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ONE TYPE OF RANDOM DIFFUSION EQUATION

Abstract: In this paper we shall solve one type of random partial differential equation, which arises in the study of the diffusion of some substances through a continuous medium, by using the notion of random operator function [2].

A radioactive gas is diffusing into the atmosphere from contaminated ground so that μg/cm² sec escape into the air. Assume the ground and the atmosphere to be semi-infinite media with $x=0$ as the boundary. Then the density of the radioactive gas in the air is governed by the relations:

$$\frac{\partial \rho(x,t)}{\partial t}=D\frac{\partial^2 \rho(x,t)}{\partial x^2}-k\rho(x,t) \qquad (k=\text{const})$$

$$\left.\frac{\partial \rho}{\partial x}\right|_{x=0}=-\chi(t), \qquad (\rho(\infty,t)<\infty)$$

$$\rho(x,0)=0 \qquad x\geqslant 0$$

where $D=$const is the diffusion coefficient which depends on the properties of the medium (D is measured in cm²/sec).

We shall solve here the random partial differential equation by the method developed in the theory of ordinary Mikusinski operators having the following form

$$\frac{\partial}{\partial t}\rho(.\,,.)=D\frac{\partial^2}{\partial x^2}\rho(.\,,.)-k\rho(.\,,.)$$

assume that

$$\rho(.\,,0)=0, \qquad x\geqslant 0$$

$$\frac{\partial}{\partial x}\rho(0,.)=-\chi, \quad \lim_{x\to\infty}\rho(x,.)\exp\left\{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}\right\}=0$$

where χ is a given strictly continuous stochastic process.

First we shall give some definitions and lemma which we shall use later.

Let (Ω,σ,P) be a given probability space. In [2] M. Ullrich introduced the notion of random Mikusinski operators as, in a certain sense, a generalization of stochastic processes and they are very important in applications, especially in the solving of random equations. The notion of a random Mikusinski operator is a generalization of the notion of an ordinary random variable and a strictly continuous stochastic process.

We shall say that a sequence of random Mikusinski operators ξ_n, $n=1, 2, \dots$ converges to a random Mikusinski operator ξ_0, and denote it by $\xi_n \to \xi_0$, if there exist a random Mikusinski operator $\alpha(\omega)$ and a sequence of strictly continuous stochastic process $f_n(\omega, .)$ $n=1, 2, \dots$ and f_0 such that for every $\omega \in \Omega$

$$\xi_n(\omega) = \alpha(\omega) f_n(\omega, .),$$

and

$$f_n(\omega, .)$$

converges uniformly on every finite interval $<0, T>$ to $f_0(\omega, .)$

An elementary application of random Mikusinski operators for solving of random differential equations is the following:

Lemma ([2]*p. 14*) *Let $x(\omega, .)$ be an arbitrary strictly continuous stochassic process all of whose realizations have all derivatives to the $(n-1)-th$ order which are absolutely continuous. Then for every $\omega \in \Omega$.*

$$s^n \{x(\omega, t)\} = \{x^{(n)}(\omega, t)\} + \sum_{j=0}^{n-1} s^j [x^{(n-1-j)}(\omega, c)]$$

where $[a]$ denotes the number operator.

The proof of this lemma is the same as in the theory of ordinary Mikusinski operators. Let I be a given set of a real number. We shall say that a mapping $a: \Omega x I \to \mathcal{M}$ is a random operator function if for every $\omega \in \Omega$, $a(\omega, .)$ is an ordinary operator function [1] and for every $x \in I$, $a(., x)$ is a random operator.

A random operator function a is continuous on the finite interval. $I = [x_1, x_2] \subset \mathbf{R}$ if there exists a strictly continuous stochastic process f depending on two parameters t, x and for every $\omega \in \Omega$ an operator $q(\omega)$ independent on x such that for every $\omega \in \Omega$ and $x \in I$

$$a(\omega, x) = q(\omega) f(\omega, x, .)$$

From this definition it follows that q is a random Mikusinski operator.

A random operator function a is continuous on an infinite interval $\mathcal{J}$ of the set of all real numbers, if it is continuous on every finite interval $I \subset \mathcal{J}$.

We can use the theory of random operator functions for solving random partial differential equations with constant coeficients.

