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ON A CLASS OF D-COMPLETE OSPK AND COMPLETE ERROR-CORRECTING CODES

In [1], the following definitions of D-complete OSPK and complete codes are given:

Definition 1. *An orthogonal system of partial quasigroups* (OSPK) $\sum = \{E, F, A_1, \ldots, A_{k-2}\}$ *defined on the finite set Q is said to be D-complete if and only if for each* OSPK $\overline{\sum} = \{\overline{E}, \overline{F}, \overline{A}_1, \ldots, \overline{A}_{k-2}\}$ *the following holds*:

(a) $$A_1 \subseteq \bar{A}_1 \wedge \ldots \wedge A_{k-2} \subseteq \bar{A}_{k-2} \Rightarrow A_1 = \bar{A}_1 \wedge \ldots \wedge A_{k-2} = \bar{A}_{k-2}.$$

Definition 2. *A code $\mathcal{K}$ of k-sequences over the alphabet Q (i. e. $\mathcal{K} \subseteq Q^k$) with the code distance d is said to be complete if and only if for arbitrary code $\overline{\mathcal{K}}$ of k-sequences with the code distance d over the alphabet Q the following holds*:

(d) $$\mathcal{K} \subseteq \overline{\mathcal{K}} \Rightarrow \mathcal{K} = \overline{\mathcal{K}}$$

The following definition is also from [1]:

Definition 3. *A partial quasigroup (Q, A) is said to be complete if and only if for arbitrary quasigroup $(Q, \bar{A})$ the following holds*:

($\bar{a}$) $$A \subseteq \bar{A} \Rightarrow A = \bar{A}.$$

OSPK is defined on the condition that all the operations are different pairwise. In particular, E is different from F. E and F are operations for which $E(x, y) = x$ and $F(x, y) = y$ hold respectively, whenever $E(x, y)$ and $F(x, y)$ are defined. A sufficient condition for E and F to be different is that for their common domain the following holds:

$$card\,\mathcal{D}E = card\,\mathcal{D}F > card\,Q.$$

Theorem 1. *For each even positive integer q there is a D-complete* OSPK *over $Q = \{0, 1, \ldots, q-1\}$ $\sum = \{E, F, A_1, A_2\}$ such that for the common domain of its operations*

$$card\,\mathcal{D}A_1 = card\,\mathcal{D}A_2 = q\,(q-1)$$

holds, where neither of the operations A_1 and A_2 is complete.

Proof. It is known ([3], p. 82) that for each even positive integer q there is a horizontally complete latin square of order q i. e. such that for any ordered pair of elements α, β $(0 \leqslant \alpha, \beta \leqslant q-1, \alpha \neq \beta)$, there exists a row of the latin square in which α and β appear as adjacent elements; in other words, all the $q(q-1)$ pairs of adjacent (in some row) elements are different from each other, considered as ordered pairs.

Let L be a horizontally complete latin square of order q based upon elements of Q. Starting from this square, two latin rectangles P_1 and P_2 of order q by q-1 can be obtained in the following way: P_1 is obtained from L when the last column of L is omitted, P_2 is obtained from L by omitting the first column of L. Since L is horizontally complete, the rectangles P_1 and P_2 thus obtained will form an orthogonal pair of latin rectangles.

The rectangles P_1 and P_2 can be viewed as the tables of partial quasigroups (Q, A_1) and (Q, A_2) respectively with the common domain for which only the products $A_1(i, q-1)$ and $A_2(i, q-1)$, $i=0,1, \ldots, q-1$, are not defined. Neither of the quasigroups (Q, A_1) and (Q, A_2) is complete, which follows from the well known Hall-s theorem.

The operations A_1 and A_2, together with the corresponding operations E and F, form OSPK $\Sigma=\{E, F, A_1, A_2\}$. We will show that this OSPK is D-complete.

Indeed, since each ordered pair of different elements from Q occurs exactly once in the form $(A_1(i, j), A_2(i, j))$, where $i, j \in \{0,1, \ldots, q-1\}$, the operations A_1 and A_2 can be completed, with their orthogonality perserved, only in such a way that for some $i \in Q$: $A_1(i, q-1)=A_2(i, q-1)$. However, by completing the operations in this fashion, the quasigroup property of at least one of the operations A_1 and A_2 is violated, i. e. in i—th row of one of these operation the same element will occur twice. Namely, in each row of the tables of both operations A_1 and A_2, exactly one element from Q is missing and, due to the way in which the tables have been obtained, it follows that the missing element in the i-th row of one table is different from the missing element in the i-th row of the other. Thus, OSPK $\sum=\{E, F, A_1, A_2\}$ is D-complete.

Corollary. *For each positive even integer q there is a complete code of $q(q-1)$ 4-sequences over the alphabet $Q=\{0,1, \ldots, q-1\}$ with the code distance $d=3$.*

The D-complete OSPK from Theorem 1, as well as the corresponding complete code, is easy to find for arbitrary even q because there is a way of finding horizontally complete latin squares for arbitrary even positive integer q (see [3], *pp* 80—83, 97).

Example. From the horizontally complete latin square of order 6, represented by Table 1, two latin rectangles are obtained (Tables 2 and 3), in the way described above, to which correspond partial quasigroups (Q, A_1) and (Q, A_2) (Tables 4 and 5) which are not complete.

0	1	5	2	4	3
1	2	0	3	5	4
2	3	1	4	0	5
3	4	2	5	1	0
4	5	3	0	2	1
5	0	4	1	3	2

Table 1

0	1	5	2	4
1	2	0	3	5
2	3	1	4	0
3	4	2	5	1
4	5	3	0	2
5	0	4	1	3

Table 2

1	5	2	4	3
2	0	3	5	4
3	1	4	0	5
4	2	5	1	0
5	3	0	2	1
0	4	1	3	2

Table 3

A_1	0	1	2	3	4	5
0	0	1	5	2	4	—
1	1	2	0	3	5	—
2	2	3	1	4	0	—
3	3	4	2	5	1	—
4	4	5	3	0	2	—
5	5	0	4	1	3	—

Table 4

A_2	0	1	2	3	4	5
0	1	5	2	4	3	—
1	2	0	3	5	4	—
2	3	1	4	0	5	—
3	4	2	5	1	0	—
4	5	3	0	2	1	—
5	0	4	1	3	2	—

Table 5

The quasigroup operations A_1 and A_2, together with the corresponding operations E and F, form a D-complete OSPK $\sum=\{E, F, A_1, A_2\}$ and to this corresponds the following complete code of 4-sequences over the alphabet $\{0, 1, 2, 3, 4, 5\}$ with the code distance $d=3$:

0001	1012	2023	3034	4045	5050
0115	1120	2131	3142	4153	5104
0252	1203	2214	3225	4230	5241
0324	1335	2340	3351	4302	5313
0443	1454	2405	3410	4421	5432

It is interesting to look at the codes obtained in this way from the point of view of perfection. First, we shall define the perfect code, according to [4].

Definition 4. *A code $\mathcal{K}$ of k-sequences over the alphabet Q with the code distance $d=2l+1$ is said to be perfect if and only if for each k-sequence X over the alphabet Q not belonging to $\mathcal{K}$ there is a k-sequence Y in $\mathcal{K}$ such that the code distance between k-sequences X and Y is not greater than l.*

Theorem 2. *Every perfect code $\mathcal{K}$ of k-sequences, k even, over the alphabet* $Q=\{0, 1, \dots, q-1\}$, *with the code distance* $d=k-1$, *is a complete code.*

Proof. Let $\mathcal{K}$ be a perfect code of k-sequences (k even) over the alphabet Q with the code distance $k-1$. Since k is even, the code distance can be written as $d=2l+1$. Let us consider any k-sequence $X=a_1a_2 \dots a_k$ over Q which is not in $\mathcal{K}$. Then the code distance of the code $\mathcal{K} \cup \{X\}$ is not greater than $l<d$. Thus, $\mathcal{K}$ is a complete code.

The converse statement is not true, which is shown by the following theorem:

Theorem 3. *The complete code of* 4-*sequences corresponding to the* OSPK $\sum=\{E, F, A_1, A_2\}$ *from Theorem* 1 *is not perfect.*

Proof. Consider any complete code $\mathcal{K}$ containing $q(q-1)$ 4-sequences constructed according to the Theorem 1, with the code distance $d=3$. The distance of the 4-sequence $0(q-1)00$, which does not belong to the code $\mathcal{K}$, from any 4-sequence $Y=a_1a_2a_3a_4$ from $\mathcal{K}$ is greater than 1. Namely, in any case $a_2 \neq q-1$, because $A_1(i, q-1)$ and $A_2(i, q-1)$ are not defined. Furthermore, either $a_3 \neq 0$ or $a_4 \neq 0$ for arbitrary $a_1 \in Q$ and $a_2 \in Q \setminus \{q-1\}$, which follows from the particular way in which the rectangles P_1 and P_2 have been obtained. Thus, $\mathcal{K}$ is not a perfect code.

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O JEDNOJ KLASI D-PUNIH OSPK I PUNIH KODOVA KOJI ISPRAVLJAJU GREŠKE

Rezime

Opisana je jedna klasa D-punih OSPK i klasa njima odgovarajućih punih kodova za koje je pokazano da nisu perfektni. Obrnuto, dokazano je da je svaki perfektan kod k-nizova, za k parno, nad azbukom $Q=\{0,1,\dots,q\text{-}1\}$, sa kodnim rastojanjem $d=k\text{-}1$ — pun kod.