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Olga Hadžić and Danica Nikolić-Despotović

FIXED POINT THEOREMS OF KRASNOSELSKI'S TYPE IN PROBABILISTIC LOCALLY CONVEX SPACES

In this paper we shall prove some fixed point theorems of Krasnoselski's type in probabilistic locally convex spaces. V. A. Istratescu introduced in [10] the notion of probabilistic locally convex space as a natural generalization of the notion of random normed space. There are many fixed point theorems in probabilistic metric and random normed spaces [1], [2], [3], [11], [15], and in [4] O. Hadžić proved a fixed point theorem in probabilistic locally convex spaces.

First, we shall give some definitions and theorems which will be used in the paper.

Let E be a linear space over the real or complex field K and for every i in the index set I is defined a function $\mathcal{F}^i: E \to \Delta^+$ where Δ^+ is the family of distribution functions F such that $F(0)=0$. We shall denote $\mathcal{F}^i(x)$ by F_x^i.

DEFINITION 1 *We say that the triplet* $(E, \{\mathcal{F}^i\}_{i\in I}, T)$ *is a probabilistic locally convex space (for short* PLC-*space) iff for every* $i \in I$, *the following conditions are satisfied*:

1. *For every* $i \in I$:

$$F_0^i = H, \quad where \quad H(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

2. *For every* $x \in E$, *every* $t \in K\,(t \neq 0)$, *every* $\varepsilon > 0$ *and every* $i \in I$:

$$F_{tx}^i(\varepsilon) = F_x^i\left(\frac{\varepsilon}{|t|}\right)$$

3. *For every* $(i, x, y, \varepsilon_1, \varepsilon_2) \in I \times E^2 \times (R^+)^2: F_{x+y}^i(\varepsilon_1+\varepsilon_2) \geqslant T(F_x^i(\varepsilon_1),\ F_y^i(\varepsilon_2))$ *where the mapping* $T: [0,1]^2 \to [0,1]$ *is a t-norm* [10].

The topology in E is introduced by the neighbourhood system of zero:

$$\mathcal{N} = \{N^i(\varepsilon, \lambda)\}_{(i,\varepsilon,\lambda)\in I\times R^+ \times (0,1)}$$

where:

$$N^i(\varepsilon, \lambda) = \{x \mid x \in E, F_x^i(\varepsilon) > 1-\lambda\} \quad ((\varepsilon, \lambda)\text{-topology})$$

and if t-norm T is continuous, then E is, in the (ε, λ)-topology, a linear topological space.

DEFINITION 2 *Let E be a linear space and X be a subset of E. The set X is starshaped iff there exists $x_0 \in X$ such that:*

$$tx+(1-t)\,x_0 \in X, \quad \textit{for every} \quad x \in X \quad \textit{and every} \quad t \in [0,1].$$

The point x_0 is then called a starpoint of X.

DEFINITION 3 *Let E be a topological vector space, $M \subset E$ and $Q: M \to E$. The mapping Q is demicompact if for every bounded sequence $\{x_n\}_{n\in N} \subset M$ for which the sequence $\{x_n - Qx_n\}_{n \in N}$ is convergent there exists a convergent subsequence $\{x_{n_k}\}_{k\in N}$.*

DEFINITION 4 *A subset M of a Hausdorff topological vector space E is admissible if for every compact subset K of M and every neighbourhood of zero V in E there exists a continuous mapping $h: K \to M$ such that:*

(i) $$x - hx \in V, \quad \textit{for every} \quad x \in K$$

(ii) $$\dim(\operatorname{span} h(K)) < \infty$$

If $K=E$, we say that E is *admissible.*

Remark: In [6] the admissibility of a class of random normed space is proved.

LEMMA [7] *Let $(E, \{\mathcal{F}^i\}_{i\in I}, T)$ be a Hausdorff sequentially complete probabilistic locally convex space with continuous t-norm T and Λ be a compact topological space. If we denote by $\mathcal{C}(\Lambda, E)$ the set of all continuous mappings from Λ into E then the triplet:*

$$(\mathcal{C}(\Lambda, E), \{\tilde{\mathcal{F}}^i\}_{i\in I}, T)$$

is a Hausdorff sequentially complete probabilistic locally convex space where the mapping $\tilde{\mathcal{F}}^i: \mathcal{C}(\Lambda, E) \to \Delta^+$ is defined by:

$$\tilde{F}^i_{\tilde{x}}(\varepsilon) = \sup_{\delta<\varepsilon} \inf_{\lambda\in\Lambda} F^i_{x(\lambda)}(\delta), \qquad \tilde{x} = \{x(\lambda)\} \in \mathcal{C}(\Lambda, E)$$

Now we shall give a probabilistic version of a fixed point theorem from [5], where Λ is a topological space.

THEOREM 1 *Let $(E, \{\mathcal{F}^i\}_{i\in I}, T)$ be a Hausdorff sequentially complete PLC-space where T is a continuous t-norm, M be a closed, probabilistic bounded subset of E, R be a compact mapping $R: M \to \Lambda, G: M \times R(M) \to M$, G is continuous so that the following conditions are satisfied:*

1. *For every $i \in I$ there exist $q(i) > 0$ and $\psi(i) \in I$ such that:*

$$F^i_{G(x_1,\lambda)-G(x_2,\lambda)}(q(i)\,\varepsilon) \geq F^{\psi(i)}_{x_1-x_2}(\varepsilon)$$

for every $x_1, x_2 \in M$, every $\lambda \in \overline{R(M)}$, and every $\varepsilon > 0$ where:

$$\lim_{n\to\infty} \prod_{k=0}^{n} q(\psi^k(i)) = 0, \quad \textit{for every} \quad i \in I.$$

2. *For every $i \in I$ there exists $g(i) \in I$ such that:*

$$F^{\psi^n(i)}_x(\varepsilon) \geq F^{g(i)}_x(\varepsilon)$$

for every $x \in E$, every $\varepsilon > 0$ and every $n \in N \cup \{0\}$.

If one of the conditions (A) and (B) is satisfied then there exists $x \in M$ such that $x = G(x, Rx)$ where:

(A) *The set M is admissible and convex,*

(B) *E is admissible and M is a starshaped closed neighbourhood retract.*

Proof: Similarly as in [7], it follows that there exists one and only one continuous mapping $\tilde{x}: \overline{R(M)} \to M$ such that:

$$\tilde{x}(y) = G(\tilde{x}(y), y), \quad \text{for every} \quad y \in \overline{R(M)}$$

Let us define the mapping R^* in the following way:

$$R^*(u) = (\tilde{x} \circ R)(u) \quad \text{for every} \quad u \in M$$

It is obvious that the mapping R^* is continuous and since R is a compact mapping we conclude that the mapping R^* is compact. If condition (A) is satisfied, then from Theorem [9] it follows that $\text{Fix}(R^*) \neq \emptyset$. If $x \in \text{Fix}(R^*)$ then:

$$x = R^*(x) = (\tilde{x} \circ R)(x) = G(\tilde{x}(R(x)), R(x)) =$$
$$= G(x, R(x))$$

If condition (B) is satisfied, then we can apply the theorem from [14] and similarly it follows that $\text{Fix}(R^*) \neq \emptyset$.

From Theorem 1 follows the Corollary which is a fixed point theorem of Krasnoselski's type.

COROLLARY *Let $(E, \{\mathcal{F}^i\}_{i \in I}, T)$ and M be as in Theorem 1, $S: M \to E$ be a compact and $Q: M \to E$ be a continuous mapping such that the following conditions are satisfied:*

1. *One of the conditions (A) and (B) from Theorem 1 is satisfied.*
2. *For every $x, y \in M: Qx + Sy \in M$*
3. *For every $i \in I$ there exist $q(i) > 0$ and $\psi(i) \in I$ such that:*

$$F^i_{Qx-Qy}(q(i)\,\varepsilon) \geq F^{\psi(i)}_{x-y}(\varepsilon), \quad \text{for every} \quad x, y \in M \quad \text{and} \quad \varepsilon > 0$$

where:

$$\lim_{n \to \infty} \prod_{k=0}^{n} q(\psi^k(i)) = 0$$

and condition 2 from Theorem 1 is satisfied.

Then there exists at least one element $x \in M$ such that $x = Qx + Sx$.

Proof: If we let $R = S$ and $G(x, y) = Qx + y$ then the equation $x = G(x, Rx)$ becomes $x = Qx + Sx$. From Theorem 1 it follows that $\text{Fix}(Q + S) \neq \emptyset$. Now we shall prove the following Theorem.

THEOREM 2 *Let $(E, \{\mathcal{F}^i\}_{i \in I}, T)$ and M be as in the Corollary, $S: M \to E$ be a compact and $Q: M \to E$ be a demicompact mapping so that conditions 1 and 2 from the Corollary are satisfied. If for every $i \in I$ there exists $q(i) > 0$ such that:*

$$F^i_{Qx-Qy}(q(i)\,\varepsilon) \geq F^{\psi(i)}_{x-y}(\varepsilon), \text{ for every } x, y \in M \text{ and } \varepsilon > 0$$

where:

$$\prod_{k=0}^{n} q(\psi^k(i)) \leqslant A(i) < \infty \text{ for every } i \in I \text{ and } n \in N$$

and condition 2 *from Theorem 1 is satisfied then* $\text{Fix}(Q+S) \neq \emptyset$.

Proof: Let $x_0 \in M$ be a starpoint of M and $\{\lambda_n\}_{n \in N}$ be a sequence of real numbers from the interval $[0,1]$ such that $\lim_{n \to \infty} \lambda_n = 1$. For every $n \in N$ let us define the mappings Q_n and S_n from M into E in the following way:

$$Q_n x = \lambda_n Q x, \quad \text{for every} \quad x \in M$$

$$S_n x = \lambda_n S x + (1-\lambda_n) x_0, \quad \text{for every} \quad x \in M$$

We shall show that the mappings Q_n and S_n satisfy all the conditions of the Corollary for every $n \in N$.

First, from the condition that $Qx + Sy \in M$ for every $x, y \in M$, it follows:

$$Q_n x + S_n y = \lambda_n (Qx + Sy) + (1-\lambda_n) x_0 \in M$$

Further, we have:

$$F^i_{Q_n x - Q_n y}(\varepsilon) = F^i_{\lambda_n Qx - \lambda_n Qy}(\varepsilon) = F^i_{Qx - Qy}\left(\frac{\varepsilon}{\lambda_n}\right) \geqslant F^{\psi(i)}_{x-y}\left(\frac{\varepsilon}{q(i)\lambda_n}\right) = F^{\psi(i)}_{x-y}\left(\frac{\varepsilon}{q_n(i)}\right)$$

where $q_n(i) = q(i)\lambda_n$. From this and the condition $\prod_{k=0}^{n} q(\psi^k(i)) \leqslant A(i) < \infty$ it follows:

$$\lim_{n \to \infty} \prod_{k=0}^{n} q_n(\psi^k(i)) = 0, \quad \text{for every } i \in I$$

and so the mapping Q_n satisfies all the conditions of the Corollary. Because the set $\overline{S(M)}$ is compact and E is a topological vector space, it follows that $\overline{S_n(M)}$ is compact, and so from the Corollary it follows that there exists, for every $n \in N$, $x_n \in M$ such that:

(1) $$x_n = Q_n x_n + S_n x_n$$

From (1) it follows that:

$$x_n = \lambda_n Q x_n + \lambda_n S x_n + (1-\lambda_n) x_0$$

and so:

$$x_n - Q x_n - S x_n = (\lambda_n - 1)(Q x_n + S x_n) + (1-\lambda_n) x_0$$

Now we shall show that:

(2) $$\lim_{n \to \infty} x_n - Q x_n - S x_n = 0$$

Namely, if we prove that the set M is bounded in the (ε, λ)-topology then we have that:

$$\lim_{n \to \infty} (\lambda_n - 1)(Q x_n + S x_n) = 0$$

because E is a topological vector space and $\lim_{n\to\infty} \lambda_n=1$. Let V be a neighbourhood of zero of the form:

$$V(i, \varepsilon, \lambda)=\{x \mid x\in E,\ F_x^i(\varepsilon)>1-\lambda\}$$

We shall show that there exists $\mu>0$ such that:

$$\mu M\subset V \tag{3}$$

which means that the set M is bounded in (ε, λ)-topology. Since t-norm T is continuous we have:

$$\sup_{\varepsilon}\inf_{x\in M} F_x^i(\varepsilon)\geqslant T\left(\sup_{\varepsilon}\inf_{x\in M} F_{x-x_0}^i\left(\frac{\varepsilon}{2}\right), \sup_{\varepsilon} F_{x0}^i\left(\frac{\varepsilon}{2}\right)\right)\geqslant$$

$$\geqslant T\left(\sup_{\varepsilon}\inf_{x,y\in M} F_{x-y}^i\left(\frac{\varepsilon}{2}\right), \sup_{\varepsilon} F_{x0}^i\left(\frac{\varepsilon}{2}\right)\right)=T(1, 1)=1$$

and so for every $\lambda\in(0, 1)$ and $i\in I$ there exists $\delta_{i,\lambda}>0$ such that:

$$\inf_{x\in M} F_x^i(\delta_{i,\lambda})>1-\lambda \tag{4}$$

From (4) we have:

$$F_x^i(\delta_{i,\lambda})>1-\lambda \text{ for every } x\in M$$

and so for $\mu(i, \varepsilon, \lambda)=\dfrac{\delta_{i,\lambda}}{\varepsilon}$ it follows:

$$F^i_{\frac{x}{\mu(i,\varepsilon,\lambda)}}(\varepsilon)>1-\lambda$$

which is relation (3).

Because the set $\overline{S(M)}$ is sequentially compact there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of the sequence $\{x_n\}_{n\in N}$ such that:

$$\lim_{k\to\infty} Sx_{n_k}=y \tag{5}$$

So from (2) it follows that $\lim_{k\to\infty}[x_{n_k}-Qx_{n_k}]=y$.

The mapping Q is demicompact and the sequence $\{x_{n_k}\}_{k\in N}$ is bounded. According to Definition 3, we conclude that there exists a convergent subsequence $\{x_{n_{k(r)}}\}_{r\in N}$ of the sequence $\{x_{n_k}\}_{k\in N}$. Let $\lim_{r\to\infty} x_{n_{k(r)}}=x^*$. Then we have from (2):

$$x^*=\lim_{r\to\infty} x_{n_{k(r)}}=\lim_{r\to\infty}(Qx_{n_{k(r)}}+Sx_{n_{k(r)}})=Qx^*+Sx^*$$

and x^* is a fixed point of the mapping $Q+S$. This completes the proof.

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Olga Hadžić i Danica Nikolić-Despotović

TEOREME O NEPOKRETNOJ TAČKI TIPA KRASNOSELJSKOG U VEROVATNOSNIM LOKALNO KONVEKSNIM PROSTORIMA

Koristeći teoremu o neprekidnoj zavisnosti nepokretne tačke od parametra u ovom radu su dokazane neke teoreme o nepokretnoj tački tipa Krasnoseljskog u verovatnosnim lokalno konveksnim prostorima.