Delta^+$ is defined by:

$$\mathcal{F}_\xi(x)=P\{\omega\mid\omega\in\Omega,\ \|\xi(\omega)\|<x\},\ x\in R$$

First, we shall give some definitions and theorems which we shall use in the following.

Let $(X, \| \|)$ be a separable Banach space and V the set of random variables on a complete probability measure space $(\Omega, \mathcal{K}, P)$ with values in X, i. e. the mappings $\xi:\Omega\to X$ such that:

$$\{\omega\mid\omega\in\Omega, \xi(\omega)\in B\}\in\mathcal{K}$$

for every Borel set B of X. For each $\xi\in V$ let:

$$F_\xi(x)=P\{\omega\mid\omega\in\Omega,\ \|\xi(\omega)\|<x\},\ x\in R$$

The mapping $F:\xi\to F_\xi(\cdot)$ is a random seminorm on V under T-norm T_m and the (ε, λ)-topology on V induced by F is the topology of the convergence in probability [1].

Let $\mathcal{U}$ be the set of all classes of random variables which are equal with probability one. The mapping $\mathcal{F}:\xi\to F_\xi$ of $\mathcal{U}$ into Δ^+ is well defined and $(\mathcal{U}, \mathcal{F}, T_m)$ is the random normed space. In [3] the following theorem is given:

Theorem A *Let $(S, \mathcal{F}, T)$ be a complete Menger space with continuous T-norm T, A be a closed subset of S and $H:A\to A$ such that:*

$$F_{Hx, Hy}(q\varepsilon)\geqslant F_{x,y}(\varepsilon),\ 0<q<1$$

for every $x, y\in A$ and every $\varepsilon>0$. If there exists $x_0\in A$ such that $\sup_\varepsilon G_{x_0}(\varepsilon)=1$, where:

$$G_{x_0}(\varepsilon)=\inf\{F_{x_n, x_0}(\varepsilon)\mid n\in N\},\ x_n=H\,x_{n-1},\ n\in N$$

then there exists one and only one fixed point x^* of the mapping H and $x^*=\lim_{n\to\infty} x_n$.

Theorem B [2] *Let $(S, \mathcal{F}, t)$ be a random normed space with continuous T-norm t, M be a closed, starconvex subset of S so that $\sup_{\varepsilon} \{\inf_{x,y \in M} F_{x-y}(\varepsilon)\} = 1$ and $H: M \to M$ such that the following conditions are satisfied:*
(i) $F_{Hx-Hy}(\varepsilon) \geq F_{x-y}(\varepsilon)$, for every $x, y \in M$ and every $\varepsilon > 0$ (ii) There exists $m \in N$ such that $\overline{H^m(M)}$ is compact in the (ε, λ)-topology.

If S is complete in the (ε, λ)-topology then there exists at least one element $x \in M$ such that $Hx = x$.

Let $(S, \mathcal{F}, t)$ be a random normed space $(\mathcal{U}, \mathcal{F}, T_m)$ and $(\Omega, \mathcal{K}, P)$ be a complete probability measure space. Using Theorem A we shall prove the following random fixed point theorem:

Theorem 1 *Let $H: \mathcal{U} \to \mathcal{U}$ so that the following conditions are sastisfied:*
a) *There exists $C > 0$ such that for every $U \in \mathcal{U}$:*

$$P\{\omega \mid \|H(U)(\omega)\| \leq C\} = 1$$

b) *There exists $q \in (0,1)$ such that for every $U, V \in \mathcal{U}$:*

$$P\{\omega \mid \|H(U)(\omega) - H(V)(\omega)\| \leq q \|U(\omega) - V(\omega)\|\} = 1$$

Then there exists one and only one element $U^* \in \mathcal{U}$ such that:

$$H(U^*) = U^* \quad i.\ e. \quad H(U^*)(\omega) = U^*(\omega), \quad for\ every \quad \omega \in \Omega_0,\ P\,\Omega_0 = 1$$

Further, $U_n \xrightarrow{P} U^*$, where $U_n = H(U_{n-1}), n \in N$ and U_0 is an arbitrary element from $\mathcal{U}$.

Proof: We shall prove that all the conditions of Theorem A are satisfied where $\mathcal{U} = M$.

In the next text we shall use the following notations:

If $U, V \in \mathcal{U}$ and $\varepsilon > 0$ then:

$$G_1(U, V, \varepsilon) \stackrel{\text{def}}{=} \{\omega \mid \|U(\omega) - V(\omega)\| < \varepsilon\}$$

$$G_2(U, V) \stackrel{\text{def}}{=} \{\omega \mid \|H(U)(\omega) - H(V)(\omega)\| \leq q \|U(\omega) - V(\omega)\|\}$$

First, let us prove that:

$$(1) \qquad F_{H(U)-H(V)}(q\varepsilon) \geq F_{U-V}(\varepsilon)$$

for every $U, V \in \mathcal{U}$ and every $\varepsilon > 0$. Let us suppose that:

$$\omega \in G_1\left(U, V, \frac{\varepsilon}{q}\right) \cap G_2(U, V)$$

Then $\omega \in G_1(H(U), H(V), \varepsilon)$ and so:

$$(2) \qquad G_1\left(U, V, \frac{\varepsilon}{q}\right) \cap G_2(U, V) \subset G_1(H(U), \text{H}(V), \varepsilon)$$

From (2) it follows that:

$$(3) \qquad P\left(G_1\left(U, V, \frac{\varepsilon}{q}\right) \cap G_2(U, V)\right) \leq P(G_1(H(U), H(V), \varepsilon))$$

Further:

$$P(G_1(U,V,\frac{\varepsilon}{q}))=P(G_1(U,V,\frac{\varepsilon}{q})\cap(G_2(U,V)\cup CG_2(U,V)))=$$

$$=P((G_1(U,V,\frac{\varepsilon}{q})\cap G_2(U,V))\cup(G_1(U,V,\frac{\varepsilon}{q})\cap CG_2(U,V)))=$$

$$=P(G_1(U,V,\frac{\varepsilon}{q})\cap G_2(U,V))+P(G_1(U,V,\frac{\varepsilon}{q})\cap CG_2(U,V))$$

Since $P(G_2(U,V))=1$, it follows that $P(CG_2(U,V))=0$ and so:

$$P(G_1(U,V,\frac{\varepsilon}{q})\cap G_2(U,V))=P(G_1(U,V,\frac{\varepsilon}{q}))$$

Now, from (3) it follows that:

$$P(G_1(U,V,\frac{\varepsilon}{q}))\leqslant P(G_1(H(U),H(V),\varepsilon))$$

i. e. the relation (1).

Let us prove that $\sup G_{x_0}(\varepsilon)=1$, where:

$$G_{x_0}(\varepsilon)=\inf\{F_{H^n(x_0)-x_0}(\varepsilon)\mid n\in N\},$$

and $x_0=U$ is an arbitrary element from $\mathcal{U}$. We shall prove that for every $U\in\mathcal{U}$:

$$\sup_{\varepsilon}\{\inf_{n\in N}P\{\omega\mid \|H^n(U)(\omega)-U(\omega)\|<\varepsilon\}\}=1$$

Let $\lambda\in(0,1)$. Let us prove that there exists $\varepsilon(\lambda)>0$ such that:

$$\inf_{n\in N}P\{\omega\mid \|H^n(U)(\omega)-U(\omega)\|<\varepsilon(\lambda)\}>1-\lambda$$

Suppose that $\lambda'<\lambda$ and $\delta(\lambda')>0$ are such that:

$$F_U(\delta(\lambda'))=P\{\omega\mid \|U(\omega)\|<\delta(\lambda')\}>1-\lambda'$$

Since $\lim_{\varepsilon}F_U(\varepsilon)=1$ such $\delta(\lambda')$ exists. Further, for every $n\in N$:

$$A_n\stackrel{\text{def}}{=}\{\omega\mid \|H^n(U)(\omega)-U(\omega)\|\leqslant C+\delta(\lambda')\}$$

$$B_n\stackrel{\text{def}}{=}\{\omega\mid \|H^n(U)(\omega)\|\leqslant C\}$$

and:

$$D\stackrel{\text{def}}{=}\{\omega\mid \|U(\omega)\|<\delta(\lambda')\}$$

Then $B_n\cap D\subset A_n$ and since $P(B_n)=1$, then $P(D)=P(B_n\cap D)\leqslant P(A_n)$. From $P(D)>1-\lambda'$, it follows that $P(A_n)>1-\lambda'$ and so:

$$\inf_{n\in N}P(A_n)\geqslant 1-\lambda'>1-\lambda$$

This means that:

$$G_U(C+\delta(\lambda'))>1-\lambda$$

i. e. $\varepsilon(\lambda)=C+\delta(\lambda')$. From Theorem A it follows that $\mathrm{Fix}(H)\neq\emptyset$

Theorem 2 *Let $\mathcal{U}$ be as in Theorem* 1, *M be a starconvex and closed subset of $\mathcal{U}$ in the* (ε, λ)*-topology so that the following conditions are satisfied*:

1. $\sup_{\varepsilon} \{ \inf_{U,V \in M} P\omega \{ \mid \|U(\omega)-V(\omega)\| < \varepsilon\}\} = 1$

2. *For every* $U, V \in \mathcal{U}$: $P\{\omega \mid \|H(U)(\omega)-H(V)(\omega)\| \leqslant \|U(\omega)-V(\omega)\|\} = 1$.

3. *There exists* $m \in N$ *such that* $\overline{H^m(M)}$ *is compact in the* (ε, λ)*-topology.*

Then there exists $W \in M$ *such that* $H(W) = W$

Proof: From Theorem *B*, similarly as in Theorem 1, it follows that $\mathrm{Fix}(H) = 0$.

Remark: Now let T be a continuous random operator on X [1]. Then one can define the Nemytskii operator $\overline{T}$ on $\mathcal{U}$ by:

$$(\overline{T}U)(\omega) = T(\omega, U(\omega)), \quad \text{for every } \omega \in \Omega$$

The fact that W is the fixed point for T i. e. that $T(\omega, W(\omega)) = W(\omega)$ P a. s. is equivalent with:

$$\overline{T}W = W$$

A continuous random operator T on separable Banach space X has a fixed point $W \in \mathcal{U}$ if and only if W is the fixed point of the Nemytskii operator $\overline{T}$ in the random normed space $(\mathcal{U}, \mathcal{F}, T_m)$ and so we can apply fixed point theorems just proved in the fixed point theory for random operators.

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DVE VEROVATNOSNE TEOREME O NEPOKRETNOJ TAČKI

U ovom radu su dokazane dve teoreme o nepokretnoj tački za preslikavanje $H{:}M \to S$ $(M \subseteq S)$ gde je $(S, \mathcal{F}, t)$ slučajan normirani prostor $(\mathcal{U}, \mathcal{F}, \top_m)$, $\mathcal{U}$ je skup klasa slučajnih promenljivih $\xi{:}\Omega \to X$ koje su jednake P skoro svuda, $(\Omega, \mathcal{K}, P)$ je kompletan verovatnosno merljiv prostor, $(X, \|\ \|)$ separabilan Banahov prostor i $\mathcal{F}{:}X \to \Delta^+$ preslikavanje definisano na sledeći način:

$$\mathcal{F}_\xi(x) = P\{\omega \mid \omega \in \Omega, \|\xi(\omega)\| < x\} \quad x \in R$$

Dobijeni rezultati se mogu primeniti u teoriji nepokretne tačke za slučajne operatore $T{:}\Omega \times X \to X$ koji su predmet ispitivanja mnogih matematičara poslednjih godina.