Namely, let us denote by $\rho(x, t)$ the density of the radioactive gas in the air which is diffusing into the atmosphere from contaminated ground Then $\rho(x, t)$ fulfills the following partial differential equation

$$\frac{\partial}{\partial t} \rho(., .) = D \frac{\partial^2}{\partial x^2} \rho(., .) - k\rho(., .) \quad (D, k = \text{const})$$

Assume that

$$\rho(.,o)=0$$

and

$$\frac{\partial}{\partial x}\rho(o,.)=-\chi$$

$$(*)\quad \lim_{x\to\infty}\rho(x,.)\exp\left\{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}\right\}=0$$

where χ is a given strictly continuous stochastic process.

The significance of the condition (*) consists only in a certain limitation of the function $\rho(x,t)$ which ensures the uniqueness of the solution.

Then for every $x>0$

$$\left\{\frac{\partial}{\partial t}\rho(x,t)\right\}=s\{\rho(x,t)\}-[\rho(x,o)]=s\{\rho(x,t)\}$$

and therefore

(1) $$s\rho(x)=[D]\,\rho''(x)-[k]\,\rho(x)$$

$$\text{where }\rho(x)=\{\rho(x,t)\},\quad \lim_{x\to\infty}\rho(x)\exp\left\{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}\right\}=0$$

Let us consider that $\rho(x)=e^{xw}$ where w is a certain Mikusinski operator. Then after substitution of e^{xw} into (1) we obtain

$$(s+[k])\,e^{xw}=[D]w^2e^{xw}$$

namely,

$$w_{1,2}=\frac{1}{[\pm\sqrt{D}]}\sqrt{s+[k]}$$

The general solution to equation (1) in question has the form

$$\rho(x)=C_1\exp\left\{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}\right\}+C_2\exp\left\{\left[\frac{x}{\sqrt{D}}\right]\sqrt{s+[k]}\right\}$$

We shall prove that if $\rho(x)=\{\rho(x,t)\}$ to satisfy condition (*) we must have $C_2=[0]$

Indeed, because of

(2) $$C_2=\rho(x)\,e^{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}}-C_1e^{\left[\frac{-2x}{\sqrt{D}}\right]\sqrt{s+[k]}}$$

but in view of

$$0\leqslant e^{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}}\leqslant 3\sqrt{\frac{6}{\pi e^3}\,\frac{[D]}{x^2(s+[k])}}$$

and

$$0 \leqslant e^{\left[\frac{-2x}{\sqrt{D}}\right]\sqrt{s+[k]}} \leqslant 3\sqrt{\frac{6}{\pi\, e^3}\,\frac{[D]}{4x^2(s+[k])}}$$

we have

$$\lim_{x\to\infty} e^{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}} = \lim_{x\to\infty} e^{\left[\frac{-2x}{\sqrt{D}}\right]\sqrt{s+[k]}} = 0$$

whence under assumption $(*)$ it follows that the right side of formula (2) tends to zero as $x\to\infty$. Therefore $C_2=[0]$.

In this manner the solution is reduced to the form

$$\rho(x) = C_1\, e^{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}}$$

where C_1 is such that the initial condition is fulfilled. This C_1 is determined by the following equation

$$-\chi = \frac{-C_1}{[\sqrt{D}]}\sqrt{s+[k]} \qquad C_1 = \frac{[\sqrt{D}]}{(s+[k])^{1/2}}\chi$$

and therefore we obtain the required solution

$$\rho(\omega, x) = \frac{[\sqrt{D}]}{(s+[k])^{1/2}}\, e^{\left[\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}} \quad \{\chi(\omega, t)\}$$

Also, we see that ρ is a random operator function.

REFERENCES

[1] J. Mikusinski: *Operational calculus*, Warszawa (1967)

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JEDNA SLUČAJNA DIFUZNA JEDNAČINA

Rezime

U ovom radu rešena je jedna slučajna parcijalna diferencijalna jednačina koja karakteriše difuziju radioaktivnog gasa u atmosferi na kontaminiranom zemljištu. Rešenje date slučajne difuzne jednačine je slučajna operatorska funkcija [2], a slučajna parcijalna diferencijalna jednačina je sledećeg oblika:

$$\frac{\partial}{\partial t}\rho(\ldots) = D\frac{\partial^2}{\partial x^2}\rho(.,.) - k\rho(.,.) \quad (D,k=\text{const})$$

$$\rho(..0)=0,\ x\geqslant 0$$

$$\frac{\partial}{\partial x}\rho(0,.) = -\chi,\ \lim_{x\to\infty}\rho(x,.)\exp\left\{\left[-\frac{-x}{\sqrt{D}}\right]\sqrt{s+[k]}\right\}=0$$

gdje je χ dati striktno neprekidni stohastički proces